Eigenvector distribution in the critical regime of BBP transition

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In this paper, we study the random matrix model of Gaussian Unitary Ensemble (GUE) with fixed-rank (aka spiked) external source. We will focus on the critical regime of the Baik-Ben Arous-Péché (BBP) phase transition and establish the distribution of the eigenvectors associated with the leading eigenvalues. The distribution is given in terms of a determinantal point process with extended Airy kernel. Our result can be regarded as an eigenvector counterpart of the BBP eigenvalue phase transition [6]. The derivation of the distribution makes use of the recently re-discovered *eigenvector-eigenvalue identity*, together with the determinantal point process representation of the GUE minor process with external source.

1 Introduction

In this paper, we consider the Gaussian Unitary Ensemble (GUE) with fixed-rank external source, also known as the spiked GUE in the literature, denoted by

$$G_{\boldsymbol{\alpha}} \equiv G_{\boldsymbol{\alpha}}^{(N)} := G + \sum_{i=1}^{k} \alpha_i \mathbf{e}_i \mathbf{e}_i^*, \tag{1.1}$$

where $G = (g_{ij})_{N,N}$ is a standard N-dimensional GUE, i.e., $g_{ii} \sim \mathrm{N}(0,1)$ $(1 \le i \le N)$; $g_{ij} \sim \mathrm{N}(0,\frac{1}{2}) + \mathrm{i}\mathrm{N}(0,\frac{1}{2})$ $(1 \le i < j \le N)$ are independent random variables with standard real/complex normal distributions, and $g_{ji} = \overline{g_{ij}}$. Here $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ is a deterministic vector with fixed dimension k, and $\{\mathbf{e}_i\}$ is the standard basis of $\mathbb{R}^N$. The entire discussion in this paper works under the following more general setting

$$G_{\boldsymbol{\alpha},\mathbf{v}} = G + \sum_{i=1}^{k} \alpha_i \mathbf{v}_i \mathbf{v}_i^* \tag{1.2}$$

with any deterministic orthonormal vectors $\mathbf{v}_i \in \mathbb{C}^N$. Nevertheless, due to the unitary invariance of GUE, it would be sufficient to focus on the model in (1.1).

Throughout the paper, we will be focusing on the critical regime of the well-known Baik-Ben Arous-Péché (BBP) phase transition [6], and thus make the following assumption on α_i's

Assumption 1. *There exist fixed constants $a_1, \ldots, a_k \in \mathbb{R}$ such that*

$$\alpha_i = \sqrt{N} + N^{\frac{1}{6}} a_{k-i+1}, \quad i = 1, \ldots, k. \tag{1.3}$$

We emphasize here that α_i's are unordered. Whenever the ordered parameters are needed in some local discussion, we will use $\alpha_{(j)}$ to denote the j-th largest α_i.

We further denote the ordered eigenvalues of $G_{\boldsymbol{\alpha}}$ by [*]

$$\sigma_1 > \cdots > \sigma_N \tag{1.4}$$

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[*]Throughout this paper, for all the random matrices we consider, the eigenvalues are distinct with probability 1. Hence we always assume the simplicity of the eigenvalues without further explanation.

and set

$$\mathbf{x}_i = (x_{i1}, \ldots, x_{iN})^\top \tag{1.5}$$

to be the unit eigenvector associated with σ_i †. In this paper, we are primarily interested in the limiting distribution of $|x_{ij}|^2$'s with bounded i, after appropriate normalization, as the dimension $N \to \infty$. Observe that $|x_{ij}|^2$ can be understood as the square of the projection of eigenvector $\mathbf{x}_i$ onto the direction $\mathbf{e}_j$. Due to the unitary invariance, our results can also be applied to the projection $|\langle \mathbf{x}_i^{\mathbf{v}}, \mathbf{v}_j \rangle|^2$, where we used $\mathbf{x}_i^{\mathbf{v}}$ to denote the i-th eigenvector of $G_{\boldsymbol{\alpha},\mathbf{v}}$ in (1.2). Before we state the main results, we first give a brief review of the literature on the eigenvalue and eigenvector of random matrices with fixed-rank deformation, in Section 1.1, and then we present the definition of the extended Airy kernel in Section 1.2, with which we will then state our main results in Section 1.3.

1.1 Random matrix with fixed-rank deformation

Our model in (1.1) is in the category of the random matrices with fixed-rank deformation, which also includes the spiked sample covariance matrix and the signal-plus-noise model as typical examples. A vast amount of work has been devoted to understanding the limiting behavior of the extreme eigenvalues and the associated eigenvectors of the deformed models. Since the seminal work of Baik, Ben Arous and Péché [6], it is now well-understood that the extreme eigenvalues undergo a so-called BBP phase transition along with the change of the strength of the deformation. Specifically, there exists a critical threshold such that the extreme eigenvalue of the deformed matrix will stick to the right end point of the limiting spectral distribution if the strength of the deformation is less than or equal to the threshold, and will otherwise jump out of the support of the limiting spectral distribution. In the latter case we often call the extreme eigenvalue as an *outlier*. Moreover, the fluctuation of the extreme eigenvalues in different regimes (subcritical, critical and supercritical) are also identified in [6] for the complex spiked covariance matrix. Particularly, for the deformed GUE in (1.1), the phase transition takes place on the scale $\alpha_i = \sqrt{N} + \mathcal{O}(N^{1/6})$. Hence, for the deformed GUE, more specifically, the regimes $\alpha_i < \sqrt{N} - N^{1/6+\varepsilon}$, $\alpha_i = \sqrt{N} + \mathcal{O}(N^{1/6})$ and $\alpha_i > \sqrt{N} + N^{1/6+\varepsilon}$ will be referred to as subcritical, critical and supercritical, respectively, in the sequel. We also refer to [4], [7], [13], [14], [23], [30], [40], [45] and the reference therein for the first-order limit of the extreme eigenvalue of various related models. The fluctuation of the extreme eigenvalues of various models have been considered in [5], [4], [8], [9], [12], [17], [16], [19], [20], [25], [26], [31], [34], [40], [41], [45], [46], [47], [54], [55].

In parallel to the results of the extreme eigenvalues, there are some corresponding results on eigenvectors in the literature. Suppose $G_{\boldsymbol{\alpha}}$ is given in (1.1) while $\alpha_1, \ldots, \alpha_k$ are significantly away from the critical threshold, say, $\min_j |\alpha_j - \sqrt{N}| \geq \varepsilon\sqrt{N}$ for some positive constant ε, it is known that (i) if $\alpha_{(i)} \leq (1-\varepsilon)\sqrt{N}$, then $\mathbf{x}_i$, the eigenvector associated with the i-th largest eigenvalue σ_i, is delocalized in the sense $\|\mathbf{x}_i\|_\infty \leq N^{-1/2+\delta}$ for any small constant $\delta > 0$ with high probability; (ii) if $\alpha_{(i)} \geq (1+\varepsilon)\sqrt{N}$, then $\mathbf{x}_i$ has an order one bias on the direction of the deformation $\mathbf{e}_{(i)}$. Here we used $\alpha_{(i)}$ to denote the i-th largest value of all α_j's and $\mathbf{e}_{(i)}$ is the canonical basis vector with the corresponding index. In [13], [14], [22], [30], [45], the behavior of the extreme eigenvectors has been studied on the level of the first order limit. A detailed discussion of eigenvector behavior in the full subcritical regime and supercritical regime can be found in the recent work [18], which was done for the spiked sample covariance matrix. Especially, the discussion in [18] indicates that in case there is an α_j close to the critical threshold, i.e, $\alpha_j = (1+o(1))\sqrt{N}$, for any fixed i such that σ_i is not an outlier, $\mathbf{x}_i$ will have a bias of small order towards the direction of $\mathbf{e}_j$. On the level of the fluctuation, the limiting behavior of the extreme eigenvectors has not been fully studied yet. By establishing a general universality result of the eigenvectors of the sample covariance matrix in the null case, the authors of [18] are able to establish the law of the eigenvectors of the spiked covariance matrices in the subcritical regime. In this regime, the eigenvector distribution is similar to (up to appropriate scaling) that of the bulk and edge regime of Wigner matrices without spikes; see [21], [39], [52], [15] and [44] for instance. More specifically, in the subcritical regime, the limiting distribution of the square of eigenvector components (after appropriate scaling) is given by χ^2 distribution, which tells the asymptotic Gaussianity of the eigenvector components themselves (without taking square). Although the result was established for sample covariance matrix only in [18], it can be extended to deformed Wigner without essential difference. In the supercritical regime, the fluctuation of the eigenvectors was recently studied in [11], [10], [24] for

† Since the eigenvalues are assumed to be distinct, $\mathbf{x}_i$ are unique up to an angular factor. We ignore the angular factor since we consider only the moduli of the components throughout the paper.

various models with generally distributed matrix entry. For generally distributed deformed Wigner matrix, the leading eigenvector distribution is non-universal in the supercritical regime; see [24]. However, if one restricts the discussion to the deformed GUE, then the limiting distribution of the square of eigenvector components (after appropriate centering and scaling) is given by Gaussian. Although the discussion in [24] has only covered the regime $\alpha_i \geq (1+\varepsilon)\sqrt{N}$, one can use the approach in [10] to extend the result to the full supercritical regime $\alpha_i \geq \sqrt{N} + N^{1/6+\varepsilon}$.

The aforementioned works leave the eigenvector distribution in the critical regime undiscussed. In this paper, we will establish the eigenvector distribution in the critical regime, i.e., $\alpha_i = \sqrt{N} + \mathcal{O}(N^{1/6})$. Here although we are dealing with the deformed GUE only, our methodology and result reveal certain universality of eigenvector distribution of random matrices with fixed-rank deformation in the critical regime of BBP transition, within the class of unitary invariant ensemble. Especially, our discussion can apply similarly to the fixed-rank deformed Laguerre Unitary Ensemble (LUE) that is also known as spiked Wishart ensemble or spiked sample covariance matrices in statistics and on which the BBP transition is most intensively studied, and the fixed-rank deformed Jacobi Unitary Ensemble (JUE, aka MANOVA ensemble in statistics) by using the corresponding correlation kernel formulas in [2]. We expect universal asymptotic results in these models. We also remark here that the universality has not yet been proved or disproved for generally distributed Wigner matrices with fixed-rank perturbation in the critical regime of BBP transition, even on the eigenvalue level.

At last, we remark that for both Hermitian type (complex) random matrices and real symmetric type random matrices, the BBP transition may happen under fixed rank deformations. The study of eigenvalues there shows that in the critical regime the two types of random matrices have different universal behaviours and are usually investigated by different methods (except for [19], [20]), while in the the supercritical and subcritical regimes, the behaviours of the two types of random matrices have more common features and are usually investigated together, by some perturbative approaches which can often reduce the problems to those of the non-perturbed models. The previous research of eigenvectors in BBP transition, which is only in the supercritical and subcritical regimes, generally works for both types of random matrices and yields similar results for them. Our approach in the critical regime, however, is non-perturbative and works only for the Hermitian type random matrices, because it depends on the determinantal property that is not available for the real symmetric ones. The study of the eigenvectors in the critical regime of BBP transition for real symmetric type random matrices is more challenging and is out of the scope of the current paper.

1.2 Extended Airy kernel

In order to state our main results, we need to first introduce the extended Airy kernel in this subsection. Recall the Airy kernel that defines the celebrated Tracy-Widom distribution that is often seen in random matrix theory and interacting particle systems of the Kardar-Parisi-Zhang (KPZ) universality class, see [3], [27] and [48], and references therein,

$$K_{\mathrm{Airy}}(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \int_\sigma \mathrm{d}u \int_\gamma \mathrm{d}v \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-yv}} \frac{1}{u-v}, \tag{1.6}$$

where the contours γ and σ are as in Figure 1. They are nonintersecting and infinite contours; γ goes from $e^{-2\pi\mathrm{i}/3}\cdot\infty$ to $e^{2\pi\mathrm{i}/3}\cdot\infty$, and σ goes from $e^{-\pi\mathrm{i}/3}\cdot\infty$ to $e^{\pi\mathrm{i}/3}\cdot\infty$. We then define the extended Airy kernel depending on real parameters $a_1, a_2, \dots$, which is the correlation kernel of a determinantal point process at discrete time $t \in \mathbb{Z}_{\geq 0}$. For any $m_1, m_2 \in \mathbb{Z}_{\geq 0}$, we let

$$K^{m_1,m_2}_{\mathrm{Airy},\mathbf{a}}(x,y) = -\mathbf{1}(m_1<m_2)\mathbf{1}(x<y)\frac{1}{2\pi\mathrm{i}}\oint \frac{e^{(y-x)w}}{\prod_{j=m_1+1}^{m_2}(w-a_j)}\mathrm{d}w + \widetilde{K}^{m_1,m_2}_{\mathrm{Airy},\mathbf{a}}(x,y), \tag{1.7}$$

where

$$\widetilde{K}^{m_1,m_2}_{\mathrm{Airy},\mathbf{a}}(x,y) = \frac{1}{(2\pi\mathrm{i})^2}\int_\sigma \mathrm{d}u \int_\gamma \mathrm{d}v \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-yv}} \frac{\prod_{j=1}^{m_1}(u-a_j)}{\prod_{j=1}^{m_2}(v-a_j)} \frac{1}{u-v}, \tag{1.8}$$

such that the contour in (1.7) encloses all the poles $a_{m_1+1},\dots,a_{m_2}$ and in (1.8) all the poles $a_1,\dots,a_{m_2}$ of v are to the left of γ. We note that in the special $m_1=m_2=0$ case, the correlation kernel $K^{m_1,m_2}_{\mathrm{Airy},\mathbf{a}}(x,y)$ is reduced to $K_{\mathrm{Airy}}(x,y)$.

The Airy kernel K_{Airy} defines a determinantal point process, with infinitely many particles, ordered as

$$+\infty > \xi_1 > \xi_2 > \cdots, \tag{1.9}$$

and the n-point correlation function

$$R_n(x_1, \dots, x_n) = \det(K_{\text{Airy}}(x_i, x_j))_{i,j=1}^n. \tag{1.10}$$

Analogously, for each $m \geq 0$, the extended Airy kernel $K^{m,m}_{\text{Airy},\mathbf{a}}$ also defines a determinantal point process, with infinitely many particles, ordered as

$$+\infty > \xi^{(m)}_1 > \xi^{(m)}_2 > \cdots, \tag{1.11}$$

with $\xi^{(0)}_i \equiv \xi_i$ in (1.9). Furthermore, if we put all the $\xi^{(m)}_i$'s ($m \geq 0$, $i \geq 1$) together, they form a determinantal point process living in space $\mathbb{R}$ and time $m \in \mathbb{Z}_{\geq 0}$. The probability meaning of the extended Airy kernel is that it defines a determinantal point process with infinitely many species of particles, $\xi^{(m)}_i$ (particle index $i = 1, 2, \dots$, species index $m = 0, 1, \dots$), such that the marginal distribution of m-species particles is given by the correlation kernels $K^{m,m}_{\text{Airy},\mathbf{a}}$, and further the mixed correlation function is given by

$$\begin{aligned} &R_n(x_1, m_1; x_2, m_2; \dots; x_n, m_n) \\ :=& \lim_{\Delta x \to 0} \frac{1}{\Delta x^n} \mathbb{P}\left(\text{there exists a particle in } [x_i, x_i + \Delta x) \text{ at time } m_i \text{ for } i = 1, \dots, n\right) \\ =& \det\left(K^{m_i,m_j}_{\text{Airy},\mathbf{a}}(x_i, x_j)\right)^n_{i,j=1}. \end{aligned} \tag{1.12}$$

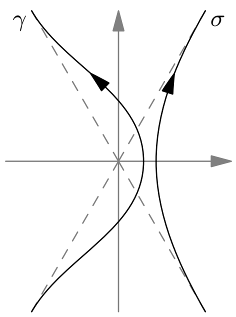

Figure 1: Contours γ and σ.

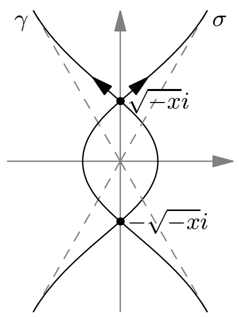

Figure 2: Double contour X consisting of γ and σ that are deformed through $\pm\sqrt{-x}i$.

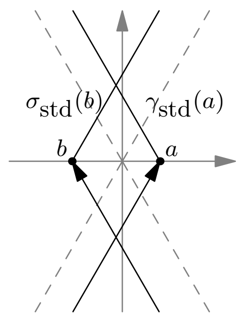

Figure 3: $\gamma_{\text{std}}(a)$ and $\sigma_{\text{std}}(b)$ that are standardized deformations of γ and σ.

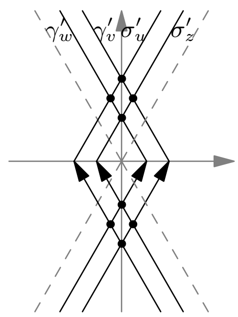

Figure 4: 4-fold contour $\mathbb{X}$ consisting of two pairs of γ and σ in standardized deformation.

Remark 1. It is not easy to check directly that the correlation functions in (1.12) are well-defined, since the kernel functions are generally non-Hermitian. (The necessary and sufficient condition for a Hermitian kernel to define a probabilistic determinantal point process is given in [50, Theorem 3].) However, since we know in Lemma 6 (see also [2]) that the extended Airy kernel is the limit of the correlation kernels of the GUE minor process with external source, we conclude thereby that the correlation functions in (1.12) are well-defined. Then it is not hard to see that the rightmost particle exists for each species. Also we have that the point process consisting of finitely many species of particles is simple (by [38, Remark 4]).

1.3 Main results

Recall the point process in (1.11) with any fixed parameter sequence $\mathbf{a} = (a_1, \dots, a_k)$. Define for any integers $n > j$ the random variable

$$\Xi^{(k)}_j(\mathbf{a}; n) := n^{\frac{1}{3}} \prod_{i=1}^{j-1} \frac{\xi^{(k)}_j - \xi^{(k-1)}_i}{\xi^{(k)}_j - \xi^{(k)}_i} \prod_{i=j+1}^{n} \frac{\xi^{(k)}_j - \xi^{(k-1)}_{i-1}}{\xi^{(k)}_j - \xi^{(k)}_i}. \tag{1.13}$$

Our first result is on the existence of the limit of $\Xi^{(k)}_j(\mathbf{a}; n)$ as $n \to \infty$.

Theorem 1. *For any $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$ with fixed components, and j a fixed positive integer,*

$$\Xi_j^{(k)}(\mathbf{a}; \infty) := \lim_{n \to \infty} \Xi_j^{(k)}(\mathbf{a}; n) \tag{1.14}$$

exists almost surely.

Our second result is on the distribution of the first k components of the eigenvectors associated with the largest eigenvalues. The theorem states for the first component, and see Remark 2 for the 2-nd, ..., k-th components.

Theorem 2. *Under Assumption 1, for any fixed j, we have*

$$N^{\frac{1}{3}} |x_{j1}|^2 \xrightarrow{d} \left(\frac{3\pi}{2}\right)^{\frac{1}{3}} \Xi_j^{(k)}(\mathbf{a}; \infty), \qquad \textit{as } N \to \infty. \tag{1.15}$$

Remark 2. Since our $\alpha_1, \ldots, \alpha_k$ are unordered, the first component corresponding to the $\alpha_1 \mathbf{e}_1 \mathbf{e}_1^*$ deformation has nothing special compared with the 2-nd, ..., k-th components, and we state the result for the first component only for notational simplicity. The result of Theorem 1 can be adapted for x_{jl} with any $l = 2, \ldots, k$ as follows: We consider, instead of $G_{\boldsymbol{\alpha}}$, the random matrix $\tilde{G}_{\boldsymbol{\alpha}}$ which is a conjugate of $G_{\boldsymbol{\alpha}}$ by switching the first row/column and the l-th row/column. Then

$$x_{jl} = \tilde{x}_{j1}, \tag{1.16}$$

where $\tilde{x}_{mn}$ means the n-th component of the normalized eigenvector of $\tilde{G}_{\boldsymbol{\alpha}}$ associated with σ_m (the m-th largest eigenvalue of $G_{\boldsymbol{\alpha}}$, which is also the m-th largest eigenvalue of $\tilde{G}_{\boldsymbol{\alpha}}$).

Remark 3. We further remark here that the result in Theorem 2 holds for any fixed j. Particularly, j can be even larger than k (but fixed). Note that σ_j is not an outlier or a critical spiked eigenvalue in case $j > k$. Hence, the result in Theorem 2 shows that a critical spike can even cause a bias of the eigenvectors associated with those non-outliers towards the direction of the spikes, since x_{j1} is typically of order $N^{-1/6}$ here while it would be of order $N^{-1/2}$ in case the deformation $\sum_{i=1}^k \alpha_i \mathbf{e}_i \mathbf{e}_i^*$ was absent, for instance. Such a phenomenon was previously observed in [18] when the spike is in the subcritical or supercritical regime, but sufficiently close to the critical regime. As $j \gg k$, this bias eventually peters out, and the decay speed deserves further study.

A consequence of Theorem 1 and Remark 2 is as follows:

Corollary 3. *Under Assumption 1, for a fixed j, the moduli of components $|x_{j,k+1}|, |x_{j,k+2}|, \ldots, |x_{jN}|$ have the same distribution, and we have, with $l > k$,*

$$N|x_{j\ell}|^2 \xrightarrow{d} \frac{1}{2}\chi^2(2), \qquad \textit{as } N \to \infty, \tag{1.17}$$

where $\chi^2(2)$ is the χ^2 random variable with parameter 2, which can be equivalently expressed as $\mathrm{Gamma}(1,1)$ *or* $\mathrm{Exp}(1)$*.*

Finally, the integrable property of the distribution of $\Xi_j^{(k)}(\mathbf{a}; \infty)$ will be the subject of further study, and here we only present the first step towards this direction: the non-degeneracy of the distribution.

Theorem 4. *Under the assumption of Theorem 1, the limit $\Xi_j^{(k)}(\mathbf{a}; \infty)$ is nondegenerate, i.e,*

$$\mathbb{P}(\Xi_j^{(k)}(\mathbf{a}; \infty) = x) < 1$$

for any $x \in \mathbb{R}$.

In the end of Section 7, we also present some simulation results, which show some features of the eigenvector distribution numerically.

1.4 Proof strategy

In order to prove Theorem 1, we turn to study the logarithm of $\Xi_j^{(k)}(\mathbf{a}; n)$ in (1.13), which can be further written as an integral of $1/(\xi_j^{(k)} - x)$ against a random measure μ, with the lower limit of the integral given by $\xi_n^{(k)}$; see (3.4) and (3.8). Specifically, the measure μ has density ϕ taking 1 on intervals $(\xi_i^{(k)}, \xi_{i-1}^{(k-1)}]$ for all i and 0 elsewhere. It is also clear that $\xi_n^{(k)} \to -\infty$ almost surely as $n \to \infty$. Hence, in order to show that the left tail of the integral in (3.8) is negligible, i.e., the integral is convergent, it suffices to study the property of the measure $\mu((x, \infty))$ when $x \to -\infty$. A key technical step in this part is Proposition 10, where we show that the random measure μ behaves like a halved Lebesgue measure when it acts on the interval (x, ∞) with $x \to -\infty$. Heuristically, since μ has density 1 on intervals $(\xi_i^{(k)}, \xi_{i-1}^{(k-1)}]$'s but 0 on $(\xi_i^{(k-1)}, \xi_i^{(k)}]$'s, Proposition 10 can be interpreted as: when i is large, $\xi_i^{(k)}$ is typically sitting neutrally between $\xi_i^{(k-1)}$ and $\xi_{i-1}^{(k-1)}$ and does not favor either side.

Technically, the complementary distribution function, $\mu((x, +\infty))$, can be expressed (approximately) in terms of the difference between two linear statistics: one of the species $\{\xi_i^{(k)}\}$ with the test function $h_x(t) = (-t + x)\mathbf{1}(t > x)$, and the other of the species $\{\xi_i^{(k-1)}\}$ with the same test function; see (4.5). The computation of the linear statistics of determinantal point processes, especially those from random matrix models, is an extensively studied area, and results are abundant. The tricky part in our case is that up to the leading term, the asymptotics of the two linear statistics are the same. We take the advantage of our model that both the mean and variance of the difference of the two linear statistics have an exact formula; see (4.13) and (4.16). For our purpose, it suffices to take limits of the mean and variance formulas. This is done in the proof of Proposition 10 via a rather delicate saddle point analysis involving a series of contour deformations. We also refer to the recent work [32] for a study on the fluctuation of the difference of linear eigenvalue statistics of a Wigner matrix and its minor, where two strongly correlated linear statistics have a significant cancellation. The work [32] indicates that exploiting the true size of the fluctuation of difference between the linear statistics of two interlacing point processes is normally more delicate than that of a single linear statistic for an individual point process. Although the result in [32] is more on the bulk regime of random matrices, while here we are discussing two interlacing species of the extended Airy process, our analysis also indicates the general complexity of the difference between two linear statistics.

For the proof of Theorem 2, we start from the celebrated *eigenvector-eigenvalue identity.* We refer the interested readers to the recent survey [29] and reference therein for a detailed discussion and a history of this identity. Here we cite the identity directly from [29] with slight modification as the following proposition.

Proposition 5. *Let $A \in \mathbb{C}^{n\times n}$ be an Hermitian matrix and let $M_j \in \mathbb{C}^{(n-1)\times(n-1)}$ be its minor obtained by deleting the j-th column and row from A. Denote by $\lambda_1(A) > \cdots > \lambda_n(A)$ the ordered eigenvalues of A and by $\lambda_1(M_j) > \cdots > \lambda_{n-1}(M_j)$ the ordered eigenvalues of M_j. Furthermore, let $\mathbf{v}_i = (v_{i1}, \dots, v_{in})^\top$ be the eigenvector of A, associated with $\lambda_i(A)$. Then one has*

$$|v_{ij}|^2 = \frac{\prod_{\ell=1}^{n-1}(\lambda_i(A) - \lambda_\ell(M_j))}{\prod_{\ell\in\{1,\dots,n\}\setminus\{i\}}(\lambda_i(A) - \lambda_\ell(A))}. \tag{1.18}$$

Applying the eigenvector-eigenvalue identity to our random matrix $G_{\boldsymbol{\alpha}} = G_{\boldsymbol{\alpha}}^{(N)}$ and its minor $G_{\boldsymbol{\alpha}}^{(N-1)}$ which is constructed by removing the first column and first row of $G_{\boldsymbol{\alpha}}$, we can write

$$|x_{j1}|^2 = \prod_{i=1}^{j-1} \frac{\sigma_j - \lambda_i}{\sigma_j - \sigma_i} \prod_{i=j+1}^{N} \frac{\sigma_j - \lambda_{i-1}}{\sigma_j - \sigma_i}, \tag{1.19}$$

where $\sigma_1 > \sigma_2 > \cdots > \sigma_N$ are eigenvalues of $G_{\boldsymbol{\alpha}}$, $\lambda_1 > \cdots > \lambda_{N-1}$ are eigenvalues of $G_{\boldsymbol{\alpha}}^{(N-1)}$, and x_{j1} is the first component of the normalized eigenvector of $G_{\boldsymbol{\alpha}}$ associated with σ_j. Equivalently, we have

$$N^{\frac{1}{3}}|x_{j1}|^2 = N^{\frac{1}{3}} \prod_{i=1}^{j-1} \frac{\sigma_j - \lambda_i}{\sigma_j - \sigma_i} \prod_{i=j+1}^{N} \frac{\sigma_j - \lambda_{i-1}}{\sigma_j - \sigma_i}, \tag{1.20}$$

which resembles formally $\Xi_j^{(k)}(\mathbf{a}; N)$ in (1.13).

First, for the fixed i terms in (1.20), we apply the result of GUE minor process with external source in [2] which shows that the two species of point process $\{\sigma_i\}$ and $\{\lambda_i\}$ together form a determinantal point process (see also [35]). We first show in Lemma 6 that the edge scaling limit of this point process is given by a determinantal point process with extended Airy kernel defined in (1.12), under Assumption 1. Consequently, we have

$$\frac{\sigma_j - \lambda_i}{\sigma_j - \sigma_i} \xrightarrow{d} \frac{\xi_j^{(k)} - \xi_i^{(k-1)}}{\xi_j^{(k)} - \xi_i^{(k)}} \tag{1.21}$$

for any fixed $i \neq j$. This gives an indication of the connection between $N^{1/3}|x_{j1}|^2$ and the limit $\Xi_j^{(k)}(\mathbf{a};\infty)$. However, the weak convergence in (1.21) cannot be applied directly to the double limit case when $i = i(N) \to \infty$ as $N \to \infty$. Hence, besides Lemma 6, we need to analyze the product of the large i terms in (1.20). To this end, we mimic the idea of the proof of Theorem 1. Again, we turn to consider the logarithm of the product in (1.20) (but over the large i terms only), which can be written as an integral of $1/(\sigma_j - x)$ against a random measure μ_N with the lower limit of the integral given by σ_N; see (5.1) and (5.3). Specifically, the measure μ_N has the density ϕ_N taking 1 on $(\sigma_i, \lambda_{i-1}]$ for all $i = 2, \ldots, N$ and 0 elsewhere. Since the domain of the integral (5.3), i.e., $[\sigma_N, \sigma_L]$ contains both the edge and bulk regimes of the semicircle law, and the point process $\{\xi_i^{(k)}\}$ only approximates the matrix eigenvalues in the edge regime (with an effective extension to certain order of intermediate regime), we need to further decompose the domain $[\sigma_N, \sigma_L]$ into two parts: $[\sigma_N, \sigma_L] = [\sigma_N, \sigma_{N_0}) \cup [\sigma_{N_0}, \sigma_L]$, with $N_0 := \lfloor 2/(3\pi) N^{1/10} \rfloor$. We will then show that under appropriate scaling, the random measure μ_N can be well approximated by μ on the domain $[\sigma_{N_0}, \sigma_L]$. Furthermore, a detailed analysis of the measure μ_N on $[\sigma_N, \sigma_{N_0}]$ shows that the integral $1/(\sigma_j - x)$ over this domain is approximately deterministic, and thus does not contribute to the randomness of the limit of $N^{1/3}|x_{j1}|^2$.

Especially, our result Theorem 2 and its proof show that the randomness of $N^{1/3}|x_{j1}|^2$ essentially depends only on the local edge regime of the eigenvalues of $G_{\boldsymbol{\alpha}}$ and $G_{\boldsymbol{\alpha}}^{(N-1)}$, and the bulk eigenvalues contribute a deterministic factor. Similar phenomenon has also shown up in the limiting theorem of some other eigenvalue statistics in the literature, which inspires our current work. For instance, in the recent work [42], the weak limit of the statistic

$$\frac{1}{\sqrt{N}} \sum_{j=2}^{N} \frac{1}{\Lambda_1 - \Lambda_j} \tag{1.22}$$

is identified to be an infinite sum given in terms of the Airy process, where $\Lambda_1 > \cdots > \Lambda_N$ are the ordered eigenvalues of Gaussian Orthogonal Ensemble (GOE). The randomness of (1.22) essentially comes from the edge part of the sum and the bulk part only contributes collectively in a deterministic manner. We also refer to [57] for a related discussion.

For the proof of the nondegeneracy of the distribution of $\Xi_j^{(k)}(\mathbf{a};\infty)$, i.e., Theorem 4, we will apply the weak convergence in Theorem 2 to translate the question to $N^{1/3}|x_{j1}|^2$. An advantage of the latter is that the eigenvalue distribution admits a log-gas representation which can facilitate our analysis. More specifically, we will show that a bounded truncation (from both below and above) of $\log(N^{1/3}|x_{j1}|^2)$ has a lower bound for variance, uniformly in N. This will finally lead to the conclusion of Theorem 4.

Finally, we remark here that in this paper we do not consider the deformed GOE since the corresponding results on the minor process used for GUE here is not available for GOE so far. But many discussions in this paper work for the deformed GOE as well.

1.5 Organization and notation

The rest of the paper is organized as follows. In Section 2, we state some preliminary results which will be used in the later sections. The main results of the paper, Theorems 1 and 2 are proved respectively in Sections 3 and 5, based on the key technical estimates in Propositions 10 and 11, whose proofs will be stated respectively in Sections 4 and 6. In addition, Corollary 3 is an easy consequence of Theorem 2 and Remark 2, and its proof will be stated at the end of Section 5. Section 7 is then devoted to the proof of Theorem 4. Finally, some proofs of technical lemmas are collected in Appendix A.

Throughout the paper, we will use $\mathcal{O}(\cdot)$ and $o(\cdot)$ for the standard big-O and little-o notations. We use C, C', C_1, etc. to denote positive constants (independent of N). For any set $\mathcal{I}$, the notation $|\mathcal{I}|$ stands for its cardinality. We use the shorthand notation $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$, for all $a, b \in \mathbb{R}$.

2 Collection of results for spiked GUE and extended Airy process

In this section, we first review the determinantal point process representation of the eigenvalue distribution of $G_{\boldsymbol{\alpha}}$ and its minors, and then state some preliminary results to be used in the later sections. Some lemmas in this section are direct consequences of existing results in the literature, while for others, only the proof strategy exists in the literature. We give the proofs and/or the references of the lemmas in Appendix A.

