

Universality for random matrices with equi-spaced external source: a case study of a biorthogonal ensemble

Tom Claeys* and Dong Wang†

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Abstract

We prove the edge and bulk universality of random Hermitian matrices with equi-spaced external source. One feature of our method is that we use neither a Christoffel-Darboux type formula, nor a double-contour formula, which are standard methods to prove universality results for exactly solvable models. This matrix model is an example of a biorthogonal ensemble, which is a special kind of determinantal point process whose kernel generally does not have a Christoffel-Darboux type formula or double-contour representation. Our methods may showcase how to handle universality problems for biorthogonal ensembles in general.

1 Introduction

The model We consider the space of $n \times n$ Hermitian matrices with a probability measure of the form

$$\frac{1}{Z_n} \exp(-n \operatorname{Tr} [V(M) - AM]) dM, \quad (1.1)$$

where

$$dM = \prod_{i < j} d\operatorname{Re} M_{ij} d\operatorname{Im} M_{ij} \prod_{j=1}^n dM_{jj}. \quad (1.2)$$

This is a random matrix model with external source A , a deterministic Hermitian matrix which can be assumed diagonal without loss of generality. Throughout the paper, we assume the technical condition that the external field $V(x)$ is real analytic and strongly convex, which means that

$$V''(x) \geq c > 0 \quad (1.3)$$

for some constant c . This condition implies that V grows sufficiently fast as $x \rightarrow \pm\infty$ for the probability measure (1.1) to be well-defined. In this paper, we assume A is given by

$$A = \operatorname{diag}(a_1, a_2, \dots, a_n), \quad (1.4)$$

with the eigenvalues a_1, \dots, a_n of A equi-spaced. Without loss of generality, we may then assume that

$$a_j = \frac{j-1}{n}, \quad j = 1, 2, \dots, n. \quad (1.5)$$

*Université Catholique de Louvain, Chemin du cyclotron 2, B-1348 Louvain-La-Neuve, Belgium
email: tom.claeys@uclouvain.be

†School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, P. R. China 100049
email: wangdong@wangd-math.xyz

Random matrix models with external source were introduced in [16, 43]. In the case of a Gaussian external field $V(x) = x^2/2$, a random matrix M from the ensemble (1.1) follows the same probability law as a Gaussian Unitary Ensemble (GUE) matrix summed with the deterministic matrix A . This model and its dynamical generalization called Dyson's Brownian motion, with deterministic and random external source A , have been important in the study of universality for Wigner matrices, see e.g. [25, 26, 27, 34, 35].

Yet another interpretation in this case is that the eigenvalues of M follow the same joint probability distribution as n non-intersecting Brownian motions with the eigenvalues of A as starting points and 0 as the common endpoint [30, 31].

For general V , although the macroscopic large n behavior of the eigenvalues has been studied in some generality [28], their microscopic behavior was only described rigorously if A has two different eigenvalues with equal multiplicity [11, 12, 1, 13, 10, 2] and if A has a sufficiently small number of non-zero eigenvalues (aka spiked model) [3, 4, 6, 7].

Biorthogonal ensembles In this paper, we use the term biorthogonal ensemble for an n -particle probability distribution defined on an interval $I \subset \mathbb{R}$, which may be semi-infinite or $(-\infty, \infty)$, such that the joint probability density is of the form

$$\frac{1}{Z_n} \prod_{i < j} (\lambda_j - \lambda_i) (f(\lambda_j) - f(\lambda_i)) \prod_{i=1}^n w(\lambda_i) d\lambda_i, \quad (1.6)$$

where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \in I$ are the positions of the particles, w is a non-negative integrable weight function on I , and $f(\lambda)$ describes one part of the two-particle interaction in the form of $|f(\lambda_j) - f(\lambda_i)|$, while we note that the other part of the two-particle interaction is given by $|\lambda_j - \lambda_i|$. To make the probability density well defined, we may require that the function $f(x)$ is increasing as $x \in I$. This is a generalization of the n -particle distribution with the joint probability density

$$\frac{1}{Z_n} \prod_{i < j} (\lambda_j - \lambda_i)^2 \prod_{i=1}^n w(\lambda_i) d\lambda_i, \quad (1.7)$$

often referred to as an orthogonal polynomial ensemble, which corresponds to (1.6) with $f(x) = x$.