2.1 Determinantal point process representation for eigenvalues of $G_{\boldsymbol{\alpha}}$

In this subsection, we re-denote the eigenvalues of $G_{\boldsymbol{\alpha}} = G_{\boldsymbol{\alpha}}^{(N)}$ and $G_{\boldsymbol{\alpha}}^{(N-1)}$ by

$$\lambda_i^{(N)} := \sigma_i, \quad \lambda_\ell^{(N-1)} := \lambda_\ell, \quad \text{where} \quad i \in [\![1, N]\!], \quad \ell \in [\![1, N-1]\!] \tag{2.1}$$

and further we denote by $\lambda_1^{(N-j)} > \cdots > \lambda_{N-j}^{(N-j)}$ the ordered eigenvalues of the matrix $G_{\boldsymbol{\alpha}}^{(N-j)}$, which is obtained from $G_{\boldsymbol{\alpha}}$ by deleting its first j rows and columns.

For each j, the eigenvalues of $G_{\boldsymbol{\alpha}}^{(N-j)}$ form a determinantal point process. In addition, the eigenvalues of all $G_{\boldsymbol{\alpha}}^{(N-j)}$, $j = 0, \ldots, k$ together form a determinantal point process, called the *GUE minor process with external source* [2]. The correlation kernel of these eigenvalues is given as follows: For $j_1, j_2 \in [\![0 \ldots, k]\!]$,

$$K_{\mathrm{GUE},\boldsymbol{\alpha}}^{j_1,j_2}(x,y) = -\mathbf{1}(j_1 > j_2)\mathbf{1}(x < y)\frac{1}{2\pi \mathrm{i}}\int_\Gamma \frac{e^{(y-x)w}}{\prod_{i=j_2+1}^{j_1}(w-\alpha_j)}\mathrm{d}w + \widetilde{K}_{\mathrm{GUE},\boldsymbol{\alpha}}^{j_1,j_2}(x,y), \tag{2.2}$$

where

$$\widetilde{K}_{\mathrm{GUE},\boldsymbol{\alpha}}^{j_1,j_2}(x,y) = \frac{1}{(2\pi \mathrm{i})^2}\int_\Sigma \mathrm{d}z \int_\Gamma \mathrm{d}w \frac{e^{\frac{z^2}{2}-xz} z^{N-k}}{e^{\frac{w^2}{2}-yw} w^{N-k}} \frac{\prod_{i=j_1+1}^{k}(z-\alpha_i)}{\prod_{i=j_2+1}^{k}(w-\alpha_i)} \frac{1}{z-w}. \tag{2.3}$$

Here Γ is a circular contour wrapping 0 and $\alpha_1, \ldots, \alpha_k$ positively, and Σ is an infinite contour from $-i \cdot \infty$ to $i \cdot \infty$, to the right of Γ; see Figure 10.

The relation between the correlation kernel $K_{\mathrm{GUE},\boldsymbol{\alpha}}^{j_1,j_2}(x,y)$ and the correlation kernel $K_{\mathrm{Airy},\mathbf{a}}^{k_1,k_2}(x,y)$ defined in (1.7) is as follows:

Lemma 6. *Let $\alpha_1, \ldots, \alpha_k$ be given in Assumption 1. If we denote, with $j_i = k - k_i \in [\![0, k]\!]$ $(i = 1, 2)$,*

$$K_{N,\mathrm{scaled}}^{j_1,j_2}(x,y) = N^{-\frac{1}{6}} N^{\frac{j_1-j_2}{6}} e^{N^{\frac{1}{3}}(x-y)} K_{\mathrm{GUE},\boldsymbol{\alpha}}^{j_1,j_2}(2\sqrt{N} + N^{-\frac{1}{6}}x, 2\sqrt{N} + N^{-\frac{1}{6}}y), \tag{2.4}$$

then

1. *For all x, y in a compact subset of $\mathbb{R}$, uniformly*

$$\lim_{N\to\infty} K_{N,\mathrm{scaled}}^{j_1,j_2}(x,y) = K_{\mathrm{Airy},\mathbf{a}}^{k_1,k_2}(x,y). \tag{2.5}$$

2. *Let $\epsilon > \max(a_1, \ldots, a_k)$. For any $C \in \mathbb{R}$, as operators on $L^2([C, +\infty))$, the ones with kernel $e^{\epsilon(x-y)}K_{N,\mathrm{scaled}}^{j_1,j_2}(x,y)$ converge, in trace norm, to the one with kernel $e^{\epsilon(x-y)}K_{\mathrm{Airy},\mathbf{a}}^{k_1,k_2}(x,y)$.*

Consequently, the above statements imply the weak convergence of the joint distribution of $\{N^{1/6}(\lambda_i^{(N-j)} - 2\sqrt{N}) \mid 0 \le j \le k,\ 1 \le i \le K\}$ to that of $\{\xi_i^{(k-j)} \mid 0 \le j \le k,\ 1 \le i \le K\}$ for any fixed positive integer K, where $\xi_i^{(k-j)}$'s are particles in the determinantal point process given by (1.12).

We note that although we discussed the joint distribution of $\lambda_i^{(N-j)}$ for all $j \in [\![0, k]\!]$, to prove Theorem 2, we only need the $j = 0, 1$ cases. Hence we recycle the notations σ_i and λ_i for $\lambda_i^{(N)}$ and $\lambda_i^{(N-1)}$ respectively in the sequel, for simplicity.

2.2 Useful estimates

By the well-known Weyl's inequality, we have that

$$\sigma_1 > \lambda_1 > \sigma_2 > \lambda_2 > \cdots > \lambda_{N-1} > \sigma_N, \quad \text{and more generally} \quad \lambda_1^{(N-j)} > \lambda_1^{(N-j-1)} > \lambda_2^{(N-j)} > \cdots > \lambda_{N-j}^{(N-j)} \tag{2.6}$$

for all $j = 0, 1, \ldots, N-1$ ‡. In parallel, for the particles in the extended Airy process defined in (1.12), we also have the following interlacing property.

Lemma 7. *Let $j = 1, \ldots, k$. With probability 1, the particles $\xi_i^{(j)}$ and $\xi_i^{(j-1)}$ ($i = 1, 2, \ldots$) whose joint distribution is given by* (1.12) *satisfy the interlacing inequality*

$$+\infty > \xi_1^{(j)} > \xi_1^{(j-1)} > \xi_2^{(j)} > \xi_2^{(j-1)} > \cdots. \tag{2.7}$$

Next, we list some rigidity result for the particles in the extended Airy process and also the eigenvalues of $G_{\boldsymbol{\alpha}}$.

Lemma 8. *Let $\xi_j^{(k)}$, $j = 1, \ldots$ be the particles defined in* (1.12)*. Then we have the following estimates.*

1. *For any fixed j, there exists a numerical $C > 0$, such that for all $t > 0$*

$$\mathbb{P}(\xi_j^{(k)} > t) < Ce^{-t/C}, \quad \mathbb{P}(\xi_j^{(k)} < -t) < Ce^{-t/C}. \tag{2.8}$$

2. *For all $n \geq 2$, there exists a numerical $c > 0$, such that*

$$\mathbb{P}\left(\left|\xi_n^{(k)} + \left(\frac{3\pi n}{2}\right)^{2/3}\right| > n^{\frac{3}{5}}\right) \leq cn^{-\frac{6}{5}} \log n. \tag{2.9}$$

Lemma 9. *Suppose $N > k$ is large enough, then:*

1. *For $j \in [\![1, k]\!]$, there exists a numerical $C > 0$, such that*

$$\mathbb{P}(\sigma_j > 2\sqrt{N} + tN^{-\frac{1}{6}}) < Ce^{-t/C}, \quad \mathbb{P}(\sigma_N < -2\sqrt{N} - tN^{-\frac{1}{6}}) < Ce^{-t/C}, \quad \textit{for all } t > 0, \tag{2.10}$$

$$\mathbb{P}(\sigma_j < 2\sqrt{N} - tN^{-\frac{1}{6}}) < Ce^{-t/C}, \quad \mathbb{P}(\sigma_N > -2\sqrt{N} + tN^{-\frac{1}{6}}) < Ce^{-t/C}, \quad \textit{for all } 2 \leq t \leq 2N^{2/3}. \tag{2.11}$$

2. *For any positive constant $C > 0$, there exists a numerical $c > 0$, such that for all $2 \leq n \leq CN^{1/10}$,*

$$\mathbb{P}\left(\left|N^{\frac{1}{6}}(\sigma_n - 2\sqrt{N}) + \left(\frac{3\pi n}{2}\right)^{2/3}\right| > n^{\frac{3}{5}}\right) \leq cn^{-\frac{6}{5}} \log n. \tag{2.12}$$

3. *Given any (small) $c \in (0, 1)$, for all $i \in [\![1, (1-c)N]\!]$, for any (small) constant $\varepsilon > 0$ and (large) constant $D > 0$*

$$\mathbb{P}\left(|\sigma_i - \Upsilon_i| \geq N^{-\frac{1}{6}+\varepsilon} i^{-\frac{1}{3}}\right) \leq N^{-D}. \tag{2.13}$$

   Here Υ_i is the scaled quantile of semicircle law defined by

$$\int_{\Upsilon_i/\sqrt{N}}^{2} \rho_{sc}(x)\mathrm{d}x = \frac{N - i + \frac{1}{2}}{N}, \quad \rho_{sc}(x) = \frac{1}{2\pi}\sqrt{(4 - x^2)_+}. \tag{2.14}$$

All the lemmas stated in this section are proved in Appendix A.

‡ Hereafter we ignore the probability 0 event that some $\lambda_i^{(N-j-1)}$ is identical to $\lambda_i^{(N-j)}$ or $\lambda_{i+1}^{(N-j)}$.

3 Existence of the limit

In this section, we prove Theorem 1. Recall (1.13). For any $1 \le j \le k < m < n$, we set the quantity

$$\Xi_j^{(k)}(\mathbf{a};n) = n^{\frac{1}{3}} \prod_{i=1}^{j-1} \frac{\xi_j^{(k)} - \xi_i^{(k-1)}}{\xi_j^{(k)} - \xi_i^{(k)}} \prod_{i=j+1}^{m} \frac{\xi_j^{(k)} - \xi_{i-1}^{(k-1)}}{\xi_j^{(k)} - \xi_i^{(k)}} \mathcal{A}_{j,m}^{(k)}(\mathbf{a};n), \quad \text{where} \quad \mathcal{A}_{j,m}^{(k)}(\mathbf{a};n) := \prod_{i=m+1}^{n} \frac{\xi_j^{(k)} - \xi_{i-1}^{(k-1)}}{\xi_j^{(k)} - \xi_i^{(k)}}. \tag{3.1}$$

In order to prove Theorem 1, it suffices to show that for any given $m > k$, $\log \mathcal{A}_{j,m}^{(k)}(\mathbf{a};n) + \frac{1}{3}\log n$ converges almost surely as $n \to \infty$, or equivalently, by Cauchy's criterion,

$$\limsup_{m\to\infty} \sup_{n>m} \left| \log \mathcal{A}_{j,m}^{(k)}(\mathbf{a};n) + \frac{1}{3}\log\frac{n}{m} \right| = 0, \quad \text{a.s.}. \tag{3.2}$$

In order to study

$$\log \mathcal{A}_{j,m}^{(k)}(\mathbf{a};n) = \sum_{i=m+1}^{n} \Big(\log(\xi_j^{(k)} - \xi_{i-1}^{(k-1)}) - \log(\xi_j^{(k)} - \xi_i^{(k)}) \Big), \tag{3.3}$$

we define a random measure $\mu = \mu^{(k)}$ on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure, and is given by a random density function $\phi(x)$

$$\phi(x) := \begin{cases} 1, & \xi_i^{(k)} < x \le \xi_{i-1}^{(k-1)} \text{ for all } i > 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Since $\xi_1^{(k)} < +\infty$ almost surely, we have that

$$M(x) := \mu((x,+\infty)) = \int_x^{\infty} \phi(t)\mathrm{d}t \tag{3.5}$$

is a well defined random, continuous and piecewise linear function of $x \in \mathbb{R}$. We also define the deterministic function

$$F(x) := \mathbb{E}\, M(x). \tag{3.6}$$

It is clear that as $x \to +\infty$, $M(x) \to 0$ almost surely, and $F(x) \to 0$. We have the following estimate of $F(x)$ and $M(x)$, whose proof will be given in the end of this section.

Proposition 10. *As the negative parameter $x \to -\infty$, we have*

$$F(x) = \frac{-x}{2} + \mathcal{O}(1), \quad \operatorname{Var} M(x) = \mathbb{E}\, M^2(x) - F^2(x) = \frac{2}{\pi}\sqrt{-x} + \mathcal{O}(|x|^{\frac{1}{4}}). \tag{3.7}$$

Then we write

$$\log \mathcal{A}_{j,m}^{(k)}(\mathbf{a};n) = \int_{\xi_n^{(k)}}^{\xi_m^{(k-1)}} \frac{-1}{\xi_j^{(k)} - x}\mathrm{d}\mu(x) = \int_{\xi_n^{(k)}}^{\xi_m^{(k-1)}} \frac{1}{\xi_j^{(k)} - x}\mathrm{d}M(x) = \int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{\xi_j^{(k)} - x}\mathrm{d}M(x), \tag{3.8}$$

where in the last step we used the fact that $\mathrm{d}M(x) = 0$ on $(\xi_m^{(k-1)}, \xi_m^{(k)}]$ by definition. Hence (3.2) is equivalent to

$$\lim_{m\to\infty} \sup_{n_2>n_1>m} D_{n_1,n_2} = 0, \quad \text{a.s.}, \quad \text{where} \quad D_{n_1,n_2} = \left| \int_{\xi_{n_2}^{(k)}}^{\xi_{n_1}^{(k)}} \frac{1}{\xi_j^{(k)} - x}\mathrm{d}M(x) + \frac{1}{3}\log\frac{n_2}{n_1} \right|. \tag{3.9}$$

Or else, we can use $F(x)$ to rewrite (3.9) as

$$\lim_{m\to\infty} \sup_{n_2>n_1>m} \tilde{D}_{n_1,n_2} = 0, \quad \text{a.s.}, \quad \text{where} \quad \tilde{D}_{n_1,n_2} = \left| \int_{\xi_{n_2}^{(k)}}^{\xi_{n_1}^{(k)}} \frac{1}{\xi_j^{(k)} - x}\mathrm{d}M(x) - \int_{-(3\pi n_2/2)^{2/3}}^{-(3\pi n_1/2)^{2/3}} \frac{1}{0 - x}\mathrm{d}F(x) \right|, \tag{3.10}$$

in light of the first identity in (3.7). It is equivalent to show: for any $\epsilon_1, \epsilon_2 > 0$, there exists $m_{\epsilon_1,\epsilon_2}$, such that for all $m > m_{\epsilon_1,\epsilon_2}$, $\mathbb{P}\left(\sup_{n_2>n_1>m} \tilde{D}_{n_1,n_2} > \epsilon_1\right) < \epsilon_2$. Furthermore, we see that it suffices to show, under the same assumption,

$$\mathbb{P}\left(\sup_{n>m} \tilde{D}_{m,n} > \epsilon_1\right) = \mathbb{P}\left(\sup_{n>m}\left|\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{\xi_j^{(k)}-x}\mathrm{d}M(x) - \int_{-(3\pi n/2)^{2/3}}^{-(3\pi m/2)^{2/3}} \frac{1}{0-x}\mathrm{d}F(x)\right| > \epsilon_1\right) < \epsilon_2. \tag{3.11}$$

Here we note that by Lemma 8, the constants 0, $-(3\pi m/2)^{2/3}$ and $-(3\pi n/2)^{2/3}$ approximate the values of $\xi_j^{(k)}$, $\xi_m^{(k)}$ and $\xi_n^{(k)}$ respectively.

Therefore, in order to prove Theorem 1, it suffices to show (3.11) in the sequel.

Remark 4. Here we remark that the proof of (3.11) relies only on the following three ingredients: (i) Proposition 10 on the properties of $F(x)$ and $M(x)$, (ii) Lemma 8 on the fluctuation of $\xi_j^{(k)}$ and $\xi_n^{(k)}$, and (iii) the property that both $\mathrm{d}F(x)$ and $\mathrm{d}M(x)$ are dominated by the Lebesgue measure. Our proof below could be potentially simplified. Nevertheless, for coherence, we keep the current presentation, since it can be easily adapted in the later proof of (5.15).

Proof of (3.11). We carry out the proof in three steps. To facilitate the proof, for $j \in [\![1,k]\!]$ and $m > k$, we denote by $\Omega_{j,m}^{(k)}$ the events that $\xi_m^{(k)}$ and $\xi_j^{(k)}$ satisfy the following rigidity properties

$$\Omega_{j,m}^{(k)} := \left\{\omega \;\middle|\; |\xi_j^{(k)}| \le m^{\frac{3}{5}},\ \left|\xi_m^{(k)} + (3\pi m/2)^{\frac{2}{3}}\right| \le m^{\frac{3}{5}}\right\}, \tag{3.12}$$

and also denote $\Omega_{j,m,n}^{(k)} = \Omega_{j,m}^{(k)} \cap \Omega_{j,n}^{(k)}$.

1. We first note that given any $\epsilon_1 > 0$, there exists m_{ϵ_1} such that for all $n > m > m_{\epsilon_1}$,

$$\begin{aligned}
&\left|\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{\xi_j^{(k)}-x}\mathrm{d}F(x) - \int_{-(3\pi n/2)^{2/3}}^{-(3\pi m/2)^{2/3}} \frac{1}{0-x}\mathrm{d}F(x)\right| \mathbf{1}(\Omega_{j,m,n}^{(k)})\\
\le &\left|\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{\xi_j^{(k)}-x}\mathrm{d}F(x) - \int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{0-x}\mathrm{d}F(x)\right| \mathbf{1}(\Omega_{j,m,n}^{(k)})\\
&+\left|\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{0-x}\mathrm{d}F(x) - \int_{-(3\pi n/2)^{2/3}}^{-(3\pi m/2)^{2/3}} \frac{1}{0-x}\mathrm{d}F(x)\right| \mathbf{1}(\Omega_{j,m,n}^{(k)})\\
< &\frac{\epsilon_1}{3} + \frac{\epsilon_1}{3} = \frac{2\epsilon_1}{3}.
\end{aligned} \tag{3.13}$$

2. We next show that given any $\epsilon_1, \epsilon_2 > 0$, there exists sufficiently large positive integer $m''_{\epsilon_1,\epsilon_2}$ such that for all $m > m''_{\epsilon_1,\epsilon_2}$

$$\mathbb{P}\left(\sup_{n>m}\left|\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{\xi_j^{(k)}-x}\mathrm{d}(M(x)-F(x))\mathbf{1}(\Omega_{j,m,n}^{(k)})\right| > \frac{\epsilon_1}{3}\right) < \frac{\epsilon_2}{2}. \tag{3.14}$$

   To see it, using integration by parts, we write

$$\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{\xi_j^{(k)}-x}\mathrm{d}(M(x)-F(x)) = \frac{M(\xi_m^{(k)})-F(\xi_m^{(k)})}{\xi_j^{(k)}-\xi_m^{(k)}} - \frac{M(\xi_n^{(k)})-F(\xi_n^{(k)})}{\xi_j^{(k)}-\xi_n^{(k)}} - \int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{M(x)-F(x)}{(\xi_j^{(k)}-x)^2}\mathrm{d}x. \tag{3.15}$$

   So to prove (3.14), we only need to show that for a large enough $m''_{\epsilon_1,\epsilon_2}$, if $m > m''_{\epsilon_1,\epsilon_2}$,

$$\mathbb{P}\left(\left|\frac{M(\xi_m^{(k)})-F(\xi_m^{(k)})}{\xi_j^{(k)}-\xi_m^{(k)}}\mathbf{1}(\Omega_{j,m}^{(k)})\right| > \frac{\epsilon_1}{9}\right) < \frac{\epsilon_2}{6}, \tag{3.16}$$

$$\mathbb{P}\left(\sup_{n>m}\left|\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{M(x)-F(x)}{(\xi_j^{(k)}-x)^2}\mathrm{d}x\mathbf{1}(\Omega_{j,m,n}^{(k)})\right| > \frac{\epsilon_1}{9}\right) < \frac{\epsilon_2}{6}. \tag{3.17}$$

(a) For (3.16), we note that since $(\xi_j^{(k)} - \xi_m^{(k)})/(3\pi m/2)^{2/3} = 1 + \mathcal{O}(m^{-1/15})$ on $\Omega_{j,m}^{(k)}$, it suffices to show

$$\mathbb{P}\left(\frac{|M(\xi_m^{(k)}) - F(\xi_m^{(k)})|}{(3\pi m/2)^{2/3}}\mathbf{1}(\Omega_{j,m}^{(k)}) > \frac{\epsilon_1}{9}\right) < \frac{\epsilon_2}{6}. \tag{3.18}$$

To prove (3.18), we first set

$$\ell_i = \left\lceil \frac{2}{3\pi} i^{3/2} \right\rceil. \tag{3.19}$$

Then for any $m \in [\![\ell_i, \ell_{i+1}]\!]$, $|M(\xi_m^{(k)}) - F(\xi_m^{(k)})| \leq A_i + B_i(m) + C_i(m)$, where

$$A_i = |M(-i) - F(-i)|, \quad B_i(m) = |M(\xi_m^{(k)}) - M(-i)|, \quad C_i(m) = |F(\xi_m^{(k)}) - F(-i)|. \tag{3.20}$$

By the estimate in Proposition 10 and Markov's inequality, we have that if i is big enough, then for any $\epsilon > 0$

$$\mathbb{P}\left(|M(-i) - F(-i)| \geq \epsilon i\right) < \epsilon^{-2} i^{-3/2}, \tag{3.21}$$

and we conclude that there exists a sufficiently large positive integer $N_{\epsilon_1,\epsilon_2}$ such that

$$\mathbb{P}\left(\sup_{i > N_{\epsilon_1,\epsilon_2}} \frac{A_i}{(3\pi\ell_i/2)^{2/3}} > \frac{\epsilon_1}{27}\right) < \frac{\epsilon_2}{6}. \tag{3.22}$$

Using the properties that $|F(x_1) - F(x_2)| \leq |x_1 - x_2|$ and $|M(x_1) - M(x_2)| \leq |x_1 - x_2|$, we have that there exists $N'_{\epsilon_1,\epsilon_2}$ such that if $i > N'_{\epsilon_1,\epsilon_2}$, then for all $m \in [\![\ell_i, \ell_{i+1}]\!]$, we have that on $\Omega_{j,m}^{(k)}$,

$$\frac{|\star_i(m)|}{(3\pi m/2)^{2/3}} \leq \frac{|\xi_m^{(k)} - (-i)|}{(3\pi m/2)^{2/3}} \leq \frac{|[-(3\pi\ell_{i+1}/2)^{2/3} - \ell_{i+1}^{3/5}] - [-(3\pi\ell_i/2)^{2/3}]|}{(3\pi m/2)^{2/3}} < \frac{\epsilon_1}{27}, \quad \star = B \text{ or } C. \tag{3.23}$$

Hence we have that if $i > \max(N_{\epsilon_1,\epsilon_2}, N'_{\epsilon_1,\epsilon_2})$ and $m \in [\![\ell_i, \ell_{i+1}]\!]$, then by (3.23)

$$\frac{|M(\xi_m^{(k)}) - F(\xi_m^{(k)})|}{(3\pi m/2)^{2/3}}\mathbf{1}(\Omega_{j,m}^{(k)}) \leq \frac{A_i}{(3\pi\ell_i/2)^{2/3}} + \frac{2}{27}\epsilon_1. \tag{3.24}$$

Then by (3.22), we prove (3.18), which further implies (3.16).

(b) To show (3.17) holds, we note that since on $\Omega_{j,m,n}^{(k)}$, $|(\xi_j^{(k)} - x)/x| = 1 + \mathcal{O}(m^{-1/15})$ if $x \geq \xi_m^{(k)}$, it suffices to show that if m is large enough,

$$\mathbb{P}\left(\sup_{n > m} \int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{|M(x) - F(x)|}{x^2} \mathrm{d}x \mathbf{1}(\Omega_{j,m,n}^{(k)}) > \frac{\epsilon_1}{9}\right) < \frac{\epsilon_2}{6}. \tag{3.25}$$

Also, since $\xi_m^{(k)}$ goes to $-\infty$ monotonically as $m \to \infty$, $\xi_m^{(k)} < -(\frac{3}{2}\pi m)^{2/3} + m^{3/5}$ on $\Omega_{j,m}^{(k)}$, and the integrand of (3.25) is non-negative, we find that it suffices to prove that there exists $K_{\epsilon_1,\epsilon_2}$, such that for all positive integers $k_2 > k_1 > K_{\epsilon_1,\epsilon_2}$,

$$\mathbb{P}\left(\sup_{k_2 > k_1} \int_{-k_2}^{-k_1} \frac{|M(x) - F(x)|}{x^2} \mathrm{d}x \geq \frac{\epsilon_1}{9}\right) < \frac{\epsilon_2}{6}. \tag{3.26}$$

We define the auxiliary functions with integer parameters $k_1 < k_2$

$$\begin{aligned} A_{k_1,k_2} &= \sum_{i=k_1}^{k_2} \frac{|M(-i) - F(-i)|}{(i-1)i}, \quad B_{k_1,k_2} = \sum_{i=k_1}^{k_2} \frac{|M(-i) - F(-i)|}{i(i+1)}, \\ C_{k_1,k_2} &= 2\sum_{i=k_1+1}^{k_2-1} \frac{F(-i)}{(i-1)i(i+1)} - \frac{F(-k_1)}{k_1(k_1+1)} + \frac{F(-k_2)}{(k_2-1)k_2}. \end{aligned} \tag{3.27}$$

Since for $x \in [-i, -(i-1)]$, we have, by the monotonicity of $F(x)$ and $M(x)$,

$$M(-(i-1)) - F(-i) \leq M(x) - F(x) \leq M(-i) - F(-(i-1)), \tag{3.28}$$

and the inequality

$$\begin{aligned}\int_{-k_2}^{-k_1} \frac{|M(x)-F(x)|}{x^2}\mathrm{d}x &\le \sum_{i=k_1+1}^{k_2} \int_{-i}^{-(i-1)} \frac{|M(-i)-F(-(i-1))|}{x^2} + \frac{|M(-(i-1))-F(-i)|}{x^2}\mathrm{d}x \\ &= \sum_{i=k_1+1}^{k_2} \frac{|M(-i)-F(-(i-1))|}{(i-1)i} + \sum_{i=k_1}^{k_2-1} \frac{|M(-i)-F(-(i+1))|}{i(i+1)} \\ &\le (A_{k_1+1,k_2}+C_{k_1,k_2}) + (B_{k_1,k_2-1}+C_{k_1,k_2}) \\ &\le A_{k_1,k_2}+B_{k_1,k_2}+2C_{k_1,k_2}. \end{aligned} \tag{3.29}$$

Hence to prove (3.26), it suffices to show that there exists $K_{\epsilon_1,\epsilon_2}$ such that for $k_1 > K_{\epsilon_1,\epsilon_2}$,

$$\mathbb{P}\left(\sup_{k_2>k_1} A_{k_1,k_2} > \frac{\epsilon_1}{27}\right) < \frac{\epsilon_2}{12}, \quad \mathbb{P}\left(\sup_{k_2>k_1} B_{k_1,k_2} > \frac{\epsilon_1}{27}\right) < \frac{\epsilon_2}{12}, \quad \text{and} \quad \sup_{k_2>k_1} 2|C_{k_1,k_2}| < \frac{\epsilon_1}{27}. \tag{3.30}$$

By the estimate of $F(x)$ in Proposition 10, we can easily verify the C_{k_1,k_2} part of (3.30) with large enough $k_1 > K_{\epsilon_1,\epsilon_2}$. On the other hand, analogous to (3.21), we have that for large enough i,

$$\mathbb{P}\left(|M(-i)-F(-i)| \ge i^{4/5}\right) < i^{-\frac{11}{10}}. \tag{3.31}$$

Also if $k_1 > 270\epsilon_1^{-1}+1$, we have

$$\sum_{i=k_1}^{\infty} i^{-6/5} < \int_{k_1-1}^{\infty} x^{-6/5}\mathrm{d}x < \frac{\epsilon_1}{54}. \tag{3.32}$$

Hence if $k_1 > 270\epsilon_1^{-1}+1$, noting that $i(i+1)^2 > (i-1)i \ge i^2/2$ for all $i \ge k_1 \ge 2$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{k_2>k_1} B_{k_1,k_2} > \frac{\epsilon_1}{27}\right) &\le \mathbb{P}\left(\sup_{k_2>k_1} A_{k_1,k_2} > \frac{\epsilon_1}{27}\right) \\ &\le \mathbb{P}\left(\sum_{i=k_1}^{\infty} \frac{|M(-i)-F(-i)|}{i^2} > \frac{\epsilon_1}{54}\right) \\ &\le \mathbb{P}\left(\bigcup_{i=k_1}^{\infty} \left\{\frac{|M(-i)-F(-i)|}{i^2} > i^{-\frac{6}{5}}\right\}\right) \\ &\le \sum_{i=k_1}^{\infty} \mathbb{P}\left(\frac{|M(-i)-F(-i)|}{i^2} > i^{-\frac{6}{5}}\right) \\ &\le \sum_{i=k_1}^{\infty} i^{-\frac{11}{10}} < \int_{k_1-1}^{\infty} x^{-\frac{11}{10}}\mathrm{d}x = 10(k_1-1)^{-\frac{1}{10}}, \end{aligned} \tag{3.33}$$

and conclude the proof of the A_{k_1,k_2} and B_{k_1,k_2} parts of (3.30). Now (3.30) is proved, and so are (3.26) and (3.17).

Thus we finish the proof of (3.14). The constant $m''_{\epsilon_1,\epsilon_2}$ can be deduced from $N_{\epsilon_1,\epsilon_2}$, $N'_{\epsilon_1,\epsilon_2}$, $K_{\epsilon_1,\epsilon_2}$ above.

3. Combining (3.13) and (3.14) we arrives at that for $m > \max(m'_{\epsilon_1}, m''_{\epsilon_1,\epsilon_2})$,

$$\mathbb{P}\left(\sup_{n>m}\left\{\left|\int_{\xi_n^{(k)}}^{\xi_m^{(k)}} \frac{1}{\xi_j^{(k)}-x}\mathrm{d}M(x) - \int_{-(3\pi n/2)^{2/3}}^{-(3\pi m/2)^{2/3}} \frac{1}{0-x}\mathrm{d}F(x)\right| \mathbf{1}(\Omega_{j,m,n}^{(k)})\right\} > \epsilon_1\right) < \frac{\epsilon_2}{2}. \tag{3.34}$$

To complete the proof of (3.2), it suffices to find $m_{\epsilon_1,\epsilon_2} > \max(m'_{\epsilon_1}, m''_{\epsilon_1,\epsilon_2})$ such that for all $m > m_{\epsilon_1,\epsilon_2}$

$$\mathbb{P}\left(\bigcup_{n\geq m} (\Omega^{(k)}_{j,m,n})^c\right) < \frac{\epsilon_2}{2}. \tag{3.35}$$

By (2.9) in Lemma 2.9, we have that if m is large enough, then $\mathbb{P}(|\xi^{(k)}_m + (3\pi m/2)^{2/3}| > m^{3/5}) < cm^{-6/5}\log m$, and the estimate also holds if m is replaced by n. Also by part 1 of Lemma 8, we have $\lim_{m\to\infty} \mathbb{P}\left(|\xi^{(k)}_j| \leq m^{3/5}\right) = 0$. Hence it is straightforward to check that (3.35) holds if m is large enough, and the desired $m_{\epsilon_1,\epsilon_2}$ exists.

Finally we complete the proof of (3.11). □

4 Analysis of random measure μ: proof of Proposition 10

We show that the estimate of the mean and variance of $M(x)$ can be transformed to the mean and variance of a linear statistic of the extended Airy process, or more precisely, the difference between the linear statistics of two species, as mentioned in Section 1.4. We observe that the mean and variance of the linear statistic have (multi)-contour integral representations. Then we prove the said estimate, first of the mean and then of the variance, via delicate saddle point analysis of the contour integrals.

Define the finite random subsets of $\mathbb{N}$

$$\mathcal{I}_x := \{i \in \mathbb{N} \mid \xi^{(k)}_i \in (x, +\infty)\}, \quad \mathcal{J}_x := \{i \in \mathbb{N} \mid \xi^{(k-1)}_i \in (x, +\infty)\}. \tag{4.1}$$

and the random variable

$$N_x = |\mathcal{I}_x| - |\mathcal{J}_x|. \tag{4.2}$$

By the interlacing property stated in Lemma 7, we see that N_x is a Bernoulli random variable for a given x. Hence,

$$\mathbb{E}\, N_x = \mathbb{P}(N_x = 1). \tag{4.3}$$

We also consider the random variable

$$S_x = -\sum_{i\in\mathcal{I}_x} \xi^{(k)}_i + \sum_{i\in\mathcal{J}_x} \xi^{(k-1)}_i. \tag{4.4}$$

We observe that if $\xi^{(k)}_1 \leq x$, then $N_x = S_x = M(x) = 0$. Under the condition that $\xi^{(k)}_1 > x$, if $N_x = 1$, then $S_x = M(x) - \xi^{(k)}_1$, otherwise $S_x = M(x) - \xi^{(k)}_1 + x$. We conclude that (noting that $N_x\mathbf{1}(\xi^{(k)}_1 > x) = N_x$)

$$M(x) = S_x + (\xi^{(k)}_1 - x(1 - N_x))\mathbf{1}(\xi^{(k)}_1 > x) = S_x + xN_x + (\xi^{(k)}_1 - x)\mathbf{1}(\xi^{(k)}_1 > x). \tag{4.5}$$

By the estimate in Lemma 8, we find that as $x \to -\infty$, $\mathbb{E}(\xi^{(k)}_1 - x)\mathbf{1}(\xi^{(k)}_1 > x) = -x + \mathcal{O}(1)$ and $\mathrm{Var}(\xi^{(k)}_1 - x)\mathbf{1}(\xi^{(k)}_1 > x) = \mathcal{O}(1)$. Hence to prove the Proposition 10, we only need the following estimates of the mean and variance of $S_x + xN_x$:

$$\mathbb{E}(S_x + xN_x) = \frac{x}{2} + \mathcal{O}(1), \tag{4.6}$$

$$\mathrm{Var}(S_x + xN_x) = \frac{2}{\pi}\sqrt{-x} + a_k + \mathcal{O}(|x|^{-1/2}), \tag{4.7}$$

in light of the linearity of expectation and the trivial inequality

$$|\mathrm{Var}(X + Y) - (\mathrm{Var}(X) + \mathrm{Var}(Y))| \leq 2\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}. \tag{4.8}$$

Below we prove (4.6) and (4.7) separately. For the proofs, we define the function

$$h_x(t) = \begin{cases} 0, & t \leq x, \\ -t + x, & t > x. \end{cases} \tag{4.9}$$

We emphasize here, all integrals in the sequel, unless the integral domain is specified, are on $\mathbb{R}$.

Proof of (4.6) We have, by the standard formula for linear statistics of a determinantal point process, that

$$
\begin{aligned}
&\mathbb{E}(S_x + xN_x)\\
&= \mathbb{E}\Big(\sum_{i\in\mathcal{J}_x}\xi_i^{(k-1)}\Big) - \mathbb{E}\Big(\sum_{i\in\mathcal{I}_x}\xi_i^{(k)}\Big) + x\Big(\mathbb{E}\,|\mathcal{I}_x| - \mathbb{E}\,|\mathcal{J}_x|\Big)\\
&= \int_x^\infty tK^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\mathrm{d}t - \int_x^\infty tK^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t)\mathrm{d}t + x\left(\int_x^\infty K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t)\mathrm{d}t - \int_x^\infty K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\mathrm{d}t\right)\\
&= \int h_x(t)\Big(K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t) - K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\Big)\mathrm{d}t.
\end{aligned}
\tag{4.10}
$$

According to (1.7) and (1.8), we have

$$
K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t) - K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t) = \frac{1}{(2\pi\mathrm{i})^2}\int_\sigma \mathrm{d}u\int_\gamma \mathrm{d}v\frac{e^{\frac{u^3}{3}-tu}}{e^{\frac{v^3}{3}-tv}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{1}{v-a_k}. \tag{4.11}
$$

Plugging (4.11) into (4.10), we have

$$
\begin{aligned}
\mathbb{E}(S_x + xN_x) &= \frac{-1}{(2\pi\mathrm{i})^2}\int_\sigma \mathrm{d}u\int_\gamma \mathrm{d}v\frac{e^{\frac{u^3}{3}}}{e^{\frac{v^3}{3}}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{1}{v-a_k}\int_x^{+\infty}(t-x)e^{-(u-v)t}\mathrm{d}t\\
&= \frac{-1}{(2\pi\mathrm{i})^2}\int_\sigma \mathrm{d}u\int_\gamma \mathrm{d}v\frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{1}{(v-a_k)(u-v)^2}.
\end{aligned}
\tag{4.12}
$$

By some standard residue calculation, we have that if $x<0$, then

$$
\begin{aligned}
&\mathbb{E}(S_x + xN_x)\\
&= \frac{-1}{(2\pi\mathrm{i})^2}\iint_X \mathrm{d}u\mathrm{d}v\frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{1}{(v-a_k)(u-v)^2} &\text{(4.13a)}\\
&\quad + \frac{-1}{2\pi\mathrm{i}}\int_{-\sqrt{-x}\mathrm{i}}^{\sqrt{-x}\mathrm{i}}\Big(\frac{v^2-x}{v-a_k} + \sum_{j=1}^{k-1}\frac{1}{(v-a_j)(v-a_k)}\Big)\mathrm{d}v, &\text{(4.13b)}
\end{aligned}
$$

where the contour X means that the two deformed contours σ and γ intersect at the two saddle points $\pm\sqrt{-x}\mathrm{i}$, see Figure 2, and the integral is understood as the Cauchy principal value, and the contour in (4.13b) goes by the right of all a_i's. By direct computation, we find the (4.13b) is $(x-a_k^2)/2 + \mathcal{O}(|x|^{-1/2})$. To evaluate (4.13a), we define two types of infinite contours:

$$
\gamma_{\mathrm{std}}(a) = \{e^{\frac{2\pi\mathrm{i}}{3}}t + a \mid t\ge 0\}\cup\{e^{\frac{\pi\mathrm{i}}{3}}t + a \mid t\le 0\},\quad \sigma_{\mathrm{std}}(b) = \{e^{\frac{\pi\mathrm{i}}{3}}t + b \mid t\ge 0\}\cup\{e^{\frac{2\pi\mathrm{i}}{3}}t + b \mid t\le 0\}, \tag{4.14}
$$

both oriented upward; see Figure 3. Assuming that $-x$ is large enough, we deform the contour X such that u is on $\sigma_{\mathrm{std}}(-\sqrt{-x/3})$, and v is on $\gamma_{\mathrm{std}}(\sqrt{-x/3})$. Direct computation shows that $\Re(u^3/3-xu)$ attains its maximum along σ at $\pm\sqrt{-x}\mathrm{i}$, and $\Re(v^3/3-xv)$ attains its minimum along γ at the same two points. Hence $\pm\sqrt{-x}\mathrm{i}$ are the saddle points. We divide the double contour X into three disjoint subsets:

(i) $X_1 = \{u,v\in X \mid |u-\sqrt{-x}\mathrm{i}|<1,\ |v-\sqrt{-x}\mathrm{i}|<1\}$,

(ii) $X_2 = \{u,v\in X \mid |u+\sqrt{-x}\mathrm{i}|<1,\ |v+\sqrt{-x}\mathrm{i}|<1\}$,

(iii) $X_3 = X\setminus(X_1\cup X_2)$.

By standard saddle point method, we have that the parts of integral (4.13a) over X_1 and X_2 are both $\mathcal{O}(|x|^{-1/2})$, and the part of integral (4.13a) over X_3 is $o(|x|^{-1/2})$. This together with the estimate of (4.13b) above completes the proof of (4.6).

Proof of (4.7) We can write, with $h_x(t)$ defined in (4.9),

$$\begin{aligned}\mathbb{E}[(S_x+xN_x)^2] = &\int h_x^2(t)K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t)\mathrm{d}t + \int h_x^2(t)K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\mathrm{d}t \\ &+\iint h_x(s)h_x(t)\begin{vmatrix}K^{k,k}_{\mathrm{Airy},\mathbf{a}}(s,s) & K^{k,k}_{\mathrm{Airy},\mathbf{a}}(s,t)\\ K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,s) & K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t)\end{vmatrix}\mathrm{d}s\mathrm{d}t \\ &+\iint h_x(s)h_x(t)\begin{vmatrix}K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(s,s) & K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(s,t)\\ K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,s) & K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\end{vmatrix}\mathrm{d}s\mathrm{d}t \\ &-\iint h_x(s)h_x(t)\begin{vmatrix}K^{k,k}_{\mathrm{Airy},\mathbf{a}}(s,s) & K^{k,k-1}_{\mathrm{Airy},\mathbf{a}}(s,t)\\ K^{k-1,k}_{\mathrm{Airy},\mathbf{a}}(t,s) & K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\end{vmatrix}\mathrm{d}s\mathrm{d}t \\ &-\iint h_x(s)h_x(t)\begin{vmatrix}K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(s,s) & K^{k-1,k}_{\mathrm{Airy},\mathbf{a}}(s,t)\\ K^{k,k-1}_{\mathrm{Airy},\mathbf{a}}(t,s) & K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t)\end{vmatrix}\mathrm{d}s\mathrm{d}t. \end{aligned} \tag{4.15}$$

Combining (4.15) with (4.10) and using (1.7) we get

$$\begin{aligned}&\mathrm{Var}\left[S_x+xN_x\right] \\ &= \int h_x^2(t)\Big(K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t)+K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\Big)\mathrm{d}t - 2\int h_x(t)\left(\int_t^\infty h_x(s)e^{a_k(s-t)}K^{k,k-1}_{\mathrm{Airy},\mathbf{a}}(s,t)\mathrm{d}s\right)\mathrm{d}t \qquad (4.16\mathrm{a})\\ &-\iint h_x(s)h_x(t)\Big(K^{k,k}_{\mathrm{Airy},\mathbf{a}}(s,t)K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,s)+K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(s,t)K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,s) \\ &\qquad - K^{k,k-1}_{\mathrm{Airy},\mathbf{a}}(s,t)\widetilde{K}^{k-1,k}_{\mathrm{Airy},\mathbf{a}}(t,s)-\widetilde{K}^{k-1,k}_{\mathrm{Airy},\mathbf{a}}(s,t)K^{k,k-1}_{\mathrm{Airy},\mathbf{a}}(t,s)\Big)\mathrm{d}s\mathrm{d}t. \qquad (4.16\mathrm{b})\end{aligned}$$

First, we consider the integrals in (4.16a). A simple change of order of integration yields

$$\begin{aligned}&\int h_x^2(t)\Big(K^{k,k}_{\mathrm{Airy},\mathbf{a}}(t,t)+K^{k-1,k-1}_{\mathrm{Airy},\mathbf{a}}(t,t)\Big)\mathrm{d}t \\ &\qquad = \frac{1}{(2\pi\mathrm{i})^2}\int_\sigma \mathrm{d}u\int_\gamma \mathrm{d}v \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}}\frac{2}{(u-v)^4}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{u+v-2a_k}{v-a_k},\end{aligned} \tag{4.17}$$

$$\begin{aligned}&\int h_x(t)\left(\int_t^\infty h_x(s)e^{a_k(s-t)}K^{k,k-1}_{\mathrm{Airy},\mathbf{a}}(s,t)\mathrm{d}s\right)\mathrm{d}t \\ &\qquad = \frac{1}{(2\pi\mathrm{i})^2}\int_\sigma \mathrm{d}u\int_\gamma \mathrm{d}v \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}}\frac{1}{(u-v)^4}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{(3u-v-2a_k)}{u-a_k}.\end{aligned} \tag{4.18}$$

So (4.16a) becomes

$$\frac{2}{(2\pi\mathrm{i})^2}\int_\sigma \mathrm{d}u\int_\gamma \mathrm{d}v \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{1}{(u-v)^2(u-a_k)(v-a_k)}. \tag{4.19}$$

Similarly to (4.13), when $x<0$, this integral can be written as

$$\frac{2}{(2\pi\mathrm{i})^2}\iint_X \mathrm{d}u\mathrm{d}v \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)\frac{1}{(u-v)^2(u-a_k)(v-a_k)} \tag{4.20a}$$

$$+\frac{2}{2\pi\mathrm{i}}\int_{-\sqrt{-x}\mathrm{i}}^{\sqrt{-x}\mathrm{i}}\left(v^2-x+\sum_{j=1}^{k-1}\frac{1}{v-a_j}-\frac{1}{v-a_k}\right)\frac{1}{(v-a_k)^2}\mathrm{d}v \tag{4.20b}$$

$$+\prod_{j=1}^{k-1}(a_k-a_j)e^{\frac{a_k^3}{3}-a_kx}\frac{2}{2\pi\mathrm{i}}\int_\gamma\frac{1}{e^{\frac{v^3}{3}-xv}}\left(\prod_{j=1}^{k-1}\frac{1}{v-a_j}\right)\frac{1}{(v-a_k)^3}\mathrm{d}v, \tag{4.20c}$$

where the contour X in (4.20a) and the contour from $-\sqrt{-x}\mathrm{i}$ to $\sqrt{-x}\mathrm{i}$ in (4.20b) are the same as those in (4.13). We have that the integral in (4.20b) is $4\sqrt{-x}/\pi + 2a_k + \mathcal{O}(|x|^{-1})$, and the integral in (4.20a) is $\mathcal{O}(|x|^{-1})$. The saddle point analysis is omitted since it is analogous to that for (4.13). The integral in (4.20c) will be cancelled out later, by a term in (4.25).

On the other hand, the double integral in (4.16b) can be expressed as

$$\frac{1}{(2\pi\mathrm{i})^4}\int_\sigma \mathrm{d}u \int_\gamma \mathrm{d}v \int_\sigma \mathrm{d}z \int_\gamma \mathrm{d}w \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}} \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \prod_{j=1}^{k-1} \frac{u-a_j}{v-a_j}\frac{z-a_j}{w-a_j} \times \frac{1}{(u-v)(z-w)(u-w)(z-v)(v-a_k)(w-a_k)}. \tag{4.21}$$

In order to estimate the integral above, we will perform several steps of contour deformation.

(I) We first deform the contour γ for w and v to $\gamma_w^{\mathrm{out}} \cup \gamma_w^{\mathrm{in}}$ and $\gamma_v^{\mathrm{out}} \cup \gamma_v^{\mathrm{in}}$, respectively, as in Figure 5, such that all a_j ($j = 1, \ldots, k$) are enclosed in γ_w^{in}, and then also in γ_v^{in}. We also slightly deform the contour σ for u and z into σ_u and σ_z, respectively, as shown in Figure 5.

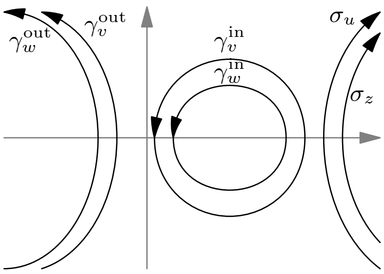

Figure 5: Deformation of contour γ for w and v, and contour σ for u and z.

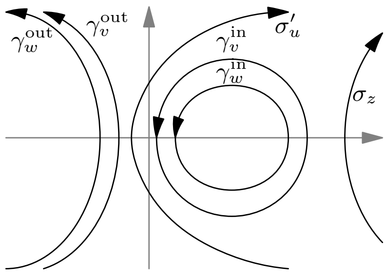

Figure 6: σ_u is deformed into σ'_u.

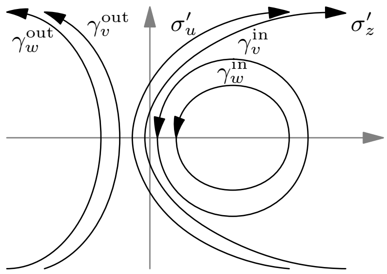

Figure 7: σ_z is deformed into σ'_z.

(II) We then further deform the contour σ_u such that it goes between γ_v^{out} and γ_v^{in}, and thus also goes between γ_w^{out} and γ_w^{in}. We denote by σ'_u the deformed σ_u; see Figure 6. By residue calculation, we write (4.21) as

$$\frac{1}{(2\pi\mathrm{i})^4}\int_{\sigma'_u} \mathrm{d}u \int_{\gamma_v^{\mathrm{out}}\cup\gamma_v^{\mathrm{in}}} \mathrm{d}v \int_{\sigma_z} \mathrm{d}z \int_{\gamma_w^{\mathrm{out}}\cup\gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}} \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \prod_{j=1}^{k-1} \frac{u-a_j}{v-a_j}\frac{z-a_j}{w-a_j} \times \frac{1}{(u-v)(z-w)(u-w)(z-v)(v-a_k)(w-a_k)} \tag{4.22a}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\sigma_z} \mathrm{d}z \int_{\gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \left(\prod_{j=1}^{k-1} \frac{z-a_j}{w-a_j}\right) \frac{1}{(w-a_k)(z-a_k)(z-w)^2} \tag{4.22b}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_v^{\mathrm{in}}} \mathrm{d}v \int_{\sigma_z} \mathrm{d}z \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{v^3}{3}-xv}} \left(\prod_{j=1}^{k-1} \frac{z-a_j}{v-a_j}\right) \frac{-1}{(v-a_k)^2(z-a_k)(z-v)} \tag{4.22c}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_v^{\mathrm{out}}} \mathrm{d}v \int_{\sigma_z} \mathrm{d}z \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{v^3}{3}-xv}} \left(\prod_{j=1}^{k-1} \frac{z-a_j}{v-a_j}\right) \frac{-1}{(v-a_k)^2(z-a_k)(z-v)} \tag{4.22d}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_w^{\mathrm{out}}} \mathrm{d}v \int_{\sigma_z} \mathrm{d}z \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \left(\prod_{j=1}^{k-1} \frac{z-a_j}{w-a_j}\right) \frac{-1}{(w-a_k)^2(z-a_k)(z-w)}. \tag{4.22e}$$

Derivation of (4.22b)–(4.22e). First we integrate u over $\sigma'_u - \sigma_u$, and the result is a 3-fold integral in v, w, z. The integral domain of z is always σ_z so we leave it alone, and divide the integral domain of

v, w into 4 subdomains: (i) $\gamma_v^{\mathrm{out}} \times \gamma_w^{\mathrm{out}}$, (ii) $\gamma_v^{\mathrm{out}} \times \gamma_w^{\mathrm{in}}$, (iii) $\gamma_v^{\mathrm{in}} \times \gamma_w^{\mathrm{out}}$ and (iv) $\gamma_v^{\mathrm{in}} \times \gamma_w^{\mathrm{in}}$. Integrating v, w on subdomain (i), the result is 0; integrating w on subdomain (ii), the result is (4.22d); integrating v on subdomain (iii), the result is (4.22e); on subdomain (iv), it is more complicated: the integrand can be divided into two parts, such that when we integrate one part with respect to v, we get (4.22b), and when we integrate the other part with respect to w, we get (4.22c). □

(III) Next, similarly to the previous step, we further deform the contour σ_z such that it goes between γ_v^{out} and γ_v^{in}, and thus also goes between γ_w^{out} and γ_w^{in}. Hence σ_z becomes σ'_z; see Figure 7. By residue calculation, the quantity in (4.22) becomes

$$\frac{1}{(2\pi \mathrm{i})^4} \int_{\sigma'_u} \mathrm{d}u \int_{\gamma_v^{\mathrm{out}} \cup \gamma_v^{\mathrm{in}}} \mathrm{d}v \int_{\sigma'_z} \mathrm{d}z \int_{\gamma_w^{\mathrm{out}} \cup \gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{u^3}{3} - xu}}{e^{\frac{v^3}{3} - xv}} \frac{e^{\frac{z^3}{3} - xz}}{e^{\frac{w^3}{3} - xw}} \prod_{j=1}^{k-1} \frac{u - a_j}{v - a_j} \frac{z - a_j}{w - a_j} \times \frac{1}{(u - v)(z - w)(u - w)(z - v)(v - a_k)(w - a_k)} \tag{4.23a}$$

$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\sigma'_u} \mathrm{d}u \int_{\gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{u^3}{3} - xu}}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{u - a_j}{w - a_j} \right) \frac{1}{(w - a_k)(u - a_k)(u - w)^2} \tag{4.23b}$$

$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\sigma'_u} \mathrm{d}u \int_{\gamma_v^{\mathrm{in}}} \mathrm{d}v \frac{e^{\frac{u^3}{3} - xu}}{e^{\frac{v^3}{3} - xv}} \left(\prod_{j=1}^{k-1} \frac{u - a_j}{v - a_j} \right) \frac{-1}{(v - a_k)^2 (u - a_k)(u - v)} \tag{4.23c}$$

$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\sigma'_u} \mathrm{d}u \int_{\gamma_v^{\mathrm{out}}} \mathrm{d}v \frac{e^{\frac{u^3}{3} - xu}}{e^{\frac{v^3}{3} - xv}} \left(\prod_{j=1}^{k-1} \frac{u - a_j}{v - a_j} \right) \frac{-1}{(v - a_k)^2 (u - a_k)(u - v)} \tag{4.23d}$$

$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\sigma'_u} \mathrm{d}u \int_{\gamma_w^{\mathrm{out}}} \mathrm{d}w \frac{e^{\frac{u^3}{3} - xu}}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{u - a_j}{w - a_j} \right) \frac{-1}{(w - a_k)^2 (u - a_k)(u - w)} \tag{4.23e}$$

$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\sigma'_z} \mathrm{d}z \int_{\gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{z^3}{3} - xz}}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{z - a_j}{w - a_j} \right) \frac{1}{(w - a_k)(z - a_k)(z - w)^2} \tag{4.23f}$$

$$+ \frac{1}{2\pi \mathrm{i}} \int_{\gamma_w^{\mathrm{in}}} \mathrm{d}w \left(w^2 - x + \sum_{j=1}^{k-1} \frac{1}{w - a_j} - \frac{1}{w - a_k} \right) \frac{1}{(w - a_k)^2} \tag{4.23g}$$

$$+ \prod_{j=1}^{k-1} (a_k - a_j) e^{\frac{a_k^3}{3} - a_k x} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{1}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{1}{w - a_j} \right) \frac{1}{(w - a_k)^3} \tag{4.23h}$$

$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_v^{\mathrm{in}}} \mathrm{d}v \int_{\sigma'_z} \mathrm{d}z \frac{e^{\frac{z^3}{3} - xz}}{e^{\frac{v^3}{3} - xv}} \left(\prod_{j=1}^{k-1} \frac{z - a_j}{v - a_j} \right) \frac{-1}{(v - a_k)^2 (z - a_k)(z - v)} \tag{4.23i}$$

$$+ \prod_{j=1}^{k-1} (a_k - a_j) e^{\frac{a_k^3}{3} - a_k x} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_v^{\mathrm{in}}} \mathrm{d}v \frac{1}{e^{\frac{v^3}{3} - xv}} \left(\prod_{j=1}^{k-1} \frac{1}{v - a_j} \right) \frac{1}{(v - a_k)^3} \tag{4.23j}$$

$$+ \frac{1}{(2\pi \mathrm{i})^2} \int_{\gamma_w^{\mathrm{out}}} \mathrm{d}w \int_{\sigma'_z} \mathrm{d}z \frac{e^{\frac{z^3}{3} - xz}}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{z - a_j}{w - a_j} \right) \frac{-2}{(w - a_k)^2 (z - a_k)(z - w)} \tag{4.23k}$$

$$+ \prod_{j=1}^{k-1} (a_k - a_j) e^{\frac{a_k^3}{3} - a_k x} \frac{1}{2\pi \mathrm{i}} \int_{\gamma_w^{\mathrm{out}}} \mathrm{d}w \frac{1}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{1}{w - a_j} \right) \frac{2}{(w - a_k)^3}. \tag{4.23l}$$

Derivation of (4.23). (i) (4.22a) is transformed to the sum of (4.23a)–(4.23e), by an argument similar to that used to transform (4.21) to the sum of (4.22a)–(4.22e).

(ii) (4.22b) is the sum of (4.23f), (4.23g) and (4.23h). To see it, we express the contours $\sigma_z = \sigma'_z + \gamma_v^{\text{in}}$, and check that if the integral domain of (4.22b) is changed into $(z, w) \in \gamma_v^{\text{in}} \times \gamma_w^{\text{in}}$, by integrating z first, we obtain (4.23g) plus (4.23h).

(iii) (4.22c) is the sum of (4.23i) and (4.23j). To see it, we express the contours $\sigma_z = \sigma'_z + \gamma_z^{\text{in}}$, where γ_z^{in} encloses γ_v^{in}, and then find that the part of (4.22c) where σ_z is replaced by σ'_z becomes (4.23i) and the part where σ_z is replaced by γ_z^{in} becomes (4.23j).

(iv) (4.22d) and (4.22e) are equal, and their sum is equal to the sum of (4.23k) and (4.23l).

$\square$

(IV) For further deformation of the contours, we introduce the following shorthand notations

$$\lozenge\!\bullet := \sqrt{-x}\mathrm{i} + \frac{1}{\sqrt{-x}}, \quad \overset{\bullet}{\lozenge} := \sqrt{-x}\mathrm{i} + \frac{\sqrt{3}\mathrm{i}}{\sqrt{-x}}, \quad \underset{\bullet}{\lozenge} := \sqrt{-x}\mathrm{i} + \frac{-\sqrt{3}\mathrm{i}}{\sqrt{-x}}, \quad \bullet\!\lozenge := \sqrt{-x}\mathrm{i} + \frac{-1}{\sqrt{-x}}. \tag{4.24}$$

For the 4-fold integral (4.23a), we perform the following operations:

(i) deform σ'_u such that it passes $\overset{\bullet}{\lozenge}$ and $\overline{\overset{\bullet}{\lozenge}}$;

(ii) deform σ'_z such that it passes $\lozenge\!\bullet$ and $\overline{\lozenge\!\bullet}$;

(iii) deform γ_v^{out} such that it goes from $e^{-2\pi\mathrm{i}/3} \cdot \infty$ to $\overline{\overset{\bullet}{\lozenge}}$, then goes along the left side of σ'_u until it reaches $\overset{\bullet}{\lozenge}$, and then goes from $\overset{\bullet}{\lozenge}$ to $e^{2\pi\mathrm{i}/3} \cdot \infty$;

(iv) deform γ_v^{in} such that it goes from $\lozenge\!\bullet$ to $\overline{\lozenge\!\bullet}$ along the right side of σ'_z, then wraps around all a_j's, and finally goes back to $\lozenge\!\bullet$;

(v) and at last add an additional contour for v, on which the contour integral vanishes: the contour goes from $\overline{\lozenge\!\bullet}$ to $\lozenge\!\bullet$ along the left side of σ'_z, then goes from $\lozenge\!\bullet$ to $\overset{\bullet}{\lozenge}$, and further goes from $\overset{\bullet}{\lozenge}$ to $\overline{\overset{\bullet}{\lozenge}}$ along the right side of σ'_u, and finally goes from $\overline{\overset{\bullet}{\lozenge}}$ to $\overline{\lozenge\!\bullet}$.

See Figure 8 for the deformation of contours.

Now we define the infinite contour γ'_v as in Figure 8 that goes from $e^{-2\pi\mathrm{i}/3} \cdot \infty$ to $\overline{\overset{\bullet}{\lozenge}}$, then to $\overline{\lozenge\!\bullet}$, then wraps a_j's until it reaches $\lozenge\!\bullet$, and then goes to $\overset{\bullet}{\lozenge}$, and finally goes to $e^{2\pi\mathrm{i}/3} \cdot \infty$. Hence the 4-fold integral (4.23a) can be simplified by the residue theorem with $\gamma_v^{\text{out}} \cup \gamma_v^{\text{in}}$ replaced by γ'_v. Then the formula (4.23) becomes

$$\frac{1}{(2\pi\mathrm{i})^4} \int_{\sigma'_u} \mathrm{d}u \int_{\sigma'_z} \mathrm{d}z \int_{\gamma_w^{\text{out}} \cup \gamma_w^{\text{in}}} \mathrm{d}w \,\mathrm{P.V.} \int_{\gamma'_v} \mathrm{d}v \frac{e^{\frac{u^3}{3} - xu}}{e^{\frac{v^3}{3} - xv}} \frac{e^{\frac{z^3}{3} - xz}}{e^{\frac{w^3}{3} - xw}} \prod_{j=1}^{k-1} \frac{u - a_j}{v - a_j} \frac{z - a_j}{w - a_j}$$
$$\times \frac{1}{(u - v)(z - w)(u - w)(z - v)(v - a_k)(w - a_k)} \tag{4.25a}$$

$$+ \frac{1}{(2\pi\mathrm{i})^3} \int_{\sigma'_z} \mathrm{d}z \int_{\gamma_w^{\text{out}} \cup \gamma_w^{\text{in}}} \mathrm{d}w \int_{\overline{\overset{\bullet}{\lozenge}}}^{\overset{\bullet}{\lozenge}} \mathrm{d}u \frac{e^{\frac{z^3}{3} - xz}}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{z - a_j}{w - a_j} \right)$$
$$\times \frac{1}{(z - w)(u - w)(z - u)(u - a_k)(w - a_k)} \tag{4.25b}$$

$$+ \frac{1}{(2\pi\mathrm{i})^3} \int_{\sigma'_u} \mathrm{d}u \int_{\gamma_w^{\text{out}} \cup \gamma_w^{\text{in}}} \mathrm{d}w \int_{\overline{\lozenge\!\bullet}}^{\lozenge\!\bullet} \mathrm{d}z \frac{e^{\frac{u^3}{3} - xu}}{e^{\frac{w^3}{3} - xw}} \left(\prod_{j=1}^{k-1} \frac{u - a_j}{w - a_j} \right)$$
$$\times \frac{1}{(u - z)(z - w)(u - w)(z - a_k)(w - a_k)} \tag{4.25c}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\gamma_w^{\mathrm{out}}}\mathrm{d}w\int_{\sigma'_z}\mathrm{d}z\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{-4}{(w-a_k)^2(z-a_k)(z-w)} \tag{4.25d}$$

$$+\prod_{j=1}^{k-1}(a_k-a_j)e^{\frac{a_k^3}{3}-a_kx}\frac{1}{2\pi \mathrm{i}}\int_{\gamma_w^{\mathrm{in}}\cup\gamma_w^{\mathrm{out}}}\mathrm{d}w\frac{1}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{1}{w-a_j}\right)\frac{2}{(w-a_k)^3} \tag{4.25e}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\sigma'_z}\mathrm{d}z\int_{\gamma_w^{\mathrm{in}}}\mathrm{d}w\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{2(2w-z-a_k)}{(w-a_k)^2(z-a_k)(z-w)^2} \tag{4.25f}$$

$$+2a_k.$$

Derivation of (4.25). (i) (4.23a) is equal to the sum of (4.25a)–(4.25c).

(ii) The sum of (4.23d), (4.23e), and (4.23k) is equal to (4.25d).

(iii) The sum of (4.23h) (4.23j) and (4.23l) is equal to (4.25e).

(iv) The sum of (4.23b), (4.23c), (4.23f), (4.23i) is equal to (4.25f).

(v) (4.23g) is equal to $2a_k$.

□

(V) Now we deform the contour $\gamma_w^{\mathrm{out}}\cup\gamma_w^{\mathrm{in}}$ for w in the way similar to our deformation of $\gamma_v^{\mathrm{out}}\cup\gamma_v^{\mathrm{in}}$ for v in Step (IV). We perform the following operations:

(i) deform σ'_u such that it passes ${\bullet}\diamond$ and $\overline{{\bullet}\diamond}$ and meanwhile still passes $\dot{\diamond}$ and $\overline{\dot{\diamond}}$;

(ii) deform σ'_z such that it passes $\underset{\bullet}{\diamond}$ and $\overline{\underset{\bullet}{\diamond}}$ and meanwhile still passes $\diamond{\bullet}$ and $\overline{\diamond{\bullet}}$;

(iii) deform γ_w^{out} such that it goes from $e^{-2\pi \mathrm{i}/3}\cdot\infty$ to $\overline{{\bullet}\diamond}$, then goes along the left side of σ'_u to ${\bullet}\diamond$, and finally goes from ${\bullet}\diamond$ to $e^{2\pi \mathrm{i}/3}\cdot\infty$;

(iv) deform γ_w^{in} such that it goes from $\underset{\bullet}{\diamond}$ to $\overline{\underset{\bullet}{\diamond}}$ along the right side of σ'_z, then wraps around all a_j's, and finally goes back to $\underset{\bullet}{\diamond}$;

(v) and at last add an additional contour for w, on which the contour integral vanishes: the contour goes from $\overline{\underset{\bullet}{\diamond}}$ to $\underset{\bullet}{\diamond}$ along the left-side of σ'_z, then goes from $\underset{\bullet}{\diamond}$ to ${\bullet}\diamond$, then from ${\bullet}\diamond$ to $\overline{{\bullet}\diamond}$ along the right side of σ'_u, and finally goes from $\overline{{\bullet}\diamond}$ to $\overline{\underset{\bullet}{\diamond}}$.

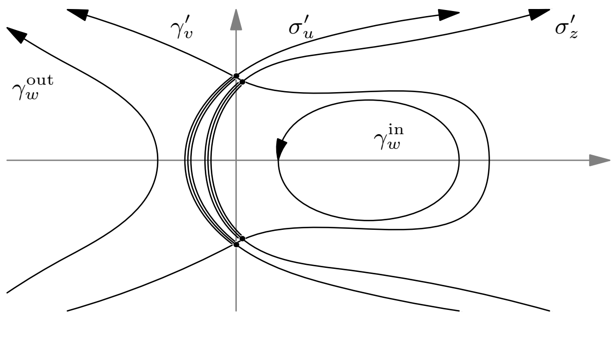

Figure 8: The deformed contours for v, u and z. The four dots are $\dot{\diamond}$, $\diamond{\bullet}$, $\overline{\dot{\diamond}}$ and $\overline{\diamond{\bullet}}$.

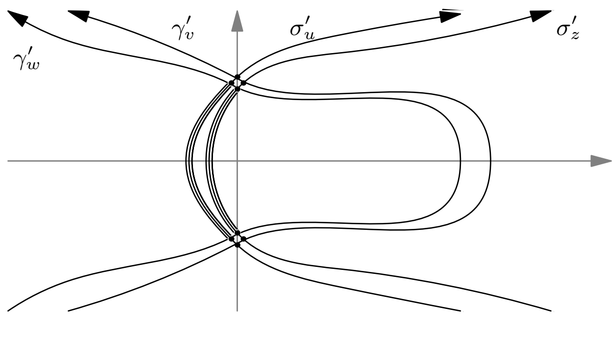

Figure 9: The deformed contours for w, u and z. The four dots on top are $\dot{\diamond}$, $\diamond{\bullet}$, ${\bullet}\diamond$ and $\underset{\bullet}{\diamond}$, and the four dots on bottom are their complex conjugates.

We have the result in Figure 9. Similar to γ'_v in Figure 8, now we define the infinite contour γ'_w that goes from $e^{-2\pi \mathrm{i}/3}\cdot\infty$ to $\overline{{\bullet}\diamond}$, then to $\overline{\underset{\bullet}{\diamond}}$, then wraps all a_j's until it reaches $\underset{\bullet}{\diamond}$, and then goes to ${\bullet}\diamond$, and finally goes to $e^{2\pi \mathrm{i}/3}\cdot\infty$. Contour γ'_w is parallel to and to the left of γ'_v. Hence by the residue theorem, the 4-fold integral (4.25a) can be simplified with $\gamma_w^{\mathrm{out}}\cup\gamma_w^{\mathrm{in}}$ replaced by γ'_w, and then formula (4.25) becomes

$$\frac{1}{(2\pi\mathrm{i})^4}\int_{\sigma'_u}\mathrm{d}u\int_{\sigma'_z}\mathrm{d}z\,\mathrm{P.V.}\int_{\gamma'_w}\mathrm{d}w\,\mathrm{P.V.}\int_{\gamma'_v}\mathrm{d}v\frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}-xv}}\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\frac{z-a_j}{w-a_j}$$
$$\times\frac{1}{(u-v)(z-w)(u-w)(z-v)(v-a_k)(w-a_k)} \tag{4.26a}$$

$$+\frac{1}{(2\pi\mathrm{i})^3}\int_{\sigma'_z}\mathrm{d}z\,\mathrm{P.V.}\int_{\gamma'_v}\mathrm{d}v\int_{\overline{\bullet\lozenge}}^{\bullet\lozenge}\mathrm{d}u\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{v^3}{3}-xv}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{v-a_j}\right)$$
$$\times\frac{1}{(u-v)(z-u)(z-v)(v-a_k)(u-a_k)} \tag{4.26b}$$

$$+\frac{1}{(2\pi\mathrm{i})^3}\int_{\sigma'_u}\mathrm{d}u\,\mathrm{P.V.}\int_{\gamma'_v}\mathrm{d}v\int_{\overline{\lozenge\bullet}}^{\lozenge\bullet}\mathrm{d}z\frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{v^3}{3}+xv}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{v-a_j}\right)$$
$$\times\frac{1}{(u-v)(u-z)(z-v)(v-a_k)(z-a_k)} \tag{4.26c}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\gamma_w^{\mathrm{in}}\cup\gamma_w^{\mathrm{out}}}\mathrm{d}w\int_{\sigma'_z}\mathrm{d}z\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{-4}{(w-a_k)^2(z-a_k)(z-w)} \tag{4.26d}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\sigma'_z}\mathrm{d}z\int_{\gamma_w^{\mathrm{in}}}\mathrm{d}w\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{2}{(w-a_k)^2(z-w)^2} \tag{4.26e}$$

$$+(4.25\mathrm{b})+(4.25\mathrm{c})+(4.25\mathrm{e})+2a_k. \tag{4.26f}$$

Derivation of (4.26). (i) (4.25a) is the sum of (4.26a)–(4.26c), and (ii) the sum of (4.25d) and (4.25f) is equal to the sum of (4.26d) and (4.26e). □

Now we can compute the contour integrals by saddle point method. Recall the contours σ_{std} and γ_{std} defined in (4.14)). To facilitate the analysis, we fix the shape of the contours as $\sigma'_u = \sigma_{\mathrm{std}}(-\sqrt{-x/3} - 1/\sqrt{-x})$, $\sigma'_z = \sigma_{\mathrm{std}}(-\sqrt{-x/3} + 1/\sqrt{-x})$, $\gamma'_v = \gamma_{\mathrm{std}}(\sqrt{-x/3} + 1/\sqrt{-x})$ and $\gamma'_w = \gamma_{\mathrm{std}}(\sqrt{-x/3} - 1/\sqrt{-x})$, and call the 4-fold contour consisting of them $\mathbb{X}$. Also when we consider contour integrals in the form of $\int_{\overline{C}}^{C}$ in (4.26) and (4.25) where $C \in \mathbb{C}_+$, we fix the shape of the contour to be the part of $\sigma_{\mathrm{std}}(a)$ between $\overline{C}$ and C, where $a = \Re(C) - \Im(C)/\sqrt{3}$. Below we also consider contour integrals denoted as $\int_{\overline{C},\mathrm{right}}^{C}$, whose contour is the part of $\gamma_{\mathrm{std}}(b)$ between $\overline{C}$ and C, where $b = \Re(C) + \Im(C)/\sqrt{3}$. (For symmetry, $\int_{\overline{C}}^{C}$ should be expressed $\int_{\overline{C},\mathrm{left}}^{C}$. But since it occurs more often than $\int_{\overline{C},\mathrm{right}}^{C}$, we omit the left subscript for simplicity.)

(1) The 4-fold integral (4.26a) can be estimated in the same way as the 2-fold integral (4.13a). To perform the saddle point analysis, we denote the subsets of $\mathbb{X}$

   (i) $\mathbb{X}_1 = \{u, v, z, w \in \mathbb{X} \mid |u - \sqrt{-x}\mathrm{i}| < 1,\ |v - \sqrt{-x}\mathrm{i}| < 1,\ |z - \sqrt{-x}\mathrm{i}| < 1,\ |w - \sqrt{-x}\mathrm{i}| < 1\}$,

   (ii) $\mathbb{X}_2 = \{u, v, z, w \in \mathbb{X} \mid |u + \sqrt{-x}\mathrm{i}| < 1,\ |v + \sqrt{-x}\mathrm{i}| < 1,\ |z + \sqrt{-x}\mathrm{i}| < 1,\ |w + \sqrt{-x}\mathrm{i}| < 1\}$,

   (iii) $\mathbb{X}_3 = \mathbb{X} \setminus (\mathbb{X}_1 \cup \mathbb{X}_2)$.

   By standard saddle point method, we have that the parts of integral (4.26a) over $\mathbb{X}_1$ and $\mathbb{X}_2$ are both $\mathcal{O}(|x|^{-1})$, and the part of integral (4.26a) over $\mathbb{X}_3$ is $o(|x|^{-1})$. Hence we have that (4.26a) is $\mathcal{O}(|x|^{-1})$.

(2) Both of the 3-fold integrals (4.26b) and (4.26c) can be evaluated similarly, and they are $\mathcal{O}(|x|^{-1})$.

(3) The 3-fold integral (4.25b) can be written as the sum of

$$\frac{1}{(2\pi \mathrm{i})^3}\int_{\sigma_z'}\mathrm{d}z\,\mathrm{P.V.}\int_{\gamma_w''}\mathrm{d}w\int_{\bullet\diamond}^{\bullet\diamond}\mathrm{d}u\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)$$

$$\times\frac{1}{(z-w)(u-w)(z-u)(u-a_k)(w-a_k)} \tag{4.27a}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\bullet\diamond}^{\bullet\diamond}\mathrm{d}u\int_{-\sqrt{-x}\mathrm{i}+e^{-\pi \mathrm{i}/6}\sqrt{-3/x}}^{\sqrt{-x}\mathrm{i}+e^{\pi \mathrm{i}/6}\sqrt{-3/x}}\mathrm{d}z\frac{-1}{(u-z)^2(u-a_k)(z-a_k)} \tag{4.27b}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\sigma_z''}\mathrm{d}z\int_{\bullet\diamond}^{\bullet\diamond}\mathrm{d}u\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{u^3}{3}-xu}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{u-a_j}\right)\frac{1}{(z-u)^2(u-a_k)^2} \tag{4.27c}$$

$$+\frac{1}{2\pi \mathrm{i}}\int_{\bullet\diamond}^{\bullet\diamond}\mathrm{d}u\left((u^2-x)+\sum_{j=1}^{k-1}\frac{1}{u-a_j}\right)\frac{1}{(u-a_k)^2}, \tag{4.27d}$$

where the contours $\gamma_w''=\gamma_{\mathrm{std}}(\sqrt{-x/3}+2/\sqrt{-x})$ and $\sigma_z''=\sigma_{\mathrm{std}}(-\sqrt{-x/3}-2/\sqrt{-x})$. We note that σ_z'' is similar to σ_z' but keeps the contour for u in (4.27c) on its right, and the contour γ_w'' is similar to γ_w' but intersects σ_z' at $\sqrt{-x}\mathrm{i}+e^{\pi \mathrm{i}/6}\sqrt{-3/x}$ and $-\sqrt{-x}\mathrm{i}+e^{-\pi \mathrm{i}/6}\sqrt{-3/x}$. On the other hand, the 3-fold (4.25c) can be written as the sum of

$$\frac{1}{(2\pi \mathrm{i})^3}\int_{\sigma_u'}\mathrm{d}u\,\mathrm{P.V.}\int_{\gamma_w''}\mathrm{d}w\int_{\diamond\bullet}^{\diamond\bullet}\mathrm{d}z\frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{w-a_j}\right)$$

$$\times\frac{1}{(u-z)(z-w)(u-w)(z-a_k)(w-a_k)} \tag{4.28a}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{-\sqrt{-x}\mathrm{i}+(1-3\sqrt{3}\mathrm{i}))/\sqrt{-4x}}^{\sqrt{-x}\mathrm{i}+(1+3\sqrt{3}\mathrm{i}))/\sqrt{-4x}}\mathrm{d}u\int_{\diamond\bullet}^{\diamond\bullet}\mathrm{d}z\frac{-1}{(u-z)^2(u-a_k)(z-a_k)} \tag{4.28b}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\sigma_u'}\mathrm{d}u\int_{\diamond\bullet}^{\diamond\bullet}\mathrm{d}z\frac{e^{\frac{u^3}{3}-xu}}{e^{\frac{z^3}{3}-xz}}\left(\prod_{j=1}^{k-1}\frac{u-a_j}{z-a_j}\right)\frac{1}{(u-z)^2(z-a_k)^2}, \tag{4.28c}$$

where the contour $\sigma_w''=\sigma_{\mathrm{std}}(-\sqrt{-x/3}-2/\sqrt{-x})$ is the same as σ_z'' in (4.27c), and it intersects σ_u' at $\sqrt{-x}\mathrm{i}+(1+3\sqrt{3}\mathrm{i}))/\sqrt{-4x}$ and $-\sqrt{-x}\mathrm{i}+(1-3\sqrt{3}\mathrm{i}))/\sqrt{-4x}$.

(4) The 2-fold integral (4.26d) can be written as the sum

$$\frac{1}{(2\pi \mathrm{i})^2}\iint_X \mathrm{d}v\mathrm{d}z\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{-4}{(w-a_k)^2(z-a_k)(z-w)} \tag{4.29a}$$

$$+\frac{1}{2\pi \mathrm{i}}\int_{-\sqrt{-x}\mathrm{i}}^{\sqrt{-x}\mathrm{i}}\mathrm{d}z\frac{4}{(z-a_k)^3}, \tag{4.29b}$$

by the same transform as (4.12) is transformed into (4.13), and the double contour X and the single contour in (4.29b) are the same as in (4.13). Note that the contour (4.13) is to the right of all a_k's while the contour in (4.29b) is to the left of all a_k's.

(5) The 2-fold integral (4.26e) can be written as the sum of

$$\frac{1}{(2\pi \mathrm{i})^2}\int_{\sigma_z''}\mathrm{d}z\int_{\bullet\diamond}^{\bullet\diamond}\mathrm{d}w\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{-1}{(w-a_k)^2(z-w)^2} \tag{4.30a}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\sigma_z''}\mathrm{d}z\int_{\bullet\diamond,\mathrm{right}}^{\bullet\diamond}\mathrm{d}w\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{1}{(w-a_k)^2(z-w)^2} \tag{4.30b}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\sigma_z''}\mathrm{d}z\int_{\overline{\diamond\bullet}}^{\diamond\bullet}\mathrm{d}w\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{-1}{(w-a_k)^2(z-w)^2} \tag{4.30c}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\sigma_z''}\mathrm{d}z\int_{\overline{\diamond\bullet},\mathrm{right}}^{\diamond\bullet}\mathrm{d}w\frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}}\left(\prod_{j=1}^{k-1}\frac{z-a_j}{w-a_j}\right)\frac{1}{(w-a_k)^2(z-w)^2}, \tag{4.30d}$$

where σ_z'' is defined the same as σ_z'' in (4.27c). We note that (4.30a) cancels with (4.27c), and (4.30c) cancels with (4.28c).

(6) The 1-fold integral (4.25e), after taking into account of the negative sign in (4.16b), cancels with (4.20c).

(7) (4.27d) can be evaluated similarly to (4.20b), and it is $2\sqrt{-x}/\pi - a_k + \mathcal{O}(|x|^{-1/2})$.

(8) (4.27b) and (4.28b) are $\mathcal{O}(|x|^{-1/2}\log|x|)$, and all the other integrals from (4.27a) to (4.30d) not mentioned above, are $\mathcal{O}(|x|^{-1/2})$, as $x \to -\infty$. Since the evaluations are all by standard saddle point analysis, we omit the details.

Hence we obtain the final proof of (4.7).

5 Eigenvector distribution for GUE with external source

Recall the notations σ_i in (1.4) and $\mathbf{x}_i$ in (1.5) for the eigenvalues and eigenvectors of $G_{\boldsymbol{\alpha}} = G_{\boldsymbol{\alpha}}^{(N)}$. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_{N-1}$ be the ordered eigenvalues of $G_{\boldsymbol{\alpha}}^{(N-1)}$, which is obtained by removing the first column and first row of $G_{\boldsymbol{\alpha}}$.

In order to prove Theorem 2, we define, analogous to (3.4), (3.5) and (3.6), the random measure $\mu_N = \mu_N^{(k)}$ with the random density function $\phi_N(x)$, (random) complementary distribution function $M_N(x)$ for μ_N, and the mean of $M_N(x)$ such that

$$\phi_N(x) = \begin{cases} 1 & \sigma_i < x \leq \lambda_{i-1} \text{ for all } i \in [\![1, N]\!] \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \begin{aligned} M_N(x) &= \mu_N((x, +\infty)) = \int_x^\infty \phi_N(t)\mathrm{d}t, \\ F_N(x) &= \mathbb{E}\, M_N(x). \end{aligned} \tag{5.1}$$

Then for any $L \in [\![j, N]\!]$,

$$|x_{j1}|^2 = \prod_{i=1}^{j-1}\frac{\sigma_j - \lambda_i}{\sigma_j - \sigma_i}\prod_{i=j+1}^{L}\frac{\sigma_j - \lambda_{i-1}}{\sigma_j - \sigma_i}\mathcal{G}_{j,L}^{(k)}(\boldsymbol{\alpha}; N), \quad \text{where} \quad \mathcal{G}_{j,L}^{(k)}(\boldsymbol{\alpha}; N) := \prod_{i=L+1}^{N}\frac{\sigma_j - \lambda_{i-1}}{\sigma_j - \sigma_i}. \tag{5.2}$$

Here the superscript k in the notation $\mathcal{G}_{j,L}^{(k)}(\boldsymbol{\alpha}; N)$ reminds us that the external source is of rank k; see $\alpha_1, \dots, \alpha_k$ in Assumption 1 that represent the external source. Analogous to (3.3) and (3.8), we write

$$\log \mathcal{G}_{j,L}^{(k)}(\boldsymbol{\alpha}; N) = \int_{\sigma_N}^{\lambda_L}\frac{1}{\sigma_j - x}\mathrm{d}M_N(x) = \int_{\sigma_N}^{\sigma_L}\frac{1}{\sigma_j - x}\mathrm{d}M_N(x). \tag{5.3}$$

Analogous to Proposition 10, we have the following key technical result, whose proof will be given in Section 6.

Proposition 11. *1. Let $\epsilon > 0$ be any small (but fixed) constant and N be large enough. For $x \in ((-2+\epsilon)\sqrt{N}, 2\sqrt{N} - N^{-1/10})$, we have*

$$F_N(x) = E_N(x) + \mathcal{O}(N^{-\frac{1}{12}}(2\sqrt{N} - x)^{\frac{1}{2}}), \quad E_N(x) = \frac{1}{2\pi}\left(-\sqrt{4N - x^2} + x\arccos\frac{x}{2\sqrt{N}}\right) + \sqrt{N} - \frac{x}{2}, \tag{5.4}$$

$$\operatorname{Var} M_N(x) = V_N(x) + \mathcal{O}(N^{-\frac{7}{24}}(2\sqrt{N} - x)^{\frac{1}{4}}), \quad V_N(x) = \frac{1}{\pi}\left(\sqrt{1 - \frac{x^2}{4N}} + \arccos\frac{x}{2\sqrt{N}}\right). \tag{5.5}$$

2. *Let C be a large enough positive constant and N be large enough. For $x \in [2\sqrt{N} - N^{-1/10}, 2\sqrt{N} - CN^{-1/6}]$, we have, with $\xi = N^{1/6}(x - 2\sqrt{N})$,*

$$F_N(x) = N^{-1/6}\left(\frac{-\xi}{2} + \mathcal{O}(1)\right), \tag{5.6}$$

$$\operatorname{Var} M_N(x) = N^{-1/3}\left(\frac{2}{\pi}\sqrt{-\xi} + \mathcal{O}(|\xi|^{1/4})\right). \tag{5.7}$$

To prove Theorem 2, we first note that by Lemma 6, upon the scaling $x \mapsto N^{1/6}(x - 2\sqrt{N})$, given a fixed L, $\sigma_1, \dots, \sigma_L$ and $\lambda_1, \dots, \lambda_L$ converge weakly and jointly to $\xi_1^{(k)}, \dots, \xi_L^{(k)}$ and $\xi_1^{(k-1)}, \dots, \xi_L^{(k-1)}$'s respectively, as $N \to \infty$. Hence for any fixed $L > j$,

$$\prod_{i=1}^{j-1} \frac{\sigma_j - \lambda_i}{\sigma_j - \sigma_i} \prod_{i=j+1}^{L} \frac{\sigma_j - \lambda_{i-1}}{\sigma_j - \sigma_i} \xrightarrow{d} \prod_{i=1}^{j-1} \frac{\xi_j^{(k)} - \xi_i^{(k-1)}}{\xi_j^{(k)} - \xi_i^{(k)}} \prod_{i=j+1}^{L} \frac{\xi_j^{(k)} - \xi_{i-1}^{(k-1)}}{\xi_j^{(k)} - \xi_i^{(k)}} = L^{-\frac{1}{3}} \Xi_j^{(k)}(\mathbf{a}; L). \tag{5.8}$$

Using the convergence result in Theorem 1, we only need to show that for any $\epsilon_1, \epsilon_2 > 0$, there is a sufficiently large $L_{\epsilon_1,\epsilon_2} > 0$ such that for any $L \geq L_{\epsilon_1,\epsilon_2}$,

$$\limsup_{N\to\infty} \mathbb{P}\left(\left|\log \mathcal{G}_{j,L}^{(k)}(\boldsymbol{\alpha}; N) - \left[\log \frac{L^{1/3}}{N^{1/3}} + \log\left(\frac{3\pi}{2}\right)^{1/3}\right]\right| > \epsilon_1\right) < \epsilon_2. \tag{5.9}$$

Here the deterministic term in (5.9) originates from the elementary identity

$$\begin{aligned}
&\int_{-2\sqrt{N}}^{2\sqrt{N}-N^{-1/6}(3\pi L/2)^{2/3}} \frac{\mathrm{d}E_N(x)}{2\sqrt{N} - x} \\
&= \frac{1}{2\pi}\int_{-1}^{1-\frac{1}{2}(3\pi L N^{-1}/2)^{2/3}} \frac{\arccos y - \pi}{1 - y}\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-1}^{1} \frac{\arccos y}{1 - y}\mathrm{d}y - \frac{1}{2}\int_{-1}^{1-\frac{1}{2}(3\pi L N^{-1}/2)^{2/3}} \frac{\mathrm{d}y}{1 - y} - \frac{1}{2\pi}\int_{1-\frac{1}{2}(3\pi L N^{-1}/2)^{2/3}}^{1} \frac{\arccos y}{1 - y}\mathrm{d}y \\
&= \log 2 - \frac{1}{2}\left(\log 2 - \log\left(\frac{1}{2}\left(\frac{3\pi L}{2N}\right)^{2/3}\right)\right) + \mathcal{O}(N^{-\frac{1}{3}}) = \log\frac{L^{1/3}}{N^{1/3}} + \log\left(\frac{3\pi}{2}\right)^{1/3} + \mathcal{O}(N^{-\frac{1}{3}}),
\end{aligned} \tag{5.10}$$

where we use the fact that $\frac{1}{2\pi}\int_{-1}^{1} \frac{\arccos y}{1-y}\mathrm{d}y = -\frac{2}{\pi}\int_0^{\pi/2} \log(\sin\theta)\mathrm{d}\theta$ by integration by parts, and then it is equal to $\log 2$ by [36, 4.224.3]. Hence it is equivalent to show that for all large enough L,

$$\limsup_{N\to\infty} \mathbb{P}\left(\left|\int_{\sigma_N}^{\sigma_L} \frac{1}{\sigma_j - x}\mathrm{d}M_N(x) - \int_{-2\sqrt{N}}^{2\sqrt{N}-N^{-1/6}(3\pi L/2)^{2/3}} \frac{1}{2\sqrt{N} - x}\mathrm{d}E_N(x)\right| > \epsilon_1\right) < \epsilon_2. \tag{5.11}$$

In order to prove (5.11), we will mainly rely on Proposition 11. There is an apparent obstacle: By part 1 of Lemma 9, σ_j and σ_N are close to $2\sqrt{N}$ and $-2\sqrt{N}$, respectively, but our estimates in Proposition 11 do not cover the domain $[-2\sqrt{N}, (-2+\epsilon)\sqrt{N})$. In order to handle the integral over $[-2\sqrt{N}, (-2+\epsilon)\sqrt{N})$, we need the observation that the measure $\mathrm{d}\mu_N(x) = -\mathrm{d}M(x)$ and also its expectation $-\mathrm{d}F(x)$ are dominated by the Lebesgue measure, according to the definition in (5.1). To be precise, we have that for any $\epsilon_3 > 0$, there is a small enough $\epsilon > 0$, such that for all large enough N

$$\left|\int_{-2\sqrt{N}}^{(-2+\epsilon)\sqrt{N}} \frac{1}{2\sqrt{N} - x}\mathrm{d}E_N(x)\right| < \epsilon_3, \quad \text{and} \quad \left|\int_{\sigma_N}^{(-2+\epsilon)\sqrt{N}} \frac{1}{\sigma_j - x}\mathrm{d}M_N(x)\right| < \epsilon_3 \quad \text{with high probability.} \tag{5.12}$$

Hence, in order to see (5.11), it remains to show that given any $\epsilon > 0$, $\epsilon_1, \epsilon_2 > 0$, for all large enough (but fixed) L, such that for all large enough N, one has

$$\mathbb{P}\left(\left|\int_{(-2+\epsilon)\sqrt{N}}^{\sigma_L} \frac{1}{\sigma_j - x}\mathrm{d}M_N(x) - \int_{(-2+\epsilon)\sqrt{N}}^{2\sqrt{N}-N^{-1/6}(3\pi L/2)^{2/3}} \frac{1}{2\sqrt{N} - x}\mathrm{d}E_N(x)\right| > \epsilon_1\right) < \epsilon_2. \tag{5.13}$$

Next, we observe that by (5.4) and (5.6), for any $\epsilon_3 > 0$, there is a small enough $\epsilon > 0$, such that for all large enough N

$$\left| \int_{(-2+\epsilon)\sqrt{N}}^{2\sqrt{N}-N^{-1/6}(3\pi L/2)^{2/3}} \frac{1}{2\sqrt{N}-x} \mathrm{d}(E_N(x) - F_N(x)) \right| < \epsilon_3. \tag{5.14}$$

Hence, instead of (5.13), we only need to show, under the same assumption,

$$\mathbb{P}\left(\left| \int_{(-2+\epsilon)\sqrt{N}}^{\sigma_L} \frac{1}{\sigma_j - x} \mathrm{d}M_N(x) - \int_{(-2+\epsilon)\sqrt{N}}^{2\sqrt{N}-N^{-1/6}(3\pi L/2)^{2/3}} \frac{1}{2\sqrt{N}-x} \mathrm{d}F_N(x) \right| > \epsilon_1 \right) < \epsilon_2. \tag{5.15}$$

Theorem 2 will then follow if (5.15) is proved.

Proof of (5.15). First, we set the integer

$$N_0 = \left\lfloor \frac{2}{3\pi} N^{\frac{1}{10}} \right\rfloor. \tag{5.16}$$

We note that by part 2 of Lemma 9, $\sigma_{N_0} = 2\sqrt{N} - N^{-1/10} + \mathcal{O}(N^{-8/75})$ with high probability. To prove (5.15), it suffices to show the following two inequalities with the same conditions:

$$\mathbb{P}\left(\left| \int_{\sigma_{N_0}}^{\sigma_L} \frac{1}{\sigma_j - x} \mathrm{d}M_N(x) - \int_{2\sqrt{N}-N^{-1/10}}^{2\sqrt{N}-N^{-1/6}(3\pi L/2)^{2/3}} \frac{1}{2\sqrt{N}-x} \mathrm{d}F_N(x) \right| > \epsilon_1 \right) < \epsilon_2, \tag{5.17}$$

$$\mathbb{P}\left(\left| \int_{(-2+\epsilon)\sqrt{N}}^{\sigma_{N_0}} \frac{1}{\sigma_j - x} \mathrm{d}M_N(x) - \int_{(-2+\epsilon)\sqrt{N}}^{2\sqrt{N}-N^{-1/10}} \frac{1}{2\sqrt{N}-x} \mathrm{d}F_N(x) \right| > \epsilon_1 \right) < \epsilon_2. \tag{5.18}$$

The proof of (5.17) is similar to that of (3.11) after a change of variable $\xi = N^{1/6}(x - 2\sqrt{N})$. Actually, we can show the following stronger estimate, which is an analogue of (3.11):

$$\mathbb{P}\left(\sup_{L<n\leq N^{1/10}} \left| \int_{\sigma_n}^{\sigma_L} \frac{1}{\sigma_j - x} \mathrm{d}M_N(x) - \int_{2\sqrt{N}-N^{-1/6}(3\pi n/2)^{2/3}}^{2\sqrt{N}-N^{-1/6}(3\pi L/2)^{2/3}} \frac{1}{2\sqrt{N}-x} \mathrm{d}F_N(x) \right| > \epsilon_1 \right) < \epsilon_2, \tag{5.19}$$

where the supremum for n is bounded by L and $N^{1/10}$, rather than in (3.11) for all n greater than m. Note that the proof of (3.11) only relies on three ingredients as explained in Remark 4, and all of them have their counterparts in the proof of (5.19): the counterpart of Proposition 10 is part 2 of Proposition 11; the counterpart of Lemma 8 is parts 1 and 2 of Lemma 9; the dominance by Lebesgue measure is the same for μ and μ_N. The only slight difference is that part 2 of Lemma 9 is only valid for n less than $N^{1/10}$ since the Airy process approximation of the GUE minor process with external source can only be extended to such an intermediate regime.

The proof of (5.18) again follows the idea of the proof of (3.11), with some necessary modifications. We describe the proof parallel to that of (3.11), and give detailed explanations on the modifications. Analogous to (3.12), we denote the events

$$\widehat{\Omega}_{j,m}^{(k)} := \left\{ \omega \;\middle|\; |N^{\frac{1}{6}}(\sigma_j - 2\sqrt{N})| \leq m^{\frac{3}{5}},\ |\sigma_m - \Upsilon_m| \leq N^{-\frac{1}{6}} m^{\frac{3}{5}} \right\}, \tag{5.20}$$

where Υ_m is defined in (2.14).

1. Analogous to (3.13), we have the estimate that if N is large enough, then

$$\left| \int_{(-2+\epsilon)\sqrt{N}}^{\sigma_{N_0}} \frac{1}{\sigma_j - x} \mathrm{d}F_N(x) - \int_{(-2+\epsilon)\sqrt{N}}^{2\sqrt{N}-N^{-1/10}} \frac{1}{2\sqrt{N}-x} \mathrm{d}F_N(x) \right| \mathbf{1}(\widehat{\Omega}_{j,N_0}^{(k)}) < \frac{2\epsilon_1}{3}. \tag{5.21}$$

2. Next, analogous to (3.14), we show

$$\mathbb{P}\left(\left| \int_{(-2+\epsilon)\sqrt{N}}^{\sigma_{N_0}} \frac{1}{\sigma_j - x} \mathrm{d}(M_N(x) - F_N(x)) \mathbf{1}(\widehat{\Omega}_{j,N_0}^{(k)}) \right| > \frac{\epsilon_1}{3} \right) < \frac{\epsilon_2}{2}. \tag{5.22}$$

Using an integration by parts decomposition like (3.15), we find that it suffices to show

$$\mathbb{P}\left(\left|\frac{M_N(\sigma_{N_0})-F_N(\sigma_{N_0})}{\sigma_j-\sigma_{N_0}}\mathbf{1}(\widehat{\Omega}^{(k)}_{j,N_0})\right|>\frac{\epsilon_1}{9}\right)<\frac{\epsilon_2}{6}, \tag{5.23}$$

$$\mathbb{P}\left(\left|\frac{M_N((-2+\epsilon)\sqrt{N})-F_N((-2+\epsilon)\sqrt{N})}{\sigma_j-(-2+\epsilon)\sqrt{N}}\mathbf{1}(\widehat{\Omega}^{(k)}_{j,N_0})\right|>\frac{\epsilon_1}{9}\right)<\frac{\epsilon_2}{6}, \tag{5.24}$$

$$\mathbb{P}\left(\left|\int_{(-2+\epsilon)\sqrt{N}}^{\sigma_{N_0}}\frac{M_N(x)-F_N(x)}{(\sigma_j-x)^2}\mathrm{d}x\mathbf{1}(\widehat{\Omega}^{(k)}_{j,N_0})\right|>\frac{\epsilon_1}{9}\right)<\frac{\epsilon_2}{6}. \tag{5.25}$$

We first see that (5.24) only needs that $M_N((-2+\epsilon)\sqrt{N})$ is close to $F_N((-2+\epsilon)\sqrt{N})$, and it is a direct consequence of (5.5) at $x=(-2+\epsilon)\sqrt{N}$, by using Markov inequality. Estimate (5.23) can be verified in the same way as for (3.16). On the other hand, to verify (5.25), it suffices to show that

$$\mathbb{P}\left(\int_{(-2+\epsilon)\sqrt{N}}^{2\sqrt{N}-N^{-1/10}}\frac{|M_N(x)-F_N(x)|}{(2\sqrt{N}-x)^2}\mathrm{d}x>\frac{\epsilon_1}{9}\right)<\frac{\epsilon_2}{6}, \tag{5.26}$$

by the arguement like that around (3.25) and (3.26). More specifically, to prove (5.26), we introduce

$$\ell_i=2\sqrt{N}-iN^{-\frac{1}{6}} \tag{5.27}$$

and analogous to (3.27) we set

$$\begin{aligned}\widehat{A}_{k_1,k_2}&=\sum_{i=k_1}^{k_2}\frac{|M_N(\ell_i)-F_N(\ell_i)|}{(i-1)i},\quad \widehat{B}_{k_1,k_2}=\sum_{i=k_1}^{k_2}\frac{|M_N(\ell_i)-F_N(\ell_i)|}{i(i+1)},\\ \widehat{C}_{k_1,k_2}&=2\sum_{i=k_1+1}^{k_2-1}\frac{F_N(\ell_i)}{(i-1)i(i+1)}-\frac{F_N(\ell_{k_1})}{k_1(k_1+1)}+\frac{F_N(\ell_{k_2})}{(k_2-1)k_2}.\end{aligned} \tag{5.28}$$

Then analogous to (3.29) and (3.30), we have (without loss of generality, assuming $N^{1/15}$ and $(4-\epsilon)N^{2/3}$ are integers)

$$\int_{(-2+\epsilon)\sqrt{N}}^{2\sqrt{N}-N^{-1/10}}\frac{|M_N(x)-F_N(x)|}{(2\sqrt{N}-x)^2}\mathrm{d}x\le N^{\frac{1}{6}}(A_{N^{1/15},(4-\epsilon)N^{2/3}}+B_{N^{1/15},(4-\epsilon)N^{2/3}}+2C_{N^{1/15},(4-\epsilon)N^{2/3}}), \tag{5.29}$$

and need only to prove

$$\mathbb{P}\left(N^{\frac{1}{6}}A_{N^{1/15},(4-\epsilon)N^{2/3}}>\frac{\epsilon_1}{27}\right)<\frac{\epsilon_2}{12},\quad \mathbb{P}\left(N^{\frac{1}{6}}B_{N^{1/15},(4-\epsilon)N^{2/3}}>\frac{\epsilon_1}{27}\right)<\frac{\epsilon_2}{12},\quad 2N^{\frac{1}{6}}C_{N^{1/15},(4-\epsilon)N^{2/3}}<\frac{\epsilon_1}{27}, \tag{5.30}$$

As N is large enough, it is straightforward to show the last inequality for $C_{N^{1/15},(4-\epsilon)N^{2/3}}$. Also, like (3.33), we have that for large enough N,

$$\begin{aligned}\mathbb{P}\left(A_{N^{1/15},(4-\epsilon)N^{2/3}}>\frac{\epsilon_1}{27}N^{-\frac{1}{6}}\right)&\le\mathbb{P}\left(A_{N^{1/15},(4-\epsilon)N^{2/3}}>\frac{\epsilon_1}{27}N^{-\frac{1}{6}}\right)\\ &\le\mathbb{P}\left(\sum_{i=N^{1/15}}^{(4-\epsilon)N^{2/3}}\frac{|M_N(\ell_i)-F_N(\ell_i)|}{i^2}>\frac{\epsilon_1}{54}N^{-\frac{1}{6}}\right)\\ &\le\mathbb{P}\left(\bigcup_{i=N^{1/15}}^{(4-\epsilon)N^{2/3}}\left\{\frac{|M_N(\ell_i)-F_N(\ell_i)|}{i^2}>i^{-\frac{6}{5}}N^{-\frac{1}{6}}\right\}\right)\\ &\le\sum_{i=N^{1/15}}^{(4-\epsilon)N^{2/3}}\mathbb{P}\left(\frac{|M_N(\ell_i)-F_N(\ell_i)|}{i^2}>i^{-\frac{6}{5}}N^{-\frac{1}{6}}\right)\\ &\le\sum_{i=N^{1/15}}^{(4-\epsilon)N^{2/3}}i^{-\frac{11}{10}}<\frac{\epsilon_2}{12}.\end{aligned} \tag{5.31}$$

We thus finish the proof of (5.26), and then (5.25), and finally finish the proof of (5.22).

3. At last, we need to show that the event $\widehat{\Omega}^{(k)}_{j,N_0}$ satisfies that $\mathbb{P}(\widehat{\Omega}^{(k)}_{j,N_0}) > 1 - \epsilon_2/2$ if N is large enough. (This is analogous to (3.35) but is much weaker.) This follows directly from the estimates of σ_j and σ_{N_0} in parts 1 and 2 of Lemma 9.

Hence, we conclude the proof of Theorem 2. □

Next, we prove Corollary 3.

Proof. Let $U = (u_{ij}) \in \mathbb{C}^{(N-k)\times(N-k)}$ be a any unitary matrix, then by the unitary invariance of GUE, the random matrix $(I_k \oplus U)G_{\boldsymbol{\alpha}}(I_k \oplus U^*)$ has the same distribution as $G_{\boldsymbol{\alpha}}$. Hence the eigenvector $\mathbf{x}_j = (x_{j1}, \dots, x_{jN})^\top$ of $G_{\boldsymbol{\alpha}}$ associated with σ_j, which is a random unit vector, satisfies that fixing $x_{j1}, \dots, x_{jk}$, the conditional distribution of the truncated random vector $(x_{j,k+1}, \dots, x_{jN})^\top$ is unitarily invariant.

Furthermore, we have $\sum_{i=k+1}^N |x_{ji}|^2 = 1 - O(kN^{-1/3})$ with high probability by Theorem 2. So the distribution of x_{ji} is approximated by that of a component of a Haar distributed complex unit random vector in $\mathbb{C}^{N-k}$, as $N \to \infty$. Combining the above facts we can conclude the proof of Corollary 3 easily. □

6 Analysis of random measure μ_N: proof of Proposition 11

To prove (5.4) and (5.6), we use the method similar to the proof of (3.7). We consider, analogous to (4.2), the random variable (with notation abused)

$$N_x = |\mathcal{I}_x| - |\mathcal{J}_x| \quad \text{where} \quad \mathcal{I}_x := \{i \in \mathbb{N} \mid \sigma_i \in (x, +\infty)\}, \quad \mathcal{J}_x := \{i \in \mathbb{N} \mid \lambda_i \in (x, +\infty)\}. \tag{6.1}$$

By the interlacing property, we note that N_x is a Bernoulli random variable. Then

$$\mathbb{E}\, N_x = \mathbb{P}(N_x = 1). \tag{6.2}$$

We also consider, analogous to (4.4) the random variable (with notation abused)

$$S_x = -\sum_{i\in\mathcal{I}_x} \sigma_i + \sum_{i\in\mathcal{J}_x} \lambda_i. \tag{6.3}$$

We observe that if $\sigma_1 \le x$, then $N_x = S_x = M_N(x) = 0$. Under the condition that $\sigma_1 > x$, we have that if $N_x = 1$, then $S_x = M_N(x) - \sigma_1$, otherwise $S_x = M_N(x) - \sigma_1 + x$. Similar to (4.5), we conclude that

$$M_N(x) = S_x + xN_x + (\sigma_1 - x)\mathbf{1}(\sigma_1 > x). \tag{6.4}$$

For the second term in (6.4) that only involves σ_1, by part 1 of Lemma 9, we have the following estimate that holds for all $x \in \mathbb{R}$:

$$\mathbb{E}[(\sigma_1 - x)\mathbf{1}(\sigma_1 > x)] = 2\sqrt{N} - x + \mathcal{O}(N^{-\frac{1}{6}}), \quad \mathrm{Var}[(\sigma_1 - x)\mathbf{1}(\sigma_1 > x)] = \mathcal{O}(N^{-\frac{1}{3}}). \tag{6.5}$$

Then by the linearity of expectation and (4.8), the mean and variance estimates in Proposition 11 follow from (6.5) and that for $x \in ((-2+\epsilon)\sqrt{N}, 2\sqrt{N} - N^{-1/10})$

$$\mathbb{E}(S_x + xN_x) = E_N(x) - (2\sqrt{N} - x) + \mathcal{O}(N^{-\frac{1}{12}}(2\sqrt{N} - x)^{\frac{1}{2}}), \tag{6.6}$$

$$\mathrm{Var}(S_x + xN_x) = V_N(x) + \mathcal{O}(N^{-\frac{1}{3}}), \tag{6.7}$$

and for $x \in [2\sqrt{N} - N^{-1/10}, 2\sqrt{N} - CN^{-1/6}]$, with $x = 2\sqrt{N} + N^{-1/6}\xi$, then we have the uniform convergence

$$N^{\frac{1}{6}}\, \mathbb{E}(S_x + xN_x) = \frac{\xi}{2} + \mathcal{O}(1), \tag{6.8}$$

$$N^{\frac{1}{3}}\, \mathrm{Var}(S_x + xN_x) = \frac{2}{\pi}\sqrt{-\xi} + \mathcal{O}(|\xi|^{\frac{1}{4}}). \tag{6.9}$$

Below we prove (6.6), (6.7) and (6.8). The proof of (6.9) is similar to that of (6.7) and we only give a sketch.

Proof of (6.6) To faciliate the saddle point analysis below, we define several types of contours. First, we recall that $\alpha_1, \dots, \alpha_k$ are defined by $a_1, \dots, a_k$ in Assumption 1. We take a constant $r \in [0, +\infty)$ such that r is bigger than all $a_1, \dots, a_k$. Then we let $\Gamma^{\circ}_{\mathrm{std},N}(a)$ be the positively oriented boundary of the open set $\{z \in \mathbb{C} \mid |z| < \sqrt{N} + a\} \cup \{z \in \mathbb{C} \mid |z - \sqrt{N}| < N^{1/6} r\}$ where $a \geq 0$, such that it is almost the circle centred at 0 with radius $\sqrt{N} + a$ and it encloses all $\alpha_1, \dots, \alpha_k$. (Actually, if $r = 0$, then Γ is the circle centred at 0 with radius $\sqrt{N} + a$.) We also define for any $b \in \mathbb{R}$ the upward vertical contour

$$\Sigma^{|}_{\mathrm{std},N}(b) = \{b + it \mid -\infty < t < +\infty\}. \tag{6.10}$$

Next, for any $\theta \in (0, \pi/6)$, we define the positively oriented contour

$$\begin{aligned}\Gamma^{>}_{\mathrm{std},N}(\theta, a) = {} & \left\{ (\sqrt{N} + a) e^{(\frac{\pi}{3} - \theta)\mathrm{i}} + e^{\frac{2\pi}{3}\mathrm{i}} t \mid -\frac{2}{\sqrt{3}} (\sqrt{N} + a) \sin\left(\frac{\pi}{3} - \theta\right) \leq t \leq 0 \right\} \\ \cup & \left\{ (\sqrt{N} + a) e^{(\frac{5\pi}{3} + \theta)\mathrm{i}} + e^{\frac{\pi}{3}\mathrm{i}} t \mid 0 \leq t \leq \frac{2}{\sqrt{3}} (\sqrt{N} + a) \sin\left(\frac{\pi}{3} - \theta\right) \right\} \cup \left\{ (\sqrt{N} + a) e^{it} \mid \frac{\pi}{3} - \theta \leq t \leq \frac{5\pi}{3} + \theta \right\}.\end{aligned} \tag{6.11}$$

We also define for any $b < \sqrt{N}$ the upward infinite polygonal contour

$$\begin{aligned}\Sigma^{<}_{\mathrm{std},N}(b) = {} & \left\{ b + e^{\frac{\pi}{3}\mathrm{i}} t \mid 0 \leq t \leq 2(\sqrt{N} - b) + \sqrt{N} \right\} \cup \left\{ b + e^{\frac{2\pi}{3}\mathrm{i}} t \mid -\sqrt{N} - 2(\sqrt{N} - b) \leq t \leq 0 \right\} \\ & \cup \left\{ \frac{3}{2}\sqrt{N} + it \mid |t| \geq \sqrt{3}(\frac{3}{2}\sqrt{N} - b) \right\}.\end{aligned} \tag{6.12}$$

See Figures 11 and 12 for their shapes.

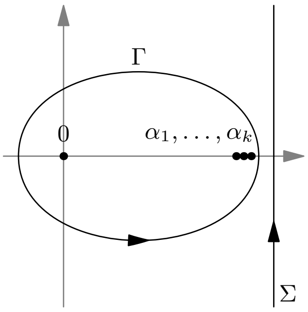

Figure 10: Contours Γ and Σ.

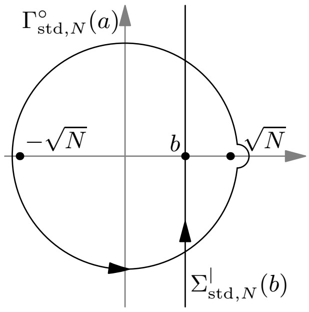

Figure 11: Shapes of $\Gamma^{\circ}_{\mathrm{std},N}(a)$ and $\Sigma^{|}_{\mathrm{std},N}(b)$.

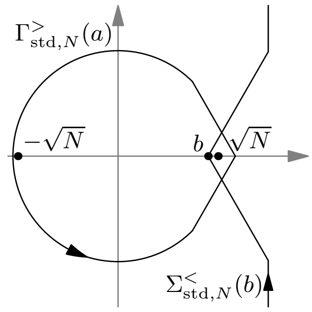

Figure 12: Shapes of $\Gamma^{>}_{\mathrm{std},N}(a)$ and $\Sigma^{<}_{\mathrm{std},N}(b)$.

We have that

$$\begin{aligned}\mathbb{E}(S_x + x N_x) &= \mathbb{E}\left(\sum_{i \in \mathcal{J}_x} \lambda_i \right) - \mathbb{E}\left(\sum_{i \in \mathcal{I}_x} \sigma_i \right) + x \,\mathbb{E}\,|\mathcal{I}_x| - x \,\mathbb{E}\,|\mathcal{J}_x| \\ &= \int_x^{\infty} t K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \mathrm{d}t - \int_x^{\infty} t K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \mathrm{d}t \\ &\quad + x \left(\int_x^{\infty} K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \mathrm{d}t - \int_x^{\infty} K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \mathrm{d}t \right) \\ &= - \int_x^{+\infty} (t - x) \left(K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) - K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \right) \mathrm{d}t.\end{aligned} \tag{6.13}$$

In light of (2.3), one has

$$K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) - K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma} \mathrm{d}z \int_{\Gamma} \mathrm{d}w \frac{e^{\frac{z^2}{2} - tz} z^{N-k}}{e^{\frac{w^2}{2} - tw} w^{N-k}} \left(\prod_{j=2}^{k} \frac{z - \alpha_j}{w - \alpha_j} \right) \frac{1}{w - \alpha_1}, \tag{6.14}$$

which implies

$$\begin{aligned}\mathbb{E}(S_x+xN_x)&=\frac{-1}{(2\pi \mathrm{i})^2}\int_\Sigma \mathrm{d}z\int_\Gamma \mathrm{d}w\frac{e^{\frac{z^2}{2}}z^{N-k}}{e^{\frac{w^2}{2}}w^{N-k}}\left(\prod_{j=2}^k\frac{z-\alpha_j}{w-\alpha_j}\right)\frac{1}{w-\alpha_1}\int_x^\infty e^{-(z-w)t}(t-x)\mathrm{d}t\\&=\frac{-1}{(2\pi \mathrm{i})^2}\int_\Sigma \mathrm{d}z\int_\Gamma \mathrm{d}w\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^k\frac{z-\alpha_j}{w-\alpha_j}\right)\frac{1}{(w-\alpha_1)(z-w)^2}.\end{aligned}\tag{6.15}$$

By some standard residue calculation, we have that

$$\mathbb{E}(S_x+xN_x)$$

$$=\frac{-1}{(2\pi \mathrm{i})^2}\iint_\Phi \mathrm{d}z\mathrm{d}w\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^k\frac{z-\alpha_j}{w-\alpha_j}\right)\frac{1}{(w-\alpha_1)(z-w)^2}\tag{6.16a}$$

$$+\frac{-1}{2\pi \mathrm{i}}\int_{\text{lowerintersect}}^{\text{upperintersect}}\frac{w-x+(N-k)/w}{w-\alpha_1}+\sum_{j=2}^k\frac{1}{(w-\alpha_j)(w-\alpha_1)}\mathrm{d}w,\tag{6.16b}$$

where (i) the contour Φ means that the Σ contour cuts into the Γ contour, and divide its interior into two parts, such that they intersect at the two saddle points $\frac{1}{2}(x+\sqrt{4N-x^2}\mathrm{i})$ and $\frac{1}{2}(x-\sqrt{4N-x^2}\mathrm{i})$, which are the upperintersect and lowerintersect respectly in (6.16b), and all $\alpha_1,\dots,\alpha_k$ are in the right part, (ii) the integral in (6.16a) is understood as the Cauchy principal value, and (iii) the contour of integral in (6.16b) goes by the right of 0 and all $\alpha_1,\dots,\alpha_k$. We then have

$$\begin{aligned}(6.16\text{b})&=\frac{-1}{2\pi \mathrm{i}}\int_{\frac{1}{2}(x-\sqrt{4N-x^2}\mathrm{i})}^{\frac{1}{2}(x+\sqrt{4N-x^2}\mathrm{i})}1+\left(\alpha_1-x+\frac{N-k}{\alpha_1}\right)\frac{1}{w-\alpha_1}-\frac{N-k}{\alpha_1}\frac{1}{w}\\&\qquad+\sum_{j=1}^k\frac{1}{\alpha_1-\alpha_j}\left(\frac{1}{w-\alpha_1}-\frac{1}{w-\alpha_j}\right)\mathrm{d}w\\&=\frac{1}{\pi}\left[-\frac{\sqrt{4N-x^2}}{2}-\left(\alpha_1-x+\frac{N-k}{\alpha_1}\right)\operatorname{arccot}\frac{x-2\alpha_1}{\sqrt{4N-x^2}}+\frac{N-k}{\alpha_1}\arccos\frac{x}{2\sqrt{N}}\right.\\&\qquad\left.-\sum_{j=2}^k\frac{1}{\alpha_1-\alpha_j}\left(\operatorname{arccot}\frac{x-2\alpha_1}{\sqrt{4N-x^2}}-\operatorname{arccot}\frac{x-2\alpha_j}{\sqrt{4N-x^2}}\right)\right]\\&=\frac{1}{\pi}\left[-\frac{\sqrt{4N-x^2}}{2}-\frac{1}{2}\left(\alpha_1-x-\frac{N-k}{\alpha_1}\right)\arccos\frac{x}{2\sqrt{N}}-\frac{\pi}{2}\left(\alpha_1-x+\frac{N-k}{\alpha_1}\right)\right.\\&\qquad-\left(\alpha_1-x+\frac{N-k}{\alpha_1}\right)\left(\operatorname{arccot}\frac{x-2\sqrt{N}-2N^{1/6}a_k}{\sqrt{4N-x^2}}-\operatorname{arccot}\frac{x-2\sqrt{N}}{\sqrt{4N-x^2}}\right)\\&\qquad\left.-\sum_{j=2}^k\frac{N^{-1/6}}{a_k-a_{k+1-j}}\left(\operatorname{arccot}\frac{x-2\sqrt{N}-2N^{1/6}a_k}{\sqrt{4N-x^2}}-\operatorname{arccot}\frac{x-2\sqrt{N}-2N^{1/6}a_{k+1-j}}{\sqrt{4N-x^2}}\right)\right]\\&=\frac{1}{\pi}\left[-\frac{\sqrt{4N-x^2}}{2}+\frac{x}{2}\arccos\frac{x}{2\sqrt{N}}-\frac{\pi}{2}(2\sqrt{N}-x)\right.\\&\qquad\left.-(2N^{1/6}a_k+\mathcal{O}(N^{-1/6}))\arccos\frac{x}{2\sqrt{N}}\right]+\mathcal{O}(N^{-1/5})\\&=E_N(x)-(2\sqrt{N}-x)+\mathcal{O}(N^{-\frac{1}{12}}(2\sqrt{N}-x)^{\frac{1}{2}}),\end{aligned}\tag{6.17}$$

where we note that $\alpha_i=\sqrt{N}+N^{1/6}a_{k-i+1}$ and $N^{-1/10}<2\sqrt{N}-x<(4-\epsilon)\sqrt{N}$, and all the $\mathcal{O}$ terms are uniform in x.

On the other hand, the integral in (6.16a) has the estimate as $\mathcal{O}((4N-x^2)^{-1/2})=\mathcal{O}(N^{-1/5})$. To see it, we deform the contours Γ and Σ into standard shapes depending on the value of x:

(i) If $(-2+\epsilon)\sqrt{N} < x < 1.9\sqrt{N}$, then let Γ be $\Gamma^{\circ}_{\mathrm{std},N}(0)$, and Σ be $\Sigma^{|}_{\mathrm{std},N}(x/2)$.

(ii) If $1.9\sqrt{N} \le x < 2\sqrt{N} - N^{-1/10}$, then let Γ be $\Gamma^{>}_{\mathrm{std},N}(\theta, 0)$ with $\theta = \arccos(x/\sqrt{4N})$, and Σ be $\Sigma^{<}_{\mathrm{std},N}(x/2 - \sqrt{N - x^2/4}/\sqrt{3})$.

Then we can estimate it by the standard saddle point analysis, similar to the analysis for the integral over X in (4.13a). We omit the detail.

Hence we obtain (6.6).

Proof of (6.8) Like in the proof of (6.6), we also make use of (6.16) and compute (6.16a) and (6.16b) separately. However, now we require that upperintersect and lowerintersect, the intersections of Σ with Γ, are $\sqrt{N} \pm (2\sqrt{N} - x)^{1/2} N^{1/4}\mathrm{i}$. (For $x \in [2\sqrt{N} - N^{-1/10}, 2\sqrt{N} - CN^{-1/6})$, they are very close to $\frac{1}{2}(x \pm \sqrt{4N - x^2}\mathrm{i})$.) Let $\xi = N^{1/6}(x - 2\sqrt{N})$, such that $-N^{1/15} < \xi < -C$. First we consider (6.16b). We have, with $v = N^{-1/6}(w - \sqrt{N})$,

$$\begin{aligned} N^{\frac{1}{6}} \times (6.16\mathrm{b}) &= \frac{-1}{2\pi\mathrm{i}} \int_{-\sqrt{-\xi}\mathrm{i}}^{\sqrt{-\xi}\mathrm{i}} \frac{v^2 - \xi}{v - a_k} + \sum_{j=2}^{k} \frac{1}{(v - a_{k+1-j})(v - a_k)} - N^{-\frac{1}{3}} \frac{v^3 + k}{1 - N^{-\frac{1}{3}} v} \mathrm{d}v \\ &= \frac{-1}{2\pi\mathrm{i}} \int_{-\sqrt{-\xi}\mathrm{i}}^{\sqrt{-\xi}\mathrm{i}} \frac{v^2 - \xi}{v - a_k} + \sum_{j=2}^{k} \frac{1}{(v - a_{k+1-j})(v - a_k)} \mathrm{d}v + \mathcal{O}(N^{-\frac{1}{3}}\xi^3). \end{aligned} \tag{6.18}$$

Here we note that $N^{-1/3}\xi^3 = \mathcal{O}(N^{-2/15})$. Like the estimate of (4.13b), we have that (6.18) is $(\xi - a_k^2)/2 + \mathcal{O}(|\xi|^{-1/2})$ for $-N^{1/15} < \xi < -C$.

Next, we consider (6.16a). We deform Γ into $\Gamma^{>}_{\mathrm{std},N}(0, (2\sqrt{N} - x)^{1/2} N^{1/4}/\sqrt{3})$, and deform Σ into $\Sigma^{<}_{\mathrm{std},N}((\sqrt{N} - 2\sqrt{N} - x)^{1/2} N^{1/4}/\sqrt{3})$. With the change of variables $\xi = N^{1/6}(x - 2\sqrt{N})$, $u = N^{-1/6}(z - \sqrt{N})$ and $v = N^{-1/6}(w - \sqrt{N})$, we have

$$N^{\frac{1}{6}} \times (6.16\mathrm{a}) = \frac{-1}{(2\pi\mathrm{i})^2} \iint_{X_N} \mathrm{d}u\mathrm{d}v \frac{e^{\frac{u^3}{3} - \xi u + f_N(u)}}{e^{\frac{v^3}{3} - \xi v + f_N(v)}} \left(\prod_{j=2}^{k} \frac{u - a_{k+1-j}}{v - a_{k+1-j}} \right) \frac{1}{(v - a_k)(u - v)^2}, \tag{6.19}$$

where

$$f_N(u) = (N - k)\log(1 + N^{-1/3}u) - N^{2/3}u + \frac{1}{2}N^{1/3}u^2 - \frac{1}{3}u^3, \tag{6.20}$$

and the contour X_N is transformed from the deformed $\Phi = \Gamma \times \Sigma$ in (6.16a) by the change of variables. We note that in the region $u, v = o(N^{1/3})$, the contour X_N in (6.19) overlaps with the contour X in (4.13a), if ξ is identified with x there. By standard saddle point analysis similar to that for (4.13a) that we discussed in Section 4, we derive that (6.19) is $\mathcal{O}(|\xi|^{-1/2})$ for $-N^{1/15} < \xi < -C$.

Combining (6.16) with the estimates of (6.18) and (6.19), we prove the identity (6.8).

Proof of (6.7) We let $h_x(t)$ be defined in (4.9), and analogous to (4.10), we have

$$\begin{aligned} \mathbb{E}[(S_x + xN_x)^2] = {} & \int h_x^2(t) K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t)\mathrm{d}t + \int h_x^2(t) K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t)\mathrm{d}t \\ & + \iint h_x(s)h_x(t) \begin{vmatrix} K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,s) & K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t) \\ K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s) & K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \end{vmatrix} \mathrm{d}s\mathrm{d}t \\ & + \iint h_x(s)h_x(t) \begin{vmatrix} K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,s) & K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t) \\ K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s) & K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \end{vmatrix} \mathrm{d}s\mathrm{d}t \\ & - \iint h_x(s)h_x(t) \begin{vmatrix} K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,s) & K^{0,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t) \\ K^{1,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s) & K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \end{vmatrix} \mathrm{d}s\mathrm{d}t \\ & - \iint h_x(s)h_x(t) \begin{vmatrix} K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,s) & K^{1,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t) \\ K^{0,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s) & K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) \end{vmatrix} \mathrm{d}s\mathrm{d}t. \end{aligned} \tag{6.21}$$

Hence, by (6.21) and (6.13), we have analogous to (4.16)

$$\begin{aligned}&\operatorname{Var}\left[S_x + xN_x\right]\\ &= \int h_x^2(t)\Big(K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) + K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t)\Big)\mathrm{d}t - 2\int h_x(t)\left(\int_t^\infty h_x(s)e^{\alpha_1(s-t)}K^{0,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t)\mathrm{d}s\right)\mathrm{d}t \qquad (6.22\mathrm{a})\\ &\quad - \iint h_x(s)h_x(t)\Big(K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t)K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s) + K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t)K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s)\\ &\qquad\qquad - K^{0,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t)\widetilde{K}^{1,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s) - \widetilde{K}^{1,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t)K^{0,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,s)\Big)\mathrm{d}s\mathrm{d}t. \qquad (6.22\mathrm{b})\end{aligned}$$

First we consider the integral in (6.22a). Like (4.17) and (4.18), we write

$$\begin{aligned}&\int h_x^2(t)\Big(K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) + K^{1,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t)\Big)\mathrm{d}t\\ &\qquad = \frac{1}{(2\pi \mathrm{i})^2}\int_\Sigma \mathrm{d}z\int_\Gamma \mathrm{d}w \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^k \frac{z-\alpha_j}{w-\alpha_j}\right)\frac{z+w-2\alpha_1}{w-\alpha_1}\frac{2}{(z-w)^4}, \qquad (6.23)\end{aligned}$$

$$\begin{aligned}&\int h_x(t)\left(\int_t^\infty h_x(s)e^{\alpha_1(s-t)}K^{0,1}_{\mathrm{GUE},\boldsymbol{\alpha}}(s,t)\mathrm{d}s\right)\mathrm{d}t\\ &\qquad = \frac{1}{(2\pi \mathrm{i})^2}\int_\Sigma \mathrm{d}z\int_\Gamma \mathrm{d}w \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^k \frac{z-\alpha_j}{w-\alpha_j}\right)\frac{1}{(z-w)^4}\frac{3z-w-2\alpha_1}{z-\alpha_1}. \qquad (6.24)\end{aligned}$$

So (6.22a) becomes

$$\frac{2}{(2\pi \mathrm{i})^2}\int_\Sigma \mathrm{d}z\int_\Gamma \mathrm{d}w \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^k \frac{z-\alpha_j}{w-\alpha_j}\right)\frac{1}{(z-w)^2(z-\alpha_1)(w-\alpha_1)}. \qquad (6.25)$$

Similarly to (6.16), when $x<0$, this integral can be written as

$$\begin{aligned}&\frac{2}{(2\pi \mathrm{i})^2}\iint_\Phi \mathrm{d}z\mathrm{d}w \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^k \frac{z-\alpha_j}{w-\alpha_j}\right)\frac{1}{(z-w)^2(z-\alpha_1)(w-\alpha_1)} \qquad (6.26\mathrm{a})\\ &+\frac{2}{2\pi \mathrm{i}}\int_{\text{lowerintersect}}^{\text{upperintersect}}\mathrm{d}w\left(w+\frac{N-k}{w}-x+\sum_{j=2}^k\frac{1}{w-\alpha_j}-\frac{1}{w-\alpha_1}\right)\frac{1}{(w-\alpha_1)^2} \qquad (6.26\mathrm{b})\\ &+e^{\frac{\alpha_1^2}{2}-a_k x}\alpha_1^{N-k}\prod_{j=2}^k(\alpha_1-\alpha_j)\frac{2}{2\pi \mathrm{i}}\int_\Gamma \mathrm{d}w\frac{1}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^k\frac{1}{w-\alpha_j}\right)\frac{1}{(w-\alpha_1)^3}, \qquad (6.26\mathrm{c})\end{aligned}$$

where the contour Φ is the same as in (6.16), and the contours Σ and Γ intersect at the two saddle points $\frac{1}{2}(x+\sqrt{4N-x^2}\mathrm{i})$ and $\frac{1}{2}(x-\sqrt{4N-x^2}\mathrm{i})$, which are the upperintersect and lowerintersect respectively. We also require that in (6.26a) the contour Σ lies to the left of all $\alpha_1,\dots,\alpha_k$, and the contour in (6.26b) lies to the right of 0 and all $\alpha_1,\dots,\alpha_k$. We evaluate (6.26b) analogous to (6.17), and have that

$$\begin{aligned}(6.26\mathrm{b}) &= \frac{2}{2\pi \mathrm{i}}\int_{\frac{1}{2}(x-\sqrt{4N-x^2}\mathrm{i})}^{\frac{1}{2}(x+\sqrt{4N-x^2}\mathrm{i})}\mathrm{d}w\left[\left(1-\frac{N-k}{\alpha_1^2}-\sum_{j=2}^k\frac{1}{(\alpha_1-\alpha_j)^2}\right)\frac{1}{w-\alpha_1}+\frac{N-k}{\alpha_1^2}\frac{1}{w}+\sum_{j=2}^k\frac{1}{(\alpha_1-\alpha_j)^2}\frac{1}{w-\alpha_j}\right.\\ &\qquad\left.+\left(\alpha_1-x+\frac{N-k}{\alpha_1}+\sum_{j=2}^k\frac{1}{\alpha_1-\alpha_j}\right)\frac{1}{(w-\alpha_1)^2}-\frac{1}{(w-\alpha_1)^3}\right]\\ &= \frac{2}{\pi}\left(\sqrt{1-\frac{x^2}{4N}}+\arccos\frac{x}{2\sqrt{N}}\right)+\mathcal{O}(N^{-\frac{1}{3}}).\end{aligned} \qquad (6.27)$$

We also evaluate the integral in (6.26a) by standard saddle point analysis similar to (4.13a), and find that it is $\mathcal{O}((4N-x^2)^{-1}) = \mathcal{O}(N^{-2/5})$. The integral in (6.26c) will be cancelled out later.

On the other hand, the double integral in (6.22b) can be expressed as

$$\frac{1}{(2\pi \mathrm{i})^4}\int_{\Sigma} \mathrm{d}u \int_{\Gamma} \mathrm{d}v \int_{\Sigma} \mathrm{d}z \int_{\Gamma} \mathrm{d}w \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \prod_{j=2}^{k} \frac{u-\alpha_j}{v-\alpha_j}\frac{z-\alpha_j}{w-\alpha_j} \\ \times \frac{1}{(u-v)(z-w)(u-w)(z-v)(v-\alpha_1)(w-\alpha_1)}. \tag{6.28}$$

In order to estimate the integral above, we will perform several steps of contour deformation. Since the arguments are parallel to those for the contour deformations of (4.21) in Section 4, we omit most justifications. Also similar to the computation of (4.21), below we assume that all contour integrals in the form of $\int_C^C$ have the contour going between 0 and $\min\{\alpha_1, \dots, \alpha_k\}$, unless they are specially marked as $\int_{C,\mathrm{right}}^C$, in which case the contour goes to the right of $\max\{\alpha_1, \dots, \alpha_k\}$.

(I) We first deform the contours for w and v to $\Gamma_w^{\mathrm{out}} \cup \Gamma_w^{\mathrm{in}}$ and $\Gamma_v^{\mathrm{out}} \cup \Gamma_v^{\mathrm{in}}$, respectively, as in Figure 13, such that all α_j ($j = 1, \dots, k$) are enclosed in Γ_w^{in}, and then also in Γ_v^{in}, and 0 is enclosed in Γ_w^{out}, and then also in Γ_v^{out}. We also slightly deform the contour Σ and denote it by Σ_u and Σ_z for the contour of u and z, respectively.

(II) We then further deform the contour Σ_u such that it goes between Γ_v^{out} and Γ_v^{in}, and thus also goes between Γ_w^{out} and Γ_w^{in}. We denote by Σ_u' the deformed Σ_u; see Figure 14. By residue calculation, we write (6.28) as

$$\frac{1}{(2\pi \mathrm{i})^4}\int_{\Sigma_u'} \mathrm{d}u \int_{\Gamma_v^{\mathrm{out}}\cup\Gamma_v^{\mathrm{in}}} \mathrm{d}v \int_{\Sigma_z} \mathrm{d}z \int_{\Gamma_w^{\mathrm{out}}\cup\Gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \prod_{j=2}^{k} \frac{u-\alpha_j}{v-\alpha_j}\frac{z-\alpha_j}{w-\alpha_j} \\ \times \frac{1}{(u-v)(z-w)(u-w)(z-v)(v-\alpha_1)(w-\alpha_1)} \tag{6.29a}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_z} \mathrm{d}z \int_{\Gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{1}{(w-\alpha_1)(z-\alpha_1)(z-w)^2} \tag{6.29b}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Gamma_v^{\mathrm{in}}} \mathrm{d}v \int_{\Sigma_z} \mathrm{d}z \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{v-\alpha_j}\right) \frac{-1}{(v-\alpha_1)^2(z-\alpha_1)(z-v)} \tag{6.29c}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Gamma_v^{\mathrm{out}}} \mathrm{d}v \int_{\Sigma_z} \mathrm{d}z \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{v-\alpha_j}\right) \frac{-1}{(v-\alpha_1)^2(z-\alpha_1)(z-v)} \tag{6.29d}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Gamma_w^{\mathrm{out}}} \mathrm{d}v \int_{\Sigma_z} \mathrm{d}z \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{-1}{(w-\alpha_1)^2(z-\alpha_1)(z-w)}. \tag{6.29e}$$

(III) Next, similarly to the previous step, we further deform the contour Σ_z such that it goes between Γ_v^{out} and Γ_v^{in}, and thus also goes between Γ_w^{out} and Γ_w^{in}. Hence Σ_z becomes Σ_z'; see Figure 15. By residue calculation, the quantity in (6.29) becomes

$$\frac{1}{(2\pi \mathrm{i})^4}\int_{\Sigma_u'} \mathrm{d}u \int_{\Gamma_v^{\mathrm{out}}\cup\Gamma_v^{\mathrm{in}}} \mathrm{d}v \int_{\Sigma_z'} \mathrm{d}z \int_{\Gamma_w^{\mathrm{out}}\cup\Gamma_w^{\mathrm{in}}} \mathrm{d}w \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \prod_{j=2}^{k} \frac{u-\alpha_j}{v-\alpha_j}\frac{z-\alpha_j}{w-\alpha_j} \\ \times \frac{1}{(u-v)(z-w)(u-w)(z-v)(v-\alpha_1)(w-\alpha_1)} \tag{6.30a}$$

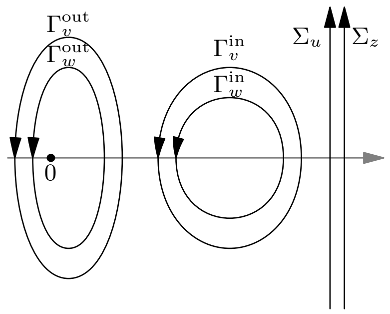

Figure 13: Separation of contours for w and v.

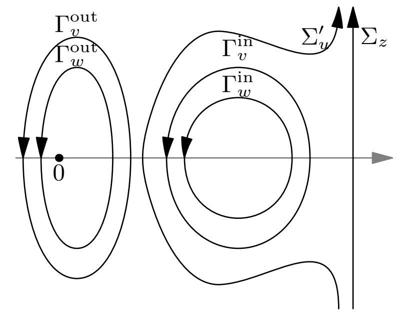

Figure 14: Σ_u is deformed into Σ_u'.

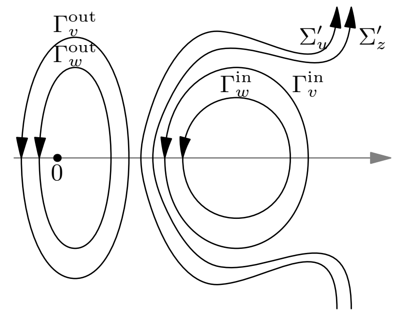

Figure 15: Σ_z is deformed into Σ_z'.

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_u'} \mathrm{d}u \int_{\Gamma_w^{\rm in}} \mathrm{d}w \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \left(\prod_{j=2}^{k} \frac{u-\alpha_j}{w-\alpha_j}\right) \frac{1}{(w-\alpha_1)(u-\alpha_1)(u-w)^2} \tag{6.30b}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_u'} \mathrm{d}u \int_{\Gamma_v^{\rm in}} \mathrm{d}v \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \left(\prod_{j=2}^{k} \frac{u-\alpha_j}{v-\alpha_j}\right) \frac{-1}{(v-\alpha_1)^2(u-\alpha_1)(u-v)} \tag{6.30c}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_u'} \mathrm{d}u \int_{\Gamma_v^{\rm out}} \mathrm{d}v \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \left(\prod_{j=2}^{k} \frac{u-\alpha_j}{v-\alpha_j}\right) \frac{-1}{(v-\alpha_1)^2(u-\alpha_1)(u-v)} \tag{6.30d}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_u'} \mathrm{d}u \int_{\Gamma_w^{\rm out}} \mathrm{d}w \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \left(\prod_{j=2}^{k} \frac{u-\alpha_j}{w-\alpha_j}\right) \frac{-1}{(w-\alpha_1)^2(u-\alpha_1)(u-w)} \tag{6.30e}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_z'} \mathrm{d}z \int_{\Gamma_w^{\rm in}} \mathrm{d}w \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{1}{(w-\alpha_1)(z-\alpha_1)(z-w)^2} \tag{6.30f}$$

$$+\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_w^{\rm in}} \mathrm{d}w \left(w-x+\frac{N-k}{w}+\sum_{j=2}^{k}\frac{1}{w-\alpha_j}-\frac{1}{w-\alpha_1}\right)\frac{1}{(w-\alpha_1)^2} \tag{6.30g}$$

$$+\prod_{j=2}^{k}(\alpha_1-\alpha_j)e^{\frac{\alpha_1^2}{2}-\alpha_1 x}\alpha_1^{N-k}\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_w^{\rm in}} \mathrm{d}w \frac{1}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{1}{w-\alpha_j}\right)\frac{1}{(w-\alpha_1)^3} \tag{6.30h}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Gamma_v^{\rm in}} \mathrm{d}v \int_{\Sigma_z'} \mathrm{d}z \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{v-\alpha_j}\right) \frac{-1}{(v-\alpha_1)^2(z-\alpha_1)(z-v)} \tag{6.30i}$$

$$+\prod_{j=2}^{k}(\alpha_1-\alpha_j)e^{\frac{\alpha_1^2}{2}-\alpha_1 x}\alpha_1^{N-k}\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_v^{\rm in}} \mathrm{d}v \frac{1}{e^{\frac{v^2}{2}-xv}v^{N-k}}\left(\prod_{j=2}^{k}\frac{1}{v-\alpha_j}\right)\frac{1}{(v-\alpha_1)^3} \tag{6.30j}$$

$$+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Gamma_w^{\rm out}} \mathrm{d}v \int_{\Sigma_z'} \mathrm{d}z \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{-2}{(w-\alpha_1)^2(z-\alpha_1)(z-w)} \tag{6.30k}$$

$$+\prod_{j=2}^{k}(\alpha_1-\alpha_j)e^{\frac{\alpha_1^2}{2}-\alpha_1 x}\alpha_1^{N-k}\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_w^{\rm out}} \mathrm{d}w \frac{1}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{1}{w-\alpha_j}\right)\frac{2}{(w-\alpha_1)^3}. \tag{6.30l}$$

(IV) For further deformation of the contours, we introduce the shorthand notations ◳, ◰, ◲ and ◱, analogous to ◇, ◇, ◇ and ◇ defined in (4.24). We delay the concrete assignment of their values to Remark 5, and only indicate that they are close to $\frac{1}{2}(x+\sqrt{4N-x^2}\mathrm{i})$, and their relative positions are shown in the subsequent figures schematically; especially see Figure 17.

For the 4-fold integral (6.30a), we perform the following operations:

(i) deform Σ_u' such that it passes ◰ and $\overline{◰}$;

(ii) deform Σ'_z such that it passes $\square^{\bullet}$ and $\overline{\square}^{\bullet}$;

(iii) deform Γ_v^{out} such that it goes from ${}^{\bullet}\overline{\square}$, along the left side of Σ'_u until it reaches ${}^{\bullet}\square$, then wraps around 0 (and Γ_w^{out}) and finally goes back to ${}^{\bullet}\overline{\square}$;

(iv) deform Γ_v^{in} such that it goes from $\square^{\bullet}$ to $\overline{\square}^{\bullet}$ along the right side of Σ'_z, then wraps around all α_j's, and finally goes back to $\square^{\bullet}$;

(v) and at last add an additional contour for v, on which the contour integral vanishes: the contour goes from $\overline{\square}^{\bullet}$ to $\square^{\bullet}$ along the left-side of Σ'_z, then goes from $\square^{\bullet}$ to ${}^{\bullet}\square$, and further goes from ${}^{\bullet}\square$ to ${}^{\bullet}\overline{\square}$ along the right-side of Σ'_u, and finally goes from ${}^{\bullet}\overline{\square}$ to $\overline{\square}^{\bullet}$.

See Figure 16 for the deformation of contours.

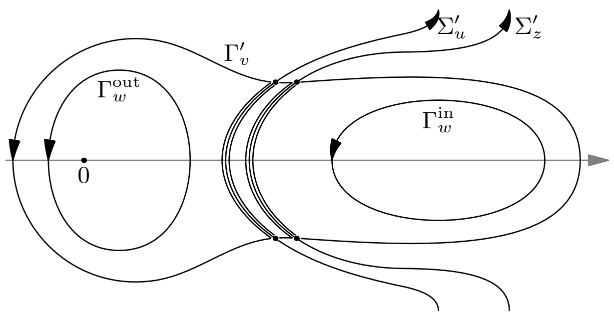

Figure 16: The deformed contours for v, u and z. The four intersections are ${}^{\bullet}\square$, $\square^{\bullet}$, ${}^{\bullet}\overline{\square}$ and $\overline{\square}^{\bullet}$.

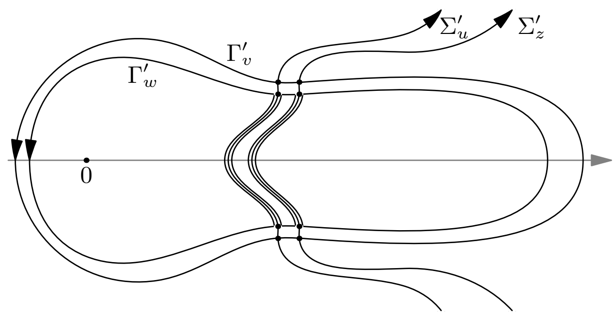

Figure 17: The deformed contours for w, u and z. The four intersections on $\mathbb{C}_+$ are ${}^{\bullet}\square$, $\square^{\bullet}$, ${}_{\bullet}\square$ and $\square_{\bullet}$, and the four intersections on $\mathbb{C}_-$ are their complex conjugates.

Now we define the contour Γ'_v as in Figure 16 that goes from ${}^{\bullet}\overline{\square}$ to $\overline{\square}^{\bullet}$, then wraps α_j's until it reaches $\square^{\bullet}$, and then goes to ${}^{\bullet}\square$, and finally wraps 0 and goes to ${}^{\bullet}\overline{\square}$. Hence the 4-fold integral (6.30a) can be simplified by the residue theorem with $\Gamma_v^{\mathrm{out}} \cup \Gamma_v^{\mathrm{in}}$ replaced by Γ'_v. Then the formula (6.30) becomes

$$\frac{1}{(2\pi\mathrm{i})^4}\int_{\Sigma'_u}\mathrm{d}u\int_{\Sigma'_z}\mathrm{d}z\int_{\Gamma_w^{\mathrm{out}}\cup\Gamma_w^{\mathrm{in}}}\mathrm{d}w\,\mathrm{P.V.}\int_{\Gamma'_v}\mathrm{d}v\frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}}\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\prod_{j=2}^{k}\frac{u-\alpha_j}{v-\alpha_j}\frac{z-\alpha_j}{w-\alpha_j}$$
$$\times\frac{1}{(u-v)(z-w)(u-w)(z-v)(v-\alpha_1)(w-\alpha_1)} \tag{6.31a}$$

$$+\frac{1}{(2\pi\mathrm{i})^3}\int_{\Sigma'_z}\mathrm{d}z\int_{\Gamma_w^{\mathrm{out}}\cup\Gamma_w^{\mathrm{in}}}\mathrm{d}w\int_{{}^{\bullet}\overline{\square}}^{{}^{\bullet}\square}\mathrm{d}u\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{w-\alpha_j}\right)$$
$$\times\frac{1}{(z-w)(u-w)(z-u)(u-\alpha_1)(w-\alpha_1)} \tag{6.31b}$$

$$+\frac{1}{(2\pi\mathrm{i})^3}\int_{\Sigma'_u}\mathrm{d}u\int_{\Gamma_w^{\mathrm{out}}\cup\Gamma_w^{\mathrm{in}}}\mathrm{d}w\int_{\overline{\square}^{\bullet}}^{\square^{\bullet}}\mathrm{d}z\frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{u-\alpha_j}{w-\alpha_j}\right)$$
$$\times\frac{1}{(u-z)(z-w)(u-w)(z-\alpha_1)(w-\alpha_1)} \tag{6.31c}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Gamma_w^{\mathrm{out}}}\mathrm{d}v\int_{\Sigma'_z}\mathrm{d}z\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{w-\alpha_j}\right)\frac{-4}{(w-\alpha_1)^2(z-\alpha_1)(z-w)} \tag{6.31d}$$

$$+\prod_{j=2}^{k}(\alpha_1-\alpha_j)e^{\frac{\alpha_1^2}{2}-\alpha_1 x}\alpha_1^{N-k}\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_w^{\mathrm{in}}\cup\Gamma_w^{\mathrm{out}}}\mathrm{d}w\frac{1}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{1}{w-\alpha_j}\right)\frac{2}{(w-\alpha_1)^3} \tag{6.31e}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma'_z}\mathrm{d}z\int_{\Gamma_w^{\mathrm{in}}}\mathrm{d}w\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{w-\alpha_j}\right)\frac{2(2w-z-\alpha_1)}{(w-\alpha_1)^2(z-\alpha_1)(z-w)^2} \tag{6.31f}$$

$$+1-\frac{N-k}{\alpha_1^2}.$$

Here we remark that the 1-fold integral (6.30g) is equal to $1-\frac{N-k}{\alpha_1^2}$.

(V) Now we deform the contour $\Gamma_w^{\mathrm{out}}\cup\Gamma_w^{\mathrm{in}}$ for w in the way similar to our deformation of $\Gamma_v^{\mathrm{out}}\cup\Gamma_v^{\mathrm{in}}$ for v in Step (IV). We perform the following operations:

- (a) deform Σ'_u such that it passes ◱ and $\overline{◱}$, and meanwhile still passes ◰ and $\overline{◰}$;
- (b) deform Σ'_z such that it passes ◲ and $\overline{◲}$ and meanwhile still passes ◳ and $\overline{◳}$;
- (c) deform Γ_w^{out} such that it goes from $\overline{◱}$, along the left side of Σ'_u until it reaches ◱, then wraps around 0 and finally goes back to $\overline{◱}$;
- (d) deform Γ_w^{in} such that it goes from ◲ to $\overline{◲}$ along the right-side of Σ'_z, then wraps around all α_j, and finally goes back to ◲;
- (e) and at last add an additional contour for w, on which the contour integral vanishes: the contour goes from $\overline{◲}$ to ◲ along the left-side of Σ'_z, then goes from ◲ to ◱, and further goes from ◱ to $\overline{◱}$ along the right-side of Σ'_u, and finally goes from $\overline{◱}$ to $\overline{◲}$.

We have the result in Figure 17. Similar to Γ'_v in Figure 16. Now we define the contour Γ'_w that goes from $\overline{◱}$ to $\overline{◲}$, then wraps α_j's until it reaches ◲, and then goes to ◱, and finally wraps 0 and goes to $\overline{◱}$. Hence by the residue theorem, the 4-fold integral (6.31a) can be simplified with $\Gamma_w^{\mathrm{out}}\cup\Gamma_w^{\mathrm{in}}$ replaced by Γ'_w, and then formula (6.31) becomes

$$\frac{1}{(2\pi\mathrm{i})^4}\int_{\Sigma'_u}\mathrm{d}u\int_{\Sigma'_z}\mathrm{d}z\,\mathrm{P.V.}\int_{\Gamma'_w}\mathrm{d}w\,\mathrm{P.V.}\int_{\Gamma'_v}\mathrm{d}v\frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}}\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\prod_{j=2}^{k}\frac{u-\alpha_j}{v-\alpha_j}\frac{z-\alpha_j}{w-\alpha_j}$$
$$\times\frac{1}{(u-v)(z-w)(u-w)(z-v)(v-\alpha_1)(w-\alpha_1)} \tag{6.32a}$$

$$+\frac{1}{(2\pi\mathrm{i})^3}\int_{\Sigma'_z}\mathrm{d}z\,\mathrm{P.V.}\int_{\Gamma'_v}\mathrm{d}v\int_{\overline{◱}}^{◰}\mathrm{d}u\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{v^2}{2}-xv}v^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{v-\alpha_j}\right)$$
$$\times\frac{1}{(u-v)(z-u)(z-v)(v-\alpha_1)(u-\alpha_1)} \tag{6.32b}$$

$$+\frac{1}{(2\pi\mathrm{i})^3}\int_{\Sigma'_u}\mathrm{d}u\,\mathrm{P.V.}\int_{\Gamma'_v}\mathrm{d}v\int_{\overline{◲}}^{◳}\mathrm{d}z\frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{v^3}{3}+xv}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{v-\alpha_j}\right)$$
$$\times\frac{1}{(u-v)(z-w)(z-v)(v-\alpha_1)(z-\alpha_1)} \tag{6.32c}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Gamma_w^{\mathrm{in}}\cup\Gamma_w^{\mathrm{out}}}\mathrm{d}w\int_{\Sigma'_z}\mathrm{d}z\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{w-\alpha_j}\right)\frac{-4}{(w-\alpha_1)^2(z-\alpha_1)(z-w)} \tag{6.32d}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma'_z}\mathrm{d}z\int_{\Gamma_w^{\mathrm{in}}}\mathrm{d}w\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{w-\alpha_j}\right)\frac{2}{(w-\alpha_1)^2(z-w)^2} \tag{6.32e}$$

$$+(6.31\mathrm{b})+(6.31\mathrm{c})+(6.31\mathrm{e})+1-\frac{N-k}{\alpha_1^2}. \tag{6.32f}$$

Now we can compute the contour integrals by saddle point method. First we need to fix the shapes of contours Γ'_v, Γ'_w, Σ'_u and Σ'_z.

(i) If $(-2+\epsilon)\sqrt{N} < x < 1.9\sqrt{N}$, then let Γ'_w be $\Gamma^{\circ}_{\mathrm{std},N}(0)$, Γ'_v be $\Gamma^{\circ}_{\mathrm{std},N}(1)$, Σ'_u be $\Sigma^{|}_{\mathrm{std},N}(x/2)$ and Σ'_z be $\Sigma^{|}_{\mathrm{std},N}(x/2+1)$.

(ii) If $1.9\sqrt{N} \le x < 2\sqrt{N} - N^{-1/10}$, then with $\theta = \arccos(x/\sqrt{4N})$, let Γ'_w be $\Gamma^{>}_{\mathrm{std},N}(\theta, 0)$, Γ'_v be $\Gamma^{>}_{\mathrm{std},N}(\theta, (2-x/\sqrt{N})^{-1/4})$, Σ'_u be $\Sigma^{<}_{\mathrm{std},N}(x/2-\sqrt{N-x^2/4}/\sqrt{3})$ and Σ'_z be $\Sigma^{<}_{\mathrm{std},N}(x/2-\sqrt{N-x^2/4}/\sqrt{3}+(2-x/\sqrt{N})^{-1/4})$.

Remark 5. We note that in either case, the four contours have four intersections around $\frac{1}{2}(x+\sqrt{4N-x}\mathrm{i})$, and they are the desired $\square^{\bullet}$, ${}^{\bullet}\square$, $\square_{\bullet}$ and ${}_{\bullet}\square$.

(1) The 4-fold integral (6.32a) can be estimated in the same way as for (4.26a), and it is $\mathcal{O}((2\sqrt{N}-x)^{-1}N^{-1/4})$.

(2) Both of the 3-fold integrals (6.32b) and (6.32c) can be estimated in the same way as for (4.26b) and (4.26c). We note that in the evaluation of (4.26b) and (4.26c), we specified the shapes of the contour from $\overline{\bullet\!\diamond}$ to $\bullet\!\diamond$ for u and the contour from $\overline{\underset{\bullet}{\diamond}}$ to $\underset{\bullet}{\diamond}$ for z. Here we can deform the contours from $\overline{{}_{\bullet}\square}$ to ${}_{\bullet}\square$ for u and the contour from $\overline{\square_{\bullet}}$ to $\square_{\bullet}$ for z in a similar way: the contour for u is the part of Σ'_u inside of Γ'_v, and the contour for z is the part of Σ'_z inside of Γ'_w. We conclude that both (6.32b) and (6.32c) are $\mathcal{O}((2\sqrt{N}-x)^{-1}N^{-1/4})$.

(3) The 3-fold integral (6.31b) can be written as the sum of

$$\frac{1}{(2\pi\mathrm{i})^3}\int_{\Sigma'_z}\mathrm{d}z\,\mathrm{P.V.}\int_{\Gamma''_w}\mathrm{d}w\int_{\overline{{}^{\bullet}\square}}^{{}^{\bullet}\square}\mathrm{d}u\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{w-\alpha_j}\right)$$
$$\times\frac{1}{(z-w)(u-w)(z-u)(u-\alpha_1)(w-\alpha_1)} \tag{6.33a}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\overline{{}^{\bullet}\square}}^{{}^{\bullet}\square}\mathrm{d}u\int_{\mathrm{lowerintersect}'}^{\mathrm{upperintersect}'}\mathrm{d}z\frac{-1}{(u-z)^2(u-\alpha_1)(z-\alpha_1)} \tag{6.33b}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma''_z}\mathrm{d}z\int_{\overline{{}^{\bullet}\square}}^{{}^{\bullet}\square}\mathrm{d}u\frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{u^2}{2}-xu}u^{N-k}}\left(\prod_{j=2}^{k}\frac{z-\alpha_j}{u-\alpha_j}\right)\frac{1}{(z-u)^2(u-\alpha_1)^2} \tag{6.33c}$$

$$+\frac{1}{2\pi\mathrm{i}}\int_{\overline{{}^{\bullet}\square}}^{{}^{\bullet}\square}\mathrm{d}u\left((u-x+\frac{N-k}{u})+\sum_{j=2}^{k}\frac{1}{u-\alpha_j}\right)\frac{1}{(u-\alpha_1)^2}, \tag{6.33d}$$

where the contours

$$\Gamma''_w = \begin{cases}\Gamma^{\circ}_{\mathrm{std},N}(2), & (-2+\epsilon)\sqrt{N} < x < 1.9\sqrt{N},\\ \Gamma^{>}_{\mathrm{std},N}(\arccos(x/\sqrt{4N}), 2(2-x/\sqrt{N})^{-1/4}), & 1.9\sqrt{N} \le x < 2\sqrt{N}-N^{-1/10},\end{cases} \tag{6.34}$$

$$\Sigma''_z = \begin{cases}\Sigma^{|}_{\mathrm{std},N}(x/2-1), & (-2+\epsilon)\sqrt{N} < x < 1.9\sqrt{N},\\ \Sigma^{<}_{\mathrm{std},N}(x/2-\sqrt{N-x^2/4}/\sqrt{3}-(2-x/\sqrt{N})^{-1/4}), & 1.9\sqrt{N} \le x < 2\sqrt{N}-N^{-1/10},\end{cases} \tag{6.35}$$

upperintersect$'$ and lowerintersect$'$ are the intersections between Γ''_w and Σ'_z. Also we take the contours from $\overline{{}^{\bullet}\square}$ to ${}^{\bullet}\square$ for u to be the part of Σ'_u inside Γ'_v, the contour from lowerintersect$'$ to upperintersect$'$ for z to be the part of Σ'_z inside Γ''_w. On the other hand, the 3-fold (6.31c) can bewritten as the sum of

$$\frac{1}{(2\pi\mathrm{i})^3}\int_{\Sigma'_u}\mathrm{d}u\,\mathrm{P.V.}\int_{\Gamma''_w}\mathrm{d}w\int_{\overline{\square^{\bullet}}}^{\square^{\bullet}}\mathrm{d}z\frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}}\left(\prod_{j=2}^{k}\frac{u-\alpha_j}{w-\alpha_j}\right)$$

$$\times \frac{1}{(u-z)(z-w)(u-w)(z-\alpha_1)(w-\alpha_1)} \tag{6.36a}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\mathrm{lowerintersect}''}^{\mathrm{upperintersect}''} \mathrm{d}u \int_{\overline{\square^{\bullet}}}^{\square^{\bullet}} \mathrm{d}z \frac{-1}{(u-z)^2(u-\alpha_1)(z-\alpha_1)} \tag{6.36b}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma'_u} \mathrm{d}u \int_{\square^{\bullet}}^{\square^{\bullet}} \mathrm{d}z \frac{e^{\frac{u^2}{2}-xu}u^{N-k}}{e^{\frac{z^2}{2}-xz}z^{N-k}} \left(\prod_{j=2}^{k} \frac{u-\alpha_j}{z-\alpha_j}\right) \frac{1}{(u-z)^2(z-\alpha_1)^2}. \tag{6.36c}$$

where the contour Γ''_w is the same as Γ''_w in (6.33a), and it intersects Σ'_u at upperintersect$''$ and lowerintersect$''$, and the contour from $\overline{\square^{\bullet}}$ to $\square^{\bullet}$ for z is the part of Σ'_z inside Γ'_v.

(4) The 2-fold integral (6.32d) can be written as the sum of

$$\frac{1}{(2\pi\mathrm{i})^2}\iint_{\Phi} \mathrm{d}v\mathrm{d}z \frac{e^{\frac{z^2}{2}-xz}z^{N-k}}{e^{\frac{w^2}{2}-xw}w^{N-k}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{-4}{(w-\alpha_1)^2(z-\alpha_1)(z-w)} \tag{6.37a}$$

$$+\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{lowerintersect}}^{\mathrm{upperintersect}} \mathrm{d}z \frac{4}{(z-\alpha_1)^3}, \tag{6.37b}$$

by the same transform as (6.15) is transformed into (6.16), and the contour Φ and the integral limits upperintersect, lowerintersect are as defined in (6.16). Note that in (6.37b) the contour is between 0 and $\min(\alpha_1,\dots,\alpha_k)$ while in (6.16b) the contour is to the right of 0 and all $\alpha_1,\dots,\alpha_k$.

(5) The 2-fold integral (6.32e) can be written as the sum of

$$\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma''_z} \mathrm{d}z \int_{\overline{{}^{\bullet}\square}}^{{}^{\bullet}\square} \mathrm{d}w \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{-1}{(w-\alpha_1)^2(z-w)^2} \tag{6.38a}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma''_z} \mathrm{d}z \int_{\overline{{}^{\bullet}\square},\mathrm{right}}^{{}^{\bullet}\square} \mathrm{d}w \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{1}{(w-\alpha_1)^2(z-w)^2} \tag{6.38b}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma''_z} \mathrm{d}z \int_{\square^{\bullet}}^{\square^{\bullet}} \mathrm{d}w \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{-1}{(w-\alpha_1)^2(z-w)^2} \tag{6.38c}$$

$$+\frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma''_z} \mathrm{d}z \int_{\square^{\bullet},\mathrm{right}}^{\square^{\bullet}} \mathrm{d}w \frac{e^{\frac{z^3}{3}-xz}}{e^{\frac{w^3}{3}-xw}} \left(\prod_{j=2}^{k} \frac{z-\alpha_j}{w-\alpha_j}\right) \frac{1}{(w-\alpha_1)^2(z-w)^2}, \tag{6.38d}$$

where Σ''_z is the same as Σ''_z in (6.33c), the contour for w in (6.38a) is the part of Σ'_u inside of Γ'_v, the contour for w in (6.38b) is the part of Γ'_v to the right of Σ'_u, the contour for w in (6.38c) is the part of Σ'_z inside of Γ'_v, and the contour for w in (6.38d) is the part of Γ'_v to the right of Σ'_z. We note that (6.38a) cancels with (6.33c), and (6.38c) cancels with (6.36c).

(6) The 1-fold integral (6.31e) cancels with (6.26c).

(7) (6.33d) can be evaluated similar to (6.26b), and it is $\frac{1}{\pi}(\sqrt{1-x^2/(4N)}+\arccos(x/\sqrt{4N})+\mathcal{O}(N^{-1/3})$.

(8) (6.33b) and (6.36b) are $\mathcal{O}(N^{-2/5}\log N)$, and all other integrals from (6.33a) to (6.38d) not mentioned above, are $\mathcal{O}(N^{-2/5})$.

Hence we obtain the final proof of (6.7).

Sketch of proof of (6.9) Like the proof of (6.7), we try to estimate (6.25) and (6.28).

For (6.25), we again use the decomposition (6.26), but the precise shapes of the contours are not to be the same as in the proof of (6.6), and the integral limits upperintersect, lowerintersect are not to be $\frac{1}{2}(x+\sqrt{4N-x^2}\mathrm{i})$ and $\frac{1}{2}(x-\sqrt{4N-x^2}\mathrm{i})$. Instead, we deform the contour $\Phi = \Gamma \times \Sigma$ as in the proof of (6.8), that is, Γ into $\Gamma^{>}_{\mathrm{std},N}(0, (2\sqrt{N}-x)^{1/2}N^{1/4}/\sqrt{3})$, and deform Σ into $\Sigma^{<}_{\mathrm{std},N}((\sqrt{N}-2\sqrt{N}-x)^{1/2}N^{1/4}/\sqrt{3})$, and then let the integral limits upperintersect, lowerintersect be $\sqrt{N} \pm (2\sqrt{N}-x)^{1/2}N^{1/4}\mathrm{i}$. Then we have that in the regime that $x = N^{-1/6}\xi + 2\sqrt{N}$ and $-N^{1/15} < \xi < -C$, (6.26a) is $\mathcal{O}(N^{-1/3}(-\xi)^{-1})$, and (6.26b) is $N^{-1/3}(\frac{4}{\pi}\sqrt{-\xi} + 2a_k) + \mathcal{O}(N^{-1/3}(-\xi)^{-1})$.

For (6.28), we take the transforms as in Steps (I) – (V), with only one methodological difference: The intersection points ◳, ◰, ◲ and ◱ now should be around $\sqrt{N} + (2\sqrt{N}-x)^{1/2}N^{1/4}\mathrm{i}$, in consistence with our choice of Φ and upperintersect, lowerintersect. However, practically we can still use the deformation of the contours in the regime $1.9\sqrt{N} \leq x < 2\sqrt{N} - N^{-1/10}$, because in the regime where we are working, $\frac{1}{2}(x+\sqrt{4N-x^2}\mathrm{i})$ and $\sqrt{N}+(2\sqrt{N}-x)^{1/2}N^{1/4}\mathrm{i}$ are very close to each other. Hence we can still use the saddle point method that is used in the proof of (6.7), especially that for the regime $1.9\sqrt{N} \leq x < 2\sqrt{N} - N^{-1/10}$.

At last, we derive (6.9) by the method delineated above, with much detail omitted.

7 Nondegeneracy of the limiting distribution

In this section, we prove that the random variable $\Xi^{(k)}_j(\mathbf{a};\infty)$ defined in (1.14) is nondegenerate, i.e., Theorem 4, which tells that the distribution is not supported on a single point. It is equivalent to show that $\log \Xi^{(k)}_j(\mathbf{a};\infty)$ is nondegenerate. It is not a trivial task, since $\log \Xi^{(k)}_j(\mathbf{a};\infty)$ is a non-linear functional on the externded Airy process. Our proof relies on that $\log \Xi^{(k)}_j(\mathbf{a};\infty)$ is the limit of $\log(N^{1/3}|x_{j1}|^2)$ by Theorem 2 (up to a constant), which is a non-linear functional on the GUE minor process with external source. The advantage of the latter is that the eigenvalue distribution of a GUE-type matrix has a log-gas representation that does not pass to their limiting processes, e.g. an Airy-type process. We make use of the log-gas representation, and prove Lemma 12, which leads to the proof of Theorem 4 in a straightforward way.

Since in the statement and proof of Lemma 12 we will condition on $\lambda_1,\dots,\lambda_{N-1},\sigma_1,\dots,\sigma_{j-1},\sigma_{j+1},\dots,\sigma_N$ and play with the randomness of σ_j only, we will denote by $\omega = (\lambda_1,\dots,\lambda_{N-1},\sigma_1,\dots,\sigma_{j-1},\sigma_{j+1},\dots,\sigma_N)$ a generic realization of the collection of the given eigenvalues, for notational simplicity.

Lemma 12. *There exist $M, \epsilon, \epsilon', \epsilon'' > 0$, such that if N is large enough, there exists an event A of ω, with $\mathbb{P}(A) > \epsilon$, and the conditional variance satisfies*

$$\mathbb{P}\left(M^{-1} < N^{\frac{1}{3}}|x_{j1}|^2 < M \mid \omega\right) > \epsilon', \quad \textit{and} \quad \mathrm{Var}\left(\log(N^{\frac{1}{3}}|x_{j1}|^2) \mid \omega \textit{ and } M^{-1} < N^{\frac{1}{3}}|x_{j1}|^2 < M\right) > \epsilon'', \tag{7.1}$$

for any fixed $\omega \in A$.

Proof of Theorem 4 by Lemma 12. Lemma 12 shows that for all large enough N, the followings hold:

(i) $\mathbb{P}(M^{-1} < N^{1/3}|x_{j1}|^2 < M) > \mathbb{P}(A \cap \{M^{-1} < N^{1/3}|x_{j1}|^2 < M\}) > \epsilon\epsilon'$, and

(ii) the conditional variance

$$\begin{aligned} &\mathrm{Var}\left(\log(N^{\frac{1}{3}}|x_{j1}|^2) \mid M^{-1} < N^{\frac{1}{3}}|x_{j1}|^2 < M\right) \\ \geq{}& \frac{\mathbb{P}(A \cap \{M^{-1} < N^{1/3}|x_{j1}|^2 < M\})}{\mathbb{P}(M^{-1} < N^{1/3}|x_{j1}|^2 < M)} \mathrm{Var}\left(\log(N^{\frac{1}{3}}|x_{j1}|^2) \mid A \cap \{M^{-1} < N^{\frac{1}{3}}|x_{j1}|^2 < M\}\right) \\ >{}& \epsilon\epsilon'\epsilon''. \end{aligned} \tag{7.2}$$

Using Theorem 2, we have $\mathbb{P}(M^{-1} < (3\pi/2)^{1/3}\Xi^{(k)}_j(\mathbf{a};\infty) < M) > \epsilon\epsilon'$ and $\mathrm{Var}(\log((3\pi/2)^{1/3}\Xi^{(k)}_j(\mathbf{a};\infty)) \mid M^{-1} < (3\pi/2)^{1/3}\Xi^{(k)}_j(\mathbf{a};\infty) < M) > \epsilon\epsilon'\epsilon''$. Then we can conclude the proof of Theorem 4 easily. □

The remaining part of this section is devoted to the proof of Lemma 12. To make notations simple, in this section we denote $\tilde{\lambda}_i = N^{1/6}(\lambda_i - 2\sqrt{N})$ and $\tilde{\sigma}_i = N^{1/6}(\sigma_i - 2\sqrt{N})$. We recall that $\tilde{\sigma}_i$'s and $\tilde{\lambda}_i$'s converge jointly in distribution to $\xi^{(k)}_i$'s and $\xi^{(k-1)}_i$'s, by Lemma 6. We further denote $F_j = \log(N^{1/3}|x_{j1}|^2)$. We will also need the following estimate:

Lemma 13. *Let $L > 0$ be any constant independent of N. If N is large enough, there is a constant C_L such that for all $x \in [-L, L]$*

$$\mathbb{E}\left(\sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - x)^2 + 1}\right) < C_L. \tag{7.3}$$

Proof. The left-hand side of (7.3) can be expressed as

$$\mathbb{E}\left(\sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - x)^2 + 1}\right) = N^{-\frac{1}{3}} \int_{-\infty}^{\infty} \rho_N(t) \frac{1}{(t - (2\sqrt{N} + N^{-1/6}x))^2 + N^{-1/3}} \mathrm{d}t, \tag{7.4}$$

where $\rho_N(t)$ is the empirical density function of $\sigma_1, \dots, \sigma_N$. By the property of determinantal processes (cf. (1.12)), $\rho_N(t) = K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t) = \widetilde{K}^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(t,t)$, where $K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}$ is defined in (2.2) and represented by a double integral formula in (2.3). The estimation of $\rho_N(t)$ with $t \in ((-2+\epsilon)\sqrt{N}, 2\sqrt{N} - N^{-1/10})$ and $t \in [2\sqrt{N} - N^{-1/10}, 2\sqrt{N} - CN^{-1/6}]$ can be done by the same saddle point analysis method as we evaluate $\mathbb{E}(S_x + xN_x)$ in (6.6) and (6.8) respectively, since $\mathbb{E}(S_x + xN_x)$ is expressed by a very similar double integral formula in (6.15). For $t > 2\sqrt{N} - CN^{-1/6}$, where $C > 0$, $\rho_N(t)$ can be estimated by using (A.20) in Appendix A and then apply the standard saddle point method to $H_{N,0}$ and $J_{N,0}$. For $t \le (-2+\epsilon)\sqrt{N}$, similar methods can be applied and we omit the detail. The estimate we need is that for large enough N:

(i) (The semicircle law) For $t \in (-2\sqrt{N} + N^{-1/10}, 2\sqrt{N} - N^{-1/10})$, $\rho_N(t) = \frac{1}{2\pi}\sqrt{4N - t^2}(1 + o(1))$.

(ii) For $t \in [2\sqrt{N} - N^{-1/10}, 2\sqrt{N} - CN^{-1/6})$ and $t \in (-2\sqrt{N} + CN^{-1/6}, -2\sqrt{N} + N^{-1/10}]$, $\rho_N(t) = \mathcal{O}(N^{1/4}(2\sqrt{N} - t)^{1/2})$ and $\rho_N(t) = \mathcal{O}(N^{1/4}(2\sqrt{N} + t)^{1/2})$ respectively.

(iii) For $t \ge 2\sqrt{N} - CN^{-1/6}$ and $t \le -2\sqrt{N} + CN^{-1/6}$, $\rho_N(t) = \mathcal{O}(N^{1/6} e^{-cN^{1/6}(t - 2\sqrt{N})})$ and $\rho_N(t) = \mathcal{O}(N^{1/6} e^{cN^{1/6}(2\sqrt{N} + t)})$, for some $c > 0$, respectively.

The estimate of $\rho_N(t)$ above and the expression (7.4) imply the desired boundedness. □

Proof of Lemma 12. We discuss first the $j = 1$ case in detail, and then extend the discussion to the $j > 1$ case.

The $j = 1$ case We note that $\xi_1^{(k-1)} > \xi_2^{(k)}$ almost surely and they are both continuous random variables. Hence there exist some $c_1 \in \mathbb{R}$ and $\epsilon_2 > 0$ such $\mathbb{P}(\xi_1^{(k)} \in (c_1, c_1 + \epsilon_2)$ and $\xi_2^{(k-1)} < c_1 - \epsilon_2) > \epsilon_1$ for some $\epsilon_1 > 0$. Hence the event A_1 defined by

$$A_1 = \{\tilde{\lambda}_1 \in (c_1, c_1 + \epsilon_2) \text{ and } \tilde{\sigma}_2 < c_1 - \epsilon_2\} \tag{7.5}$$

satisfies that $\mathbb{P}(A_1) > \epsilon_1$ for large enough N.

Let $p(\omega)$ be the marginal density of ω, whose formula is not relevant. If ω is fixed, the conditional density of σ_1 given ω is, by [2, Theorem 1]

$$p_\omega(\sigma) = \frac{1}{C_\omega} \exp(-f_{\omega,\alpha_1}(\sigma)) \mathbf{1}(\sigma > \lambda_1), \quad \text{where} \quad f_{\omega,\alpha_1}(\sigma) = \frac{\sigma^2}{2} - \alpha_1 \sigma - \sum_{i=2}^{N} \log(\sigma - \sigma_i), \tag{7.6}$$

for some constant C_ω, or equivalently, the conditional density for $\tilde{\sigma}_1$ is

$$\tilde{p}_\omega(\tilde{\sigma}) = \frac{p(\sigma)}{N^{1/6}} = \frac{1}{\tilde{C}_{\omega,a_k}} \exp(-\tilde{f}_{\omega,a_k}(\tilde{\sigma})) \mathbf{1}(\tilde{\sigma} > \tilde{\lambda}_1), \quad \text{where} \quad \tilde{f}_{\omega,a_k}(\tilde{\sigma}) = N^{\frac{1}{3}}\tilde{\sigma} - a_k \tilde{\sigma} + N^{-\frac{1}{3}} \frac{\tilde{\sigma}^2}{2} - \sum_{i=2}^{N} \log(\tilde{\sigma} - \tilde{\sigma}_i), \tag{7.7}$$

and $\tilde{C}_{\omega,a_k} = N^{-N/6} \exp(2N^{2/3} a_k) C_\omega$. Hence

$$\tilde{f}'_{\omega,a_k}(\tilde{\sigma}) = N^{\frac{1}{3}} - a_k + N^{-\frac{1}{3}}\tilde{\sigma} - \sum_{i=2}^{N} \frac{1}{\tilde{\sigma} - \tilde{\sigma}_i}, \quad \tilde{f}''_{\omega,a_k}(\tilde{\sigma}) = N^{-\frac{1}{3}} + \sum_{i=2}^{N} \frac{1}{(\tilde{\sigma} - \tilde{\sigma}_i)^2}, \quad \text{and} \quad \int_{\tilde{\lambda}_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma}) \mathrm{d}\tilde{\sigma} = 1. \tag{7.8}$$

We note that since the random variable $\xi_1^{(k)} < +\infty$ and as $N \to \infty$, we have that there exists $M_1 > c_1 + \epsilon_2$ such that $\mathbb{P}(\xi_1^{(k)} > M_1) < \frac{\epsilon_1}{6}$, and then for large enough N,

$$\mathbb{P}(\tilde{\sigma}_1 > M_1) = \int p(\omega)\mathrm{d}\omega \int_{M_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} < \frac{\epsilon_1}{6}. \tag{7.9}$$

Then the event A_2 defined by

$$A_2 = \left\{ \omega \in A_1 \mid \int_{\tilde{\lambda}_1}^{M_1} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} > \frac{2}{3} \right\} \tag{7.10}$$

satisfies $\mathbb{P}(A_2) > \epsilon_1/2$ for large enough N. Otherwise we will have

$$\int p(\omega)\mathrm{d}\omega \int_{M_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} \geq \int_{A_1 \setminus A_2} p(\omega)\mathrm{d}\omega \int_{M_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} > \frac{1}{3}\mathbb{P}(A_1 \setminus A_2) = \frac{\epsilon_1}{6}, \tag{7.11}$$

contradictory to (7.9).

Next, since $\xi_1^{(k)} > \xi_1^{(k-1)}$ almost surely and $\xi_1^{(k)}, \xi_1^{(k-1)}$ are both continuous random variables, there exist $\epsilon_3 > 0$ such that $\mathbb{P}(\xi_1^{(k)} > \xi_1^{(k-1)} + \epsilon_3) > 1 - \epsilon_1/12$, and then for large enough N,

$$\mathbb{P}(\tilde{\sigma}_1 \leq \tilde{\lambda}_1 + \epsilon_3) = \int p(\omega)\mathrm{d}\omega \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1 + \epsilon_3} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} < \frac{\epsilon_1}{12}. \tag{7.12}$$

Then the event A_3 defined by

$$A_3 = \left\{ \omega \in A_2 \mid \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1 + \epsilon_3} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} < \frac{1}{3} \right\} \tag{7.13}$$

satisfies $\mathbb{P}(A_3) > \epsilon_1/4$ for large enough N. Otherwise, we will have

$$\int p(\omega)\mathrm{d}\omega \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1 + \epsilon_3} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} \geq \int_{A_2 \setminus A_3} p(\omega)\mathrm{d}\omega \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1 + \epsilon_3} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} > \frac{1}{3}\mathbb{P}(A_2 \setminus A_3) = \frac{\epsilon_1}{12}, \tag{7.14}$$

contradictory to (7.12). We note that if $\omega \in A_3$, then

$$\begin{gathered} \mathbb{P}(\tilde{\sigma}_1 > M_1 \mid \omega) = \int_{M_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} < \frac{1}{3}, \quad \mathbb{P}(\tilde{\sigma}_1 \in (\tilde{\lambda}_1, \tilde{\lambda}_1 + \epsilon_3) \mid \omega) = \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1 + \epsilon_3} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} < \frac{1}{3}, \\ \text{and} \quad \mathbb{P}(\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1] \mid \omega) = \int_{\tilde{\lambda}_1 + \epsilon_3}^{M_1} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} > \frac{1}{3}. \end{gathered} \tag{7.15}$$

Recall $L > 0$ in Lemma 13. Now we choose L sufficiently large such that $[-L, L] \supseteq (c_1 - \epsilon_2, M_1)$, and define

$$A_4 = \left\{ \omega \in A_3 \mid \int_{\tilde{\lambda}_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma}_1) \sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - c_1)^2 + 1}\mathrm{d}\tilde{\sigma}_1 < \frac{C_L}{\epsilon_1/8} \right\}. \tag{7.16}$$

We have that $\mathbb{P}(A_4) > \epsilon_1/8$ for all large enough N. Otherwise,

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - c_1)^2 + 1} \right) &= \int p(\omega)\mathrm{d}\omega \int_{\tilde{\lambda}_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma}_1) \sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - c_1)^2 + 1}\mathrm{d}\tilde{\sigma}_1 \\ &\geq \int_{A_3 \setminus A_4} p(\omega)\mathrm{d}\omega \int_{\tilde{\lambda}_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma}_1) \sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - c_1)^2 + 1} d\tilde{\sigma}_1 \\ &\geq \mathbb{P}(A_3 \setminus A_4) \frac{C_L}{\epsilon_1/8} \\ &\geq C_L, \end{aligned} \tag{7.17}$$

contradictory to (7.3).

Now consider the function $F_1 = \log(N^{1/3}|x_{11}|^2)$ defined by (1.20). For a fixed $\omega \in A_4$, F_1 depends on σ_1, or equivalently $\tilde{\sigma}_1$. Below we express it as $F_1(\tilde{\sigma}_1; \omega)$ as a function of $\tilde{\sigma}_1$. $F_1(\tilde{\sigma}_1; \omega)$ is an increasing function as $\tilde{\sigma}_1 \in (\tilde{\lambda}_1, +\infty)$. In Theorem 2 we have shown that the random variable F_1 converges weakly to the random variable $(3\pi/2)^{1/3}\Xi_j^{(k)}(\mathbf{a}; \infty)$, so there exists $M_2 > 1$ such that for large enough N,

$$\mathbb{P}(F_1 < -\log M_2 \text{ or } F_1 > \log M_2) < \frac{\epsilon_1}{16}. \tag{7.18}$$

Then the event A_5 defined by

$$A_5 = \left\{\omega \in A_4 \mid [-\log M_2, \log M_2] \cap [F_1(\tilde{\lambda}_1 + \epsilon_3; \omega), F_1(M_1; \omega)] \neq \emptyset\right\} \tag{7.19}$$

satisfies $\mathbb{P}(A_5) > \epsilon_1/16$ for large enough N. Otherwise, we will have

$$\mathbb{P}(F_1 < -\log M_2 \text{ or } F_1 > \log M_2) > \int_{A_4 \setminus A_5} p(\omega)\mathrm{d}\omega \mathbb{P}(\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1] \mid \omega) \geq \frac{\epsilon_1}{16}\frac{1}{3} = \frac{\epsilon_1}{48}. \tag{7.20}$$

We have, by (5.2) and (5.3),

$$F_1(\tilde{\sigma}; \omega) = \int_{\bigcup_{i=2}^{N}(\tilde{\sigma}_i, \tilde{\lambda}_{i-1}]} \frac{-1}{\tilde{\sigma} - x}\mathrm{d}x + \frac{1}{3}\log N. \tag{7.21}$$

Hence with $\omega \in A_4$, we have

$$F_1(M_1; \omega) - F_1(\tilde{\lambda}_1 + \epsilon_3; \omega) < \int_{(-\infty, \tilde{\lambda}_1]} \left(\frac{1}{\tilde{\lambda}_1 + \epsilon_3 - x} - \frac{1}{M_1 - x}\right)\mathrm{d}x = \log\frac{M_1 - \tilde{\lambda}_1}{\epsilon_3} < \log\frac{M_1 - c_1}{\epsilon_3}, \tag{7.22}$$

and for $\tilde{\sigma}$ between $\tilde{\lambda}_1 + \epsilon_3$ and M_1,

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{\sigma}}F_1(\tilde{\sigma}; \omega) > \int_{[c_1 - \epsilon_2, c_1]} \frac{1}{(\tilde{\sigma} - x)^2}\mathrm{d}x = \frac{1}{M_1 - c_1} - \frac{1}{M_1 - c_1 + \epsilon_2}, \tag{7.23}$$

where we recalled the domain in (7.5).

By (7.22), with $M_3 = M_2 \cdot ((M_1 - c_1)/\epsilon_3)$, we have that

$$[F_1(\tilde{\lambda}_1 + \epsilon_3; \omega), F_1(M_1; \omega)] \subseteq [-\log M_3, \log M_3], \quad \text{if } \omega \in A_5. \tag{7.24}$$

Hence we have for all $\omega \in A_5$,

$$\begin{aligned} \mathbb{P}(-\log M_3 < F_1(\tilde{\sigma}_1; \omega) < \log M_3 \mid \omega) &\geq \mathbb{P}(F_1(\tilde{\lambda}_1 + \epsilon_3; \omega) < F_1(\tilde{\sigma}_1; \omega) < F_1(M_1; \omega) \mid \omega) \\ &= \mathbb{P}(\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1] \mid \omega) > \frac{1}{3}. \end{aligned} \tag{7.25}$$

We note that by (7.8), $\tilde{f}''_{\omega, a_k}$ is positive and decreasing on $(\tilde{\lambda}_1, +\infty)$. For $\omega \in A_5$, we have that if $\tilde{\sigma} > \tilde{\lambda}_1$, then for large enough N,

$$\begin{aligned} 0 < \tilde{f}''_{\omega, a_k}(\tilde{\sigma}) &< N^{-\frac{1}{3}} + \sum_{i=2}^{N} \frac{1}{(\tilde{\sigma}_i - c_1)^2} \\ &< N^{-\frac{1}{3}} + \frac{1 + \epsilon_2^2}{\epsilon_2^2} \sum_{i=2}^{N} \frac{1}{(\tilde{\sigma}_i - c_1)^2 + 1} \\ &< N^{-\frac{1}{3}} + \frac{1 + \epsilon_2^2}{\epsilon_2^2} \int_{\tilde{\lambda}_1}^{\infty} \tilde{p}_\omega(\tilde{\sigma}_1) \sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - c_1)^2 + 1}\mathrm{d}\tilde{\sigma}_1 \\ &< \frac{8C_L}{\epsilon_1 \epsilon_2^2}, \end{aligned} \tag{7.26}$$

where we use the property that $\tilde{\sigma}_i + \epsilon_2 < c_1 < \tilde{\sigma}$ $(i = 2, \ldots, N)$.

Now with the aid of (7.26), we claim that for all $\omega \in A_5$, if we take C_A to be a constant bigger than both $\epsilon_3^{-1} + 8C_L(\epsilon_1\epsilon_2^2)^{-1}(M_1 - c_1)$ and $16C_L(\epsilon_1\epsilon_2^2)^{-1}(M_1 - c_1)$, then

$$|\tilde{f}'_{\omega,a_k}(\tilde{\sigma})| < C_A \quad \text{if } \tilde{\sigma} \in [\tilde{\lambda}_1 + \epsilon_3, M_1]. \tag{7.27}$$

Since $\tilde{f}'_{\omega,a_k}(\tilde{\sigma})$ is increasing on $(\tilde{\lambda}_1, +\infty)$, to prove (7.27), it suffices to check that

$$\tilde{f}'_{\omega,a_k}(\tilde{\lambda}_1 + \epsilon_3) > -C_A, \tag{7.28}$$

$$\tilde{f}'_{\omega,a_k}(M_1) < C_A. \tag{7.29}$$

If (7.28) does not hold, we have that, in light of (7.26),

$$\begin{aligned}
\tilde{f}'_{\omega,a_k}(2M_1 - c_1) &= \tilde{f}'_{\omega,a_k}(\tilde{\lambda}_1 + \epsilon_3) + \int_{\tilde{\lambda}_1+\epsilon_3}^{2M_1-c_1} \tilde{f}''_{\omega,a_k}(\tilde{\sigma})\mathrm{d}\tilde{\sigma} \\
&\leq \tilde{f}'_{\omega,a_k}(\tilde{\lambda}_1 + \epsilon_3) + [(2M_1 - c_1) - (\tilde{\lambda}_1 + \epsilon_3)]\frac{8C_L}{\epsilon_1\epsilon_2^2} \\
&\leq -C_A + 2(M_1 - c_1)\frac{8C_L}{\epsilon_1\epsilon_2^2} < 0.
\end{aligned} \tag{7.30}$$

It implies that $\tilde{f}'_{\omega,a_k}(\tilde{\sigma})$ is negative on $(\tilde{\lambda}_1, 2M_1 - c_1)$, and then $\tilde{f}_{\omega,a_k}(\tilde{\sigma})$ is decreasing there. By (7.7), $\tilde{p}_\omega(\tilde{\sigma})$ is increasing there. We hence have

$$\frac{\mathbb{P}(\tilde{\sigma}_1 \in (M_1, +\infty) \mid \omega)}{\mathbb{P}(\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1] \mid \omega)} \geq \frac{\mathbb{P}(\tilde{\sigma}_1 \in (M_1, 2M_1 - c_1) \mid \omega)}{\mathbb{P}(\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1] \mid \omega)} = \frac{\int_{M_1}^{2M_1-c_1} \tilde{p}_\omega(\tilde{\sigma})d\tilde{\sigma}}{\int_{\tilde{\lambda}_1+\epsilon_3}^{M_1} \tilde{p}_\omega(\tilde{\sigma})d\tilde{\sigma}} > 1, \tag{7.31}$$

which is contradictory to (7.15). On the other hand, if (7.29) does not hold, then

$$\tilde{f}'_{\omega,a_k}(\tilde{\lambda}_1) = \tilde{f}'_{\omega,a_k}(M_1) - \int_{\tilde{\lambda}_1}^{M_1} \tilde{f}''_{\omega,a_k}(\tilde{\sigma})\mathrm{d}\tilde{\sigma} \geq \tilde{f}'_{\omega,a_k}(M_1) - (M_1 - \tilde{\lambda}_1)\frac{8C_L}{\epsilon_1\epsilon_2^2} \geq C_A - (M_1 - c_1)\frac{8C_L}{\epsilon_1\epsilon_2^2} \geq \epsilon_3^{-1}. \tag{7.32}$$

Hence we have, by the monotonicity of $\tilde{f}'_{\omega,a_k}(\tilde{\sigma})$, that $\tilde{f}'_{\omega,a_k}(\tilde{\sigma}) \geq \epsilon_3^{-1}$ for all $\tilde{\sigma} \in [\tilde{\lambda}_1, M_1]$. Hence we have

$$\begin{aligned}
\int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1+\epsilon_3} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} &= \frac{1}{\tilde{C}_{\omega,a_k}} \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1+\epsilon_3} \exp(-\tilde{f}_{\omega,a_k}(\tilde{\sigma}))\mathrm{d}\tilde{\sigma} \\
&\geq \frac{1}{\tilde{C}_{\omega,a_k}} \int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1+\epsilon_3} \exp[-\tilde{f}_{\omega,a_k}(\tilde{\lambda}_1 + \epsilon_3) + \epsilon_3^{-1}(\tilde{\lambda}_1 + \epsilon_3 - \tilde{\sigma})]\mathrm{d}\tilde{\sigma} \\
&= (e-1)\epsilon_3\frac{\exp[-\tilde{f}_{\omega,a_k}(\tilde{\lambda}_1 + \epsilon_3)]}{\tilde{C}_{\omega,a_k}},
\end{aligned} \tag{7.33}$$

and

$$\begin{aligned}
\int_{\tilde{\lambda}_1+\epsilon_3}^{M_1} \tilde{p}_\omega(\tilde{\sigma})\mathrm{d}\tilde{\sigma} &= \frac{1}{\tilde{C}_{\omega,a_k}} \int_{\tilde{\lambda}_1+\epsilon_3}^{M_1} \exp(-\tilde{f}_{\omega,a_k}(\tilde{\sigma}))\mathrm{d}\tilde{\sigma} \\
&\leq \frac{1}{\tilde{C}_{\omega,a_k}} \int_{\tilde{\lambda}_1+\epsilon_3}^{M_1} \exp[-\tilde{f}_{\omega,a_k}(\tilde{\lambda}_1 + \epsilon_3) - \epsilon_3^{-1}(\tilde{\sigma} - \tilde{\lambda}_1 - \epsilon_3)]\mathrm{d}\tilde{\sigma} \\
&< \epsilon_3\frac{\exp[-\tilde{f}_{\omega,a_k}(\tilde{\lambda}_1 + \epsilon_3)]}{\tilde{C}_{\omega,a_k}}.
\end{aligned} \tag{7.34}$$

Hence we have

$$\frac{\mathbb{P}(\tilde{\sigma}_1 \in (\tilde{\lambda}_1, \tilde{\lambda}_1 + \epsilon_3) \mid \omega)}{\mathbb{P}(\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1] \mid \omega)} = \frac{\int_{\tilde{\lambda}_1}^{\tilde{\lambda}_1+\epsilon_3} \tilde{p}_\omega(\tilde{\sigma})d\tilde{\sigma}}{\int_{\tilde{\lambda}_1+\epsilon_3}^{M_1} \tilde{p}_\omega(\tilde{\sigma})d\tilde{\sigma}} > 1, \tag{7.35}$$

which is also contradictory to (7.15). Hence we have (7.27).

We have that if $\omega \in A_5$ and $\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1]$, then (i) $\tilde{\sigma}_1$ spreads fairly even due to (7.27); and (ii) $F_1(\tilde{\sigma}_1;\omega)$ varies monotonically and significantly, by (7.23). We therefore conclude that for all $\omega \in A_5$, for some $\epsilon_4 > 0$ independent of ω,

$$\operatorname{Var}(F_1(\tilde{\sigma}_1;\omega) \mid \omega \text{ and } \tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1]) > \epsilon_4. \tag{7.36}$$

Then using (7.25), we have

$$\begin{aligned} &\operatorname{Var}(F_1(\tilde{\sigma}_1;\omega) \mid \omega \text{ and } -\log M_3 < F_1(\tilde{\sigma}_1;\omega) < \log M_3) \\ \geq &\frac{\mathbb{P}(\tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1] \mid \omega)}{\mathbb{P}(-\log M_3 < F_1(\tilde{\sigma}_1;\omega) < \log M_3 \mid \omega)} \operatorname{Var}(F_1(\tilde{\sigma}_1;\omega) \mid \omega \text{ and } \tilde{\sigma}_1 \in [\tilde{\lambda}_1 + \epsilon_3, M_1]) \\ > &\frac{1}{3}\epsilon_4. \end{aligned} \tag{7.37}$$

Taking $A = A_5$, $M = M_3$, $\epsilon = \epsilon_1/16$, $\epsilon' = 1/3$ and $\epsilon'' = \epsilon_4$, we see that Lemma 12 is verified in the $j = 1$ case by (7.25) and (7.37).

The $j > 1$ case Similar to A_1 in the proof of the $j = 1$ case, we can define

$$B_1 = \{\tilde{\lambda}_j \in (d_1, d_1 + \delta_2) \text{ and } \tilde{\lambda}_{j-1} \in (d_2 - \delta_2, d_2) \text{ and } \tilde{\sigma}_{j+1} < d_1 - \delta_2 \text{ and } \tilde{\sigma}_{j-1} > d_2 + \delta_2 \text{ and } \tilde{\sigma}_1 < N_1\}, \tag{7.38}$$

such that $\mathbb{P}(B_1) > \delta_1$ for large enough N, where d_1, d_2, N_1 are real numbers, δ_1, δ_2 are positive numbers, and $d_1 + \delta_2 < d_2 - \delta_2$, $d_2 + \delta_2 < N_1$. We additionally require that $\delta_2 < (d_2 - d_1)/6$ for later use.

Next, let $p(\omega)$ be the marginal density of ω, and the conditional density of σ_j, as ω is fixed, is

$$p_\omega(\sigma) = \frac{1}{C_\omega} \exp(-g_{\omega,\alpha_1}(\sigma)) \mathbf{1}(\lambda_j < \sigma < \lambda_{j-1}) \quad \text{where} \quad g_{\omega,\alpha_1} = \frac{\sigma^2}{2} - \alpha_1 \sigma - \sum_{i=j+1}^{N} \log(\sigma - \sigma_i) - \sum_{i=1}^{j-1} \log(\sigma_i - \sigma), \tag{7.39}$$

for some constant C_ω, or equivalently, the conditional density for $\tilde{\sigma}_j$ is

$$\begin{gathered} \tilde{p}_\omega(\tilde{\sigma}) = \frac{p_\omega(\sigma)}{N^{1/6}} = \frac{1}{\tilde{C}_{\omega,a_k}} \exp(-\tilde{g}_{\omega,a_k}(\tilde{\sigma})) \mathbf{1}(\tilde{\lambda}_j < \tilde{\sigma} < \tilde{\lambda}_{j-1}), \\ \text{where} \quad \tilde{g}_{\omega,a_k}(\tilde{\sigma}) = N^{\frac{1}{3}} \tilde{\sigma} - a_k \tilde{\sigma} + N^{-\frac{1}{3}} \frac{\tilde{\sigma}^2}{2} - \sum_{i=j+1}^{N} \log(\tilde{\sigma} - \tilde{\sigma}_i) - \sum_{i=1}^{j-1} \log(\tilde{\sigma}_i - \tilde{\sigma}), \end{gathered} \tag{7.40}$$

and $\tilde{C}_{\omega,a_k} = N^{-N/6} \exp(2N^{2/3} a_k) C_\omega$. Hence

$$\begin{gathered} \tilde{g}'_{\omega,a_k}(\tilde{\sigma}) = N^{\frac{1}{3}} - a_k + N^{-\frac{1}{3}} \tilde{\sigma} - \sum_{i=j+1}^{N} \frac{1}{\tilde{\sigma} - \tilde{\sigma}_i} + \sum_{i=1}^{j-1} \frac{1}{\tilde{\sigma}_i - \tilde{\sigma}}, \quad \tilde{g}''_{\omega,a_k}(\tilde{\sigma}) = N^{-\frac{1}{3}} + \sum_{i \in [\![1,N]\!] \setminus \{j\}} \frac{1}{(\tilde{\sigma} - \tilde{\sigma}_i)^2}, \\ \text{and} \quad \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} \tilde{p}_\omega(\tilde{\sigma}) \mathrm{d}\tilde{\sigma} = 1. \end{gathered} \tag{7.41}$$

Then analogous to A_2 and A_3 in the proof of the $j = 1$ case, we can define

$$B_3 = \left\{ \omega \in B_1 \mid \int_{\tilde{\lambda}_{j-1} - \delta_3}^{\tilde{\lambda}_{j-1}} \tilde{p}_\omega(\tilde{\sigma}) \mathrm{d}\tilde{\sigma} < \frac{1}{3} \text{ and } \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_j + \delta_3} \tilde{p}_\omega(\tilde{\sigma}) \mathrm{d}\tilde{\sigma} < \frac{1}{3} \right\}, \tag{7.42}$$

such that $\mathbb{P}(B_3) > \delta_1/4$ for large enough N, where $\delta_3 > 0$. We additionally require that $\delta_3 < \delta_2/2$ for later use. It is straightforward to see that for all $\omega \in B_3$,

$$\mathbb{P}(\tilde{\sigma}_j \in [\tilde{\lambda}_j + \delta_3, \tilde{\lambda}_{j-1} - \delta_3]) > \frac{1}{3}. \tag{7.43}$$

Analogous to the definition of A_4, we choose L in Lemma 13 sufficiently large such that $[-L, L] \supseteq (d_1 - \delta_2, d_2 + \delta_2)$, and let

$$B_4 = \Bigg\{\omega \in B_3 \mid \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} \tilde{p}_\omega(\tilde{\sigma}_j) \sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - d_1)^2 + 1} \mathrm{d}\tilde{\sigma}_j < \frac{C_L}{\delta_1/16} \tag{7.44a}$$

$$\text{and} \int_{\tilde{\lambda}_j}^{\tilde{\lambda}_{j-1}} \tilde{p}_\omega(\tilde{\sigma}_j) \sum_{i=1}^{N} \frac{1}{(\tilde{\sigma}_i - d_2)^2 + 1} \mathrm{d}\tilde{\sigma}_j < \frac{C_L}{\delta_1/16} \Bigg\}. \tag{7.44b}$$

We have that $\mathbb{P}(B_4) > \delta_1/8$ for large enough N. Otherwise, we have that with probability no less than $\delta_1/8$, at least one of the two inequalities in (7.44a) and (7.44b) fails. Without of loss of generality, we assume that in probability $\geq \delta_1/16$ inequality (7.44a) fails. Then like (7.17), we can derive $\mathbb{E}\left(\sum_{i=1}^N \frac{1}{(\tilde{\sigma}_i - d_1)^2+1}\right) \geq C_L$, contradictory to (7.3).

Now consider the function $F_j = \log(N^{1/3}|x_{j1}|^2)$ defined by (1.20). For a fixed $\omega \in B_4$, F_j depends on σ_j, or equivalently $\tilde{\sigma}_j$. Below we express it as a function of $\tilde{\sigma}_j$ and further decompose it into two parts $F_j(\tilde{\sigma}_j; \omega) = F_j^{(1)}(\tilde{\sigma}_j; \omega) + F_j^{(2)}(\tilde{\sigma}_j; \omega)$, where

$$F_j^{(1)}(\tilde{\sigma}_j; \omega) := \sum_{i=1}^{j-1} \log \frac{\sigma_j - \lambda_i}{\sigma_j - \sigma_i}, \quad F_j^{(2)}(\tilde{\sigma}_j; \omega) := \sum_{i=j+1}^{N} \log \frac{\sigma_j - \lambda_{i-1}}{\sigma_j - \sigma_i} + \frac{1}{3} \log N. \tag{7.45}$$

We also note that $F_j^{(2)}(\tilde{\sigma}; \omega)$ is an increasing function of $\tilde{\sigma} \in (\tilde{\lambda}_j, \tilde{\lambda}_{j-1})$, analogous to $F_1(\tilde{\sigma}; \omega)$ as $\tilde{\sigma} \in (\tilde{\lambda}_1, +\infty)$. Similar to A_5 in the proof of the $j = 1$ case, we have that for a large enough $N_2 > 0$, the event

$$B_5 = \left\{\omega \in B_4 \mid [-\log N_2, \log N_2] \cap [F_j^{(2)}(\tilde{\lambda}_j + \delta_3; \omega), F_j^{(2)}(\tilde{\lambda}_{j-1} - \delta_3; \omega)] \neq \emptyset\right\} \tag{7.46}$$

satisfies $\mathbb{P}(B_5) > \delta_1/16$ for large enough N.

We have, by (5.2) and (5.3),

$$F_j^{(2)}(\tilde{\sigma}; \omega) = \int_{\bigcup_{i=j+1}^N (\tilde{\sigma}_i, \tilde{\lambda}_{i-1}]} \frac{-1}{\tilde{\sigma} - x} \mathrm{d}x + \frac{1}{3} \log N. \tag{7.47}$$

Hence with $\omega \in A_4$, we have, analogous to (7.22) and (7.23),

$$F_j^{(2)}(\tilde{\lambda}_{j-1} - \delta_3; \omega) - F_j^{(2)}(\tilde{\lambda}_j + \delta_3; \omega) < \int_{(-\infty, \tilde{\lambda}_j]} \left(\frac{1}{\tilde{\lambda}_j + \delta_3 - x} - \frac{1}{\tilde{\lambda}_{j-1} - \delta_3 - x}\right) \mathrm{d}x < \log \frac{d_2 - d_1}{\delta_3}, \tag{7.48}$$

By (7.48), with $N_3 = N_2 \cdot ((d_2 - d_1)/\delta_3)$, we have, analogous to (7.24),

$$[F_j^{(2)}(\tilde{\lambda}_j + \delta_3; \omega), F_j^{(2)}(\tilde{\lambda}_{j-1} - \delta_3; \omega)] \subseteq [-\log N_3, \log N_3], \quad \text{if } \omega \in B_5. \tag{7.49}$$

On the other hand, since $\tilde{\lambda}_{j-1} > d_2 - \delta_2$ and $\tilde{\sigma}_1 < N_1$, we have

$$-\log \frac{N_1 - d_2 + \delta_2 + \delta_3}{\delta_3} < \log \frac{\tilde{\lambda}_{j-1} - \tilde{\sigma}_j}{\tilde{\sigma}_1 - \tilde{\sigma}_j} < F_j^{(1)}(\tilde{\sigma}_j; \omega) < 0, \quad \text{for all } \tilde{\sigma}_j \in [\tilde{\lambda}_j + \delta_3, \tilde{\lambda}_{j-1} - \delta_3]. \tag{7.50}$$

Hence by setting $N_4 = N_3 \cdot ((N_1 - d_2 + \delta_2 + \delta_3)/\delta_3)$, we have for any $\tilde{\sigma} \in [\tilde{\lambda}_j + \delta_3, \tilde{\lambda}_{j-1} - \delta_3]$,

$$[F_j(\tilde{\lambda}_j + \delta_3; \omega), F_j(\tilde{\lambda}_{j-1} - \delta_3; \omega)] \subseteq [-\log N_4, \log N_4], \quad \text{if } \omega \in B_5. \tag{7.51}$$

We then have, analogous to (7.25) in the $j = 1$ case,

$$\mathbb{P}(-\log N_4 < F_j(\tilde{\sigma}_j; \omega) < \log N_4 \mid \omega) > \frac{1}{3}. \tag{7.52}$$

Further, note that if $\tilde{\sigma} \in (\tilde{\lambda}_j, d_2 - \delta_2)$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tilde{\sigma}} F_j(\tilde{\sigma}; \omega) &= \int_{\bigcup_{i=j+1}^N (\tilde{\sigma}_i, \tilde{\lambda}_{i-1}]} \frac{\mathrm{d}x}{(\tilde{\sigma} - x)^2} - \int_{\bigcup_{i=1}^{j-1} (\tilde{\lambda}_i, \tilde{\sigma}_i]} \frac{\mathrm{d}x}{(\tilde{\sigma} - x)^2} \\ &> \int_{[\tilde{\lambda}_j - \delta_2, \tilde{\lambda}_j]} \frac{\mathrm{d}x}{(\tilde{\sigma} - x)^2} - \int_{[d_2 - \delta_2, +\infty)} \frac{\mathrm{d}x}{(\tilde{\sigma} - x)^2} = \frac{\delta_2}{(\tilde{\sigma} - \tilde{\lambda}_j)(\tilde{\sigma} - \tilde{\lambda}_j + \delta_2)} - \frac{1}{d_2 - \delta_2 - \tilde{\sigma}}. \end{aligned} \tag{7.53}$$

By direct computation, and based on our assumptions on $d_1, d_2, \delta_2, \delta_3$, we have that for all $\omega \in B_5$

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{\sigma}} F_j(\tilde{\sigma};\omega) \geq \frac{1}{6\delta_2} > 0, \quad \tilde{\sigma} \in [\lambda_j + \delta_3, \lambda_j + \delta_2], \tag{7.54}$$

which is a counterpart of (7.23).

We note that by (7.41), $\tilde{g}''_{\omega,a_k}$ is positive and concave on $(\tilde{\lambda}_j, \tilde{\lambda}_{j-1})$, so we have for $\tilde{\sigma} \in (\tilde{\lambda}_j, \tilde{\lambda}_{j-1})$,

$$0 < \tilde{g}''_{\omega,a_k}(\tilde{\sigma}) < \max\{\tilde{g}''_{\omega,a_k}(\tilde{\lambda}_j), \tilde{g}''_{\omega,a_k}(\tilde{\lambda}_{j-1})\}, \tag{7.55}$$

and then for large enough N, we can derive in a way parallel to (7.26)

$$0 < \tilde{g}''_{\omega,a_k}(\tilde{\sigma}) < \frac{16C_L}{\delta_1\delta_2^2}. \tag{7.56}$$

Moreover, we can show that for all $\omega \in B_5$, we can find a large enough $C_B > 0$ such that

$$|\tilde{g}'_{\omega,a_k}(\tilde{\sigma})| < C_B \quad \text{if } \tilde{\sigma} \in [\tilde{\lambda}_j + \delta_3, \tilde{\lambda}_{j-1} - \delta_3]. \tag{7.57}$$

The proof relies on that $\tilde{g}_{\omega,a_k}(\tilde{\sigma})$ is increasing on $(\tilde{\lambda}_j, \tilde{\lambda}_{j-1})$ and we can show both $\tilde{g}'_{\omega,a_k}(\tilde{\lambda}_j + \delta_3) > -C_N$ and $\tilde{g}'_{\omega,a_k}(\tilde{\lambda}_{j-1} - \delta_3) < C_N$. Since the proof techniques are similar to those used for (7.28) and (7.29), we omit the detail.

We have, analogous to (7.36),

$$\mathrm{Var}(F_j(\tilde{\sigma}_j;\omega) \mid \omega \text{ and } \tilde{\sigma}_1 \in [\tilde{\lambda}_j + \delta_3, \tilde{\lambda}_j + \delta_2]) > \epsilon_4. \tag{7.58}$$

Also because of the probability inequality (7.43) and the boundedness of $\tilde{g}'_{\omega,a_k}(\tilde{\sigma})$ given in (7.57), we have that

$$\mathbb{P}(\tilde{\sigma}_j \in [\tilde{\lambda}_j + \delta_3, \tilde{\lambda}_j + \delta_2] \mid \omega) > \frac{\delta_5}{3}. \tag{7.59}$$

for some $\delta_4 > 0$. Hence we have, analogous to (7.37),

$$\begin{aligned} &\mathrm{Var}(F_j(\tilde{\sigma}_j;\omega) \mid \omega \text{ and } -\log N_4 < F_j(\tilde{\sigma}_1;\omega) < \log N_4) \\ \geq &\frac{\mathbb{P}(\tilde{\sigma}_j \in [\tilde{\lambda}_j + \delta_3, \tilde{\lambda}_j + \delta_2] \mid \omega)}{\mathbb{P}(-\log N_4 < F_j(\tilde{\sigma}_j;\omega) < \log N_4 \mid \omega)} \mathrm{Var}(F_j(\tilde{\sigma}_j;\omega) \mid \omega \text{ and } \tilde{\sigma}_j \in [\tilde{\lambda}_1 + \delta_3, \delta_2]) \\ > &\frac{\delta_5}{3}\delta_4. \end{aligned} \tag{7.60}$$

Taking $A = B_5$, $M = N_4$, $\epsilon = \delta_1/16$, $\epsilon' = 1/3$ and $\epsilon'' = \delta_4\delta_5/3$, we see that Lemma 12 is verified in the $j > 1$ case by (7.52) and (7.60). □

In the end of this section, we state some simulation results on the distribution of the eigenvector components; see Figures 18 and 19 below. The simulation is done under the following setting for $G_{\boldsymbol{\alpha}}$ in (1.1): $N = 1000$, $k = 2$, $\alpha_1 = \sqrt{N}$, $\alpha_2 = \sqrt{N} - N^{1/6}$. The simulation results are based on 6000 replications. In Figure 18, we plot the kernel density estimates (smooth approximations of histograms), p_X, for $X = N^{1/3}|x_{11}|^2$, $N^{1/3}|x_{12}|^2$, and $N|x_{13}|^2$. In Figure 19, we plot the negative logarithm of the tail function, $-\log \mathbb{P}(X > t), t \geq 0.5$, for $X = N^{1/3}|x_{11}|^2$, $N^{1/3}|x_{12}|^2$, and $N|x_{13}|^2$. Observe that the limiting distribution of $N|x_{13}|^2$ shall be Exp(1), in light of Corollary 3. However, the yellow curve is apparently above 1 at $t = 0$ in Figure 18. This is due to a finite N effect, since according to our proof of Corollary 3, the difference between the distribution of $N|x_{13}|^2$ and the limiting one, Exp(1), is of order $O(N^{-1/3})$. We also remark here that Figure 19 shows (numerically) the difference between the tail behavior of the laws in Theorem 2 and that of Exp(1). A theoretical study of the tails of these laws will be deferred to future study.

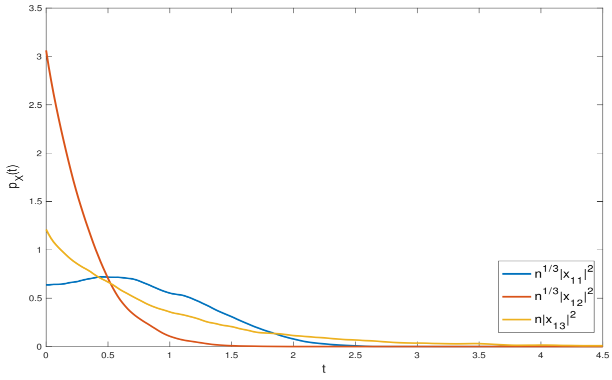

Figure 18: Kernel Density Estimate

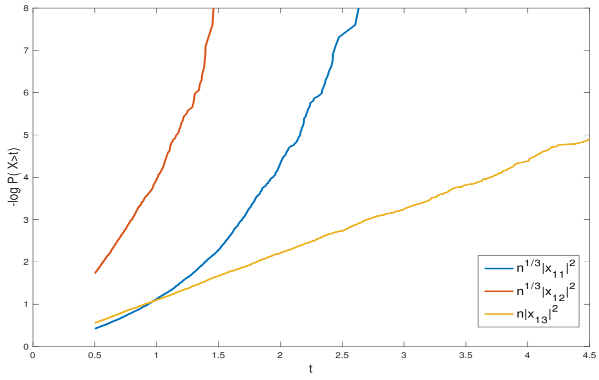

Figure 19: Negative Logarithm of Tail Probability

A Proof of results in Section 2

Proof of Lemma 6 We only give a sketch of the proof of Lemma 6, because the case that $j_1 = j_2$, and all $\alpha_j = \sqrt{N}$ (equivalent to $a_{k-j+1} = 0$) for $j = 1, \dots, k$ is already proved in [46] which follows closely the method used in [6]. Our proof is an adaption of that in [46]. The main difference between our lemma and the results in [6] and [46] is that we require ϵ to be large enough while in [6] and [46], ϵ is only required to be positive, because they essentially assume all $a_j = 0$. Our assumption on ϵ implies that the operators $e^{\epsilon(x-y)} K^{k_1,k_2}_{\text{Airy},\mathbf{a}}(x, y)$ and $e^{\epsilon(x-y)} K^{j_1,j_2}_{N,\text{scaled}}(x, y)$ are both trace class.

Proof. Below in this proof we are going to use notation in [46] that is quite different from the notation used elsewhere in our paper.

We define, analogous to [46, Formula (16)],

$$K'_{N,j_1,j_2}(x, y) = N^{-1/6} N^{(j_1-j_2)/6} e^{N^{1/3}(y-x)} \widetilde{K}^{j_1,j_2}_{\text{GUE},\boldsymbol{\alpha}}(2\sqrt{N} + N^{-1/6} y, 2\sqrt{N} + N^{-1/6} x). \tag{A.1}$$

We only need to consider the convergence of $e^{\epsilon(y-x)} K'_{N,j_1,j_2}(x, y)$ to $e^{\epsilon(y-x)} \widetilde{K}^{k_1,k_2}_{\text{Airy},\mathbf{a}}(y, x)$ (pointwise and in trace norm). Analogous to [46, Formulas (18) and (19)], we denote $F(z) = z^2/2 - 2z + \log z$ (see [46, Formula (17)]), $\tilde{w}_c = 1 + \epsilon N^{-1/3}$ (see [46, Formula (16)]), and define

$$H_{N,j_2}(x) = \frac{N^{\frac{1}{3}}}{2\pi} \int_\Gamma \left[z^k \prod_{i=j_2+1}^{k} \frac{1}{z - \frac{\alpha_i}{\sqrt{N}}} \right] \exp(-NF(z)) \exp(N^{\frac{1}{3}} x (z - \tilde{w}_c)) \mathrm{d}z, \tag{A.2}$$

$$J_{N,j_1}(y) = \frac{N^{\frac{1}{3}}}{2\pi} \int_\gamma \left[w^{-k} \prod_{i=j_1+1}^{k} \left(w - \frac{\alpha_i}{\sqrt{N}} \right) \right] \exp(NF(w)) \exp(-N^{\frac{1}{3}} y (w - \tilde{w}_c)) \mathrm{d}w, \tag{A.3}$$

where the contours Γ and γ are defined as in [46, Formula (14)]. Then we have, analogous to [46, Proposition 2.1],

$$N^{(j_1-j_2)/3} e^{\epsilon(y-x)} K'_{N,j_1,j_2}(x, y) = -\int_0^\infty H_{N,j_2}(x + t) J_{N,j_1}(y + t) \mathrm{d}t. \tag{A.4}$$

Next, analogous to [46, Formulas (21) and (22)], define

$$H_{\infty,k_2}(x) = \frac{\exp(-\epsilon x)}{2\pi} \int_{\Gamma_\infty} \exp\left(xz - \frac{z^3}{3} \right) \prod_{i=1}^{k_2} \frac{1}{z - a_i} \mathrm{d}z, \tag{A.5}$$

$$J_{\infty,k_1}(y) = \frac{\exp(\epsilon y)}{2\pi} \int_{\gamma_\infty} \exp\left(-yw + \frac{w^3}{3} \right) \prod_{i=1}^{k_1} (w - a_i) \mathrm{d}w, \tag{A.6}$$

where the contours Γ_∞ and γ_∞ are defined in [46, Figure 1]. By the arguments in [46, Sections 2.1 and 2.2], we have, analogous to [46, Proposition 2.2], that for any fixed $y_0 \in \mathbb{R}$, there exists $C > 0$, $c > 0$, an integer

$N_0 > 0$ such that

$$\left|Z_{N,j_2}H_{N,j_2}(x) - H_{\infty,k_2}(x)\right| \leq \frac{C\exp(-cx)}{N^{1/3}}, \qquad \text{for any } x > y_0,\ N \geq N_0, \tag{A.7}$$

$$\left|Z_{N,j_1}^{-1}J_{N,j_1}(y) - J_{\infty,k_1}(y)\right| \leq \frac{C\exp(-cy)}{N^{1/3}}, \qquad \text{for any } y > y_0,\ N \geq N_0, \tag{A.8}$$

where $Z_{N,j} = N^{(j-k)/3}\exp(NF(1))$. On the other hand, we have

$$e^{\epsilon(y-x)}\widetilde{K}^{k_2,k_1}_{\mathrm{Airy},\mathbf{a}}(y,x) = -\int_0^\infty H_{\infty,k_2}(x+t)J_{\infty,k_1}(y+t)\mathrm{d}t. \tag{A.9}$$

The proof is finished by using the argument in [6, Section 3.3]. □

Proof of Lemma 7 By Lemma 6, for any n, the joint distribution of $\lambda_1^{(N-j)}, \lambda_1^{(N-j-1)}, \lambda_2^{(N-j)}, \lambda_2^{(N-j-1)}, \dots, \lambda_n^{(N-j)}$ converges weakly to that of $\xi_1^{(k-j)}, \xi_1^{(k-j-1)}, \xi_2^{(k-j)}, \xi_2^{(k-j-1)}, \dots, \xi_n^{(k-j)}$ up to a scaling transform. Hence we have that the interlacing inequality (2.6) implies the weak interlacing property

$$+\infty > \xi_1^{(k-j)} \geq \xi_1^{(k-j-1)} \geq \xi_2^{(k-j)} \geq \xi_2^{(k-j-1)} \geq \cdots \geq \xi_n^{(k-j)}. \tag{A.10}$$

On the other hand, the determinantal structure requires that the point process consisting of $\xi_i^{(k-j)}$ and $\xi_l^{(k-j-1)}$ is simple, so with probability 1 the inequalities in (A.10) are all strict. So with probability 1 we have (2.7) by letting $n \to \infty$.

Proof of Lemma 8 We prove the lemma in three steps: First the right tail estimate of $\xi_j^{(k)}$ in part 1, then the left tail estimate of $\xi_j^{(k)}$, and at last we prove part 2 about the rigidity of $\xi_n^{(k)}$.

Proof of the right tail estimate of $\xi_j^{(k)}$. We note that

$$\mathbb{P}(\xi_j^{(k)} > t) \leq \mathbb{P}(\xi_1^{(k)} > t) \leq \mathbb{E}(\#\text{ of } \xi_i^{(k)} \text{ on } [t,+\infty)) = \int_t^{+\infty} K^{k,k}_{\mathrm{Airy},\mathbf{a}}(x,x)\mathrm{d}x. \tag{A.11}$$

Then by (1.7),

$$\int_t^{+\infty} K^{k,k}_{\mathrm{Airy},\mathbf{a}}(x,x)\mathrm{d}x = \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma \mathrm{d}u \int_\sigma \mathrm{d}v \frac{e^{\frac{u^3}{3}-tu}}{e^{\frac{v^3}{3}-tv}} \frac{\prod_{j=1}^k (u-a_j)}{\prod_{j=1}^k (v-a_j)} \frac{1}{(u-v)^2}. \tag{A.12}$$

Let γ and σ be deformed into $\gamma_{\mathrm{std}}(\sqrt{t})$ and $\sigma_{\mathrm{std}}(-\sqrt{t})$ (c.f. (4.14)). By standard saddle point analysis, we find that as $t \to +\infty$, the integral (A.12) concentrates on the region $u \in \gamma_{\mathrm{std}}(\sqrt{t}) \cap \{u - \sqrt{t} = \mathcal{O}(t^{-1/4})\}$ and $v \in \sigma_{\mathrm{std}}(-\sqrt{t}) \cap \{v + \sqrt{t} = \mathcal{O}(t^{-1/4})\}$. Then we conclude that as $t \to +\infty$,

$$\int_t^{+\infty} K^{k,k}_{\mathrm{Airy},\mathbf{a}}(x,x)\mathrm{d}x = \mathcal{O}\left(t^{-3/2}\exp\left(-\frac{4}{3}t^{3/2}\right)\right). \tag{A.13}$$

Hence by choosing C properly, we have $\mathbb{P}(\xi_j^{(k)} > t) < \int_t^{+\infty} K^{k,k}_{\mathrm{Airy},\mathbf{a}}(x,x)dx < Ce^{-t/C}$. □

Proof of the left tail estimate of $\xi_j^{(k)}$. We note that by the interlacing property in Lemma 7, $\mathbb{P}(\xi_j^{(k)} < -t) < \mathbb{P}(\xi_j^{(0)} < -t)$, where $\xi_j^{(0)}$ is the j-th particle in the determinantal point process defined by the Airy kernel (1.6). Then by [53], with any $\lambda \in (0,1)$, we have

$$\mathbb{P}(\xi_j^{(0)} < -t) = \sum_{n=0}^{j-1} E(n;-t) < (1-\lambda)^{1-j}\sum_{n=0}^{\infty}(1-\lambda)^n E(n;-t) = 2^{j-1}D(-t,\lambda), \tag{A.14}$$

where $E(n;-t)$ is the probability that exactly n particles are in $[-t,\infty)$ as denoted in [53, Section ID], and $D(-t,\lambda)$ is defined by [53, Formula (1.17)] as

$$D(-t,\lambda) = \exp\left(-\int_{-t}^\infty (x+t)q(x;\lambda)^2\mathrm{d}x\right), \quad \text{where} \quad \begin{aligned} &\frac{\mathrm{d}q(s;\lambda)}{\mathrm{d}s^2} = sq(s;\lambda) + 2q^3(s;\lambda), \\ &q(s;\lambda) \sim \sqrt{\lambda}\,\mathrm{Ai}(s) \text{ as } s \to \infty. \end{aligned} \tag{A.15}$$

The function $q(s;\lambda)$ is the Ablowitz-Segur solution to the Painlevé II equation [1], [49], its asymptotics at $+\infty$ is given by the Airy function multiplied by constant $\sqrt{\lambda}$. The asymptotic behaviour of $q(s;\lambda)$ has been extensively studied, see [28] for a rigorous and systematic discussion. We then derive the upper bound of $D(-t,\lambda)$ for large t from the asymptotics of $q(s;\lambda)$, and finally justify the estimate $\mathbb{P}(\xi_j^{(k)} < -t) < Ce^{-t/C}$ for some properly chosen C. □

Proof of the rigidity of $\xi_n^{(k)}$. We note that by the interlacing property (2.6), for all $n > k$,

$$\mathbb{P}\left(\left|\xi_n^{(k)} + \left(\frac{3\pi n}{2}\right)^{2/3}\right| > n^{\frac{3}{5}}\right) \leq \mathbb{P}\left(\# \text{ of } \xi_l^{(0)} \text{ in } \left(-\left(\frac{3\pi n}{2}\right)^{2/3} + n^{\frac{3}{5}}, \infty\right) \text{ is } \geq n-k\right)$$
$$+ \mathbb{P}\left(\# \text{ of } \xi_l^{(0)} \text{ in } \left(-\left(\frac{3\pi n}{2}\right)^{2/3} - n^{\frac{3}{5}}, \infty\right) \text{ is } < n\right). \quad \text{(A.16)}$$

Since $\xi_n^{(0)}$ are the n-th particle in the determinantal point process with the Airy kernel, so the problem is reduced to the rigidity of particles in this determinantal point process. The desired regidity can be deduced from the mean and variance of the number of $\xi_l^{(0)}$ in $(-T,\infty)$ and the Markov inequality. If we denote the number of $\xi_l^{(0)}$ in $(-T,\infty)$ as $v_1(T)$, in the notation of [51], then

$$\mathbb{E}(v_1(T)) = 2T^{3/2}/(3\pi) + \mathcal{O}(1), \quad \text{and} \quad \operatorname{Var}(v_1(T)) = \mathcal{O}(\log T) \quad \text{(A.17)}$$

as $T \to +\infty$, see [51, Theorem 1 and the paragraph above Theorem 1] [§]. That is enough to show that as $l \to \infty$,

$$\mathbb{P}\left(v_1\left(\left(\frac{3\pi n}{2}\right)^{2/3} - n^{\frac{3}{5}}\right) \geq n-k\right) = \mathcal{O}\left(\frac{\log n}{n^{6/5}}\right), \quad \mathbb{P}\left(v_1\left(\left(\frac{3\pi n}{2}\right)^{2/3} + n^{\frac{3}{5}}\right) < n\right) = \mathcal{O}\left(\frac{\log n}{n^{6/5}}\right). \quad \text{(A.18)}$$

By choosing the constant c properly, we obtain (2.9) for all $n \geq 2$. □

Proof of Lemma 9 This lemma is analogous to Lemma 8. We prove it in four steps, with the first three steps parallel to those in the proof of Lemma 8: First, the right tail estimate of σ_j (part 1), next the left tail estimate of σ_j (part 1), and then the rigidity for σ_n close to the edge (part 2), and at last the rigidity of σ_n in the bulk (part 3). In part 1 we also need to consider σ_N, but we omit it, because the estimates for σ_N are analogous to the estimate for σ_1.

Proof of the right tail estimate of σ_j. We use the same idea as in (A.11), and write

$$\mathbb{P}(\sigma_j > 2\sqrt{N} + tN^{-\frac{1}{6}}) \leq \mathbb{P}(\sigma_1 > 2\sqrt{N} + tN^{-\frac{1}{6}}) = \int_{2\sqrt{N}+tN^{-1/6}}^{\infty} K^{0,0}_{\mathrm{GUE},\boldsymbol{\alpha}}(x,x)\mathrm{d}x = \int_t^{\infty} K'_{N,0,0}(x,x)\mathrm{d}x, \quad \text{(A.19)}$$

where $K'_{N,0,0}(x,x)$ is defined in (A.1). Although we can evaluate the right-hand side of (A.19) like (A.12), we prefer an indirect method that relies on result and proof of Lemma 6. We recall that as a special case of (A.4),

$$K'_{N,0,0}(x,x) = -\int_0^{\infty} H_{N,0}(x+t)J_{N,0}(x+t)\mathrm{d}t, \quad \text{(A.20)}$$

and then by (A.7) and (A.8), there exists $N_0 > 0$ and $C > 0$ such that for $x > 0$, $N > N_0$,

$$|Z_{N,0}H_{N,0}(x) - H_{\infty,k}(x)| \leq \frac{C\exp(-cx)}{N^{1/3}}, \quad \left|Z^{-1}_{N,j_1}J_{N,0}(x) - J_{\infty,k}(x)\right| \leq \frac{C\exp(-cy)}{N^{1/3}}, \quad \text{(A.21)}$$

where $Z_{N,0} = N^{-k/3}\exp(-3N/2)$ and $H_{\infty,k}, J_{\infty,k}$ are defined in (A.5) and (A.6). Hence by the very rough estimate (whose proof is omitted) that $H_{\infty,k}(x) = \mathcal{O}(1)$ and $J_{\infty,k}(x) = \mathcal{O}(1)$ for all $x > 0$, and with the help of (A.9), we have that

$$K'_{N,0,0}(x,x) - K^{k,k}_{\mathrm{Airy},\mathbf{a}}(x,x) = \mathcal{O}(N^{-1/3}e^{-cx}), \quad \text{for all } x > 0 \text{ and } N > N_0. \quad \text{(A.22)}$$

[§] It is pointed out in [42] that [51, Theorem 1] has a calculational error. See [42, Theorem 6.2]. Since we only need the magnitude of the variance, this mistake does not affect our argument. We also note that the variance of $\mathbb{P}(\# \text{ of } \xi_l^{(0)} \text{ in } (-T,-\infty))$ as $T \to +\infty$ can be computed by the contour integral method that is used in the proof of our Proposition 10.

Therefore, the desired right tail estimate of σ_j is implied by the estimate (A.13) for the right tail estimate of $\xi_j^{(k)}$. ☐

Proof of the left tail estimate of $\sigma_j = \lambda_j^{(N)}$. We use the same idea as in the proof of the left tail estimate of $\xi_j^{(k)}$, that $\mathbb{P}(\sigma_j < 2\sqrt{N} - tN^{-1/6}) \leq \mathbb{P}(\lambda_j^{(N-k)} < 2\sqrt{N} - tN^{-1/6})$, so it is not hard to see that it suffices to prove that there exists $C > 0$ such that

$$\mathbb{P}(\lambda_j^{(N-k)} < 2\sqrt{N-k} - t(N-k)^{-\frac{1}{6}}) < Ce^{-t/C}, \quad \text{for all } 2 \leq t \leq 2(N-k)^{2/3}. \tag{A.23}$$

where $\lambda_j^{(N-k)}$ is the j-th largest eigenvalue of a GUE random matrix with dimension $N-k$. The $j=1$ case of (A.23) exists in literature, see [43, Section 5.3, especially Formula (5.16)], where a stronger version of (A.23) is derived in a very accessible way. The $j > 1$ case of (A.23) is not found in literature, to the best knowledge of the authors. However, we can extend the method in [43, Section 5.3] to solve this case. To see it, we note that like (A.14), with $\lambda \in (0,1)$, we have

$$\begin{aligned} \mathbb{P}(\lambda_j^{(N-k)} < 2\sqrt{N-k} - t(N-k)^{-\frac{1}{6}}) ={}& \sum_{n=0}^{j-1} E(n; 2\sqrt{N-k} - t(N-k)^{-\frac{1}{6}}) \\ &< (1-\lambda)^{1-j} \sum_{n=0}^{\infty} (1-\lambda)^n E(n; 2\sqrt{N-k} - t(N-k)^{-\frac{1}{6}}) \\ &= (1-\lambda)^{1-j} \det(\mathrm{Id} - \lambda K), \end{aligned} \tag{A.24}$$

where K is the $N \times N$ matrix whose (m,n) entry is

$$\langle P_{m-1}, P_{n-1} \rangle_{L^2((1-\frac{t}{2}(N-k)^{-2/3}, \infty), d\mu)}, \tag{A.25}$$

such that the meanings of P_m and $d\mu$ are the same as in [43, Formula (1.11)]. Then by the same arguments that leads to [43, Formula (5.14)], we have

$$\det(\mathrm{Id} - \lambda K) = \prod_{i=1}^{N} (1 - \lambda \rho_i) \leq e^{-\frac{1}{2}\sum_{i=1}^{N} \rho_i} = \exp\left(-\lambda N \mu^N \left((1 - \frac{t}{2}(N-k)^{-2/3}, \infty) \right) \right), \tag{A.26}$$

where ρ_i are the eigenvalues of K, and μ^N is the measure defined in [43, Formula (1.4)]. We note that if we let $\lambda = 1$ in (A.26), then (A.26) is equivalent to [43, Formula (5.14)]. At last, using the estimate of $\mu^N((1 - \frac{t}{2}(N-k)^{-2/3}, \infty))$ given in [43, Section 5.3], we derive an estimate of $\det(\mathrm{Id} - \lambda K)$, which yields the desired estimate of $\mathbb{P}(\lambda_j^{(N-k)} < 2\sqrt{N-k} - t(N-k)^{-1/6})$ and $\mathbb{P}(\sigma_j < 2\sqrt{N} - tN^{-1/6})$. Finally, we note that essentially the idea of the proof above is in [56]. ☐

Proof of the rigidity of σ_n *for* $n \leq CN^{1/10}$. As in (2.12), we note that (2.12) is analogous to (2.9), and can be proved by an analogous argument. Instead of (A.17), we have that if $v_1^{(n)}(T)$ is the number of eigenvalues of an n-dimensional GUE random matrix in the interval $(2\sqrt{n} - n^{-1/6}T, +\infty)$, then as $n \to \infty$, $T \geq T_0$ a positive constant, and $T/n = o(1)$, by the result of [37] ¶

$$\mathbb{E}(v_1^{(n)}(T)) = 2T^{3/2}/(3\pi) + \mathcal{O}(1), \quad \text{and} \quad \mathrm{Var}(v_1^{(n)}(T)) = \mathcal{O}(\log T). \tag{A.27}$$

Then we prove (2.12) by the same argument as the proof of (2.9). ☐

The proof of the regidity of σ_n *as in* (2.13). This is a direct consequence of the interlacing property $\lambda_n^{(N-k)} \leq \sigma_n \leq \lambda_{n-k}^{(N-k)}$ and the rigidity of GUE eigenvalues in [33, Theorem 2.2], which states the rigidity of eigenvalues for Wigner matrices that of which the GUE random matrices are a special case. ☐

¶The mean estimate is given in [37, Lemma 2.2], and the variance estimate is given in [37, Lemma 2.3] under an additional condition that $T \to \infty$ as $n \to \infty$. However, as pointed out by [42, Remark under Theorem 6.3], if we only need a crude estimate as in (A.27), then the argument in [37] works for all $T > T_0 > 0$.

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