The biorthogonal ensembles defined by (1.6) were systematically studied by Borodin in [15], in particular for $f(x) = x^\theta$ and $w(x)$ one of the classical Hermite, Laguerre and Jacobi weights. Before [15], the special case of (1.6) with $f(x) = x^\theta$ and $w(x) = x^\alpha e^{-x}$ was introduced by Muttalib in [39] from a physical point of view. Hence, the special case of (1.6) with $f(x) = x^\theta$ is also called the Muttalib-Borodin ensemble, see [29, 36, 38, 20, 19] for example. Biorthogonal ensembles and variations thereof are also considered in [5, 37, 14, 17, 22, 24, 9, 8, 30, 31].

For the biorthogonal ensemble given by (1.6), we define the monic biorthogonal polynomials p_i and q_i with $i = 0, 1, 2, \dots$ being the degree of p_i and q_i , such that

$$\int_a^b p_i(x) q_j(f(x)) w(x) dx = h_i \delta_{ij}. \quad (1.8)$$

We note that by the positivity of the distribution (1.6), p_i and q_i are all uniquely defined. Thus, there is a well defined sum

$$K_n(x, y) = \sum_{i=0}^{n-1} \frac{1}{h_i} p_i(x) q_i(f(y)) \sqrt{w(x)w(y)}. \quad (1.9)$$

Then by the general theory of determinantal point processes (see [41] for example), the n -particle distribution (1.6) is a determinantal point process, whose correlation kernel can be taken to be $g(x)K_n(x, y)g(y)^{-1}$ for any nonzero function $g(x)$ on I .

We remark that the term “biorthogonal ensemble” was, to the best of the authors’ knowledge, first introduced by Borodin in [15], and we use the term in a slightly more general sense than [15]. Other authors have also used the term in more general contexts, like [23]. We stick to the definition given in (1.6), so that the biorthogonal polynomials p_i, q_i can be defined by (1.8).

The model as a biorthogonal ensemble For general eigenvalues a_1, \dots, a_n , the Harish-Chandra-Itzykson-Zuber formula can be used to express the joint probability distribution for the eigenvalues of the random matrices M with distribution (1.1) in the form [16, 32, 33]

$$\frac{1}{Z'_n} \det(e^{na_i \lambda_j})_{i,j=1}^n \prod_{i<j} (\lambda_j - \lambda_i) \prod_{j=1}^n e^{-nV(\lambda_j)} \prod_{j=1}^n d\lambda_j, \quad (1.10)$$

where $Z'_n = \text{const} \cdot Z_n \cdot \prod_{i<j} (a_j - a_i)$. For our special choice of A , this becomes

$$\frac{1}{Z'_n} \prod_{i<j} (\lambda_j - \lambda_i) \prod_{i<j} (e^{\lambda_j} - e^{\lambda_i}) \prod_{j=1}^n e^{-nV(\lambda_j)} \prod_{j=1}^n d\lambda_j, \quad (1.11)$$

which is the biorthogonal ensemble (1.6) with $f(\lambda) = e^\lambda$ and varying weight function

$$w(\lambda) = e^{-nV(\lambda)}. \quad (1.12)$$

Let $p_i^{(n)}$ and $q_i^{(n)}$ be monic biorthogonal polynomials for the model as defined in (1.8). Then, we have the specialization of (1.9)

$$K_n^{(n)}(x, y) = \sum_{j=0}^{n-1} \frac{1}{h_j^{(n)}} p_j^{(n)}(x) q_j^{(n)}(y) e^{-\frac{n}{2}(V(x)+V(y))}, \quad h_j^{(n)} = \int_{\mathbb{R}} p_j^{(n)}(x) q_j^{(n)}(e^x) e^{-nV(x)} dx, \quad (1.13)$$

and we recall that the correlation kernel of the model can be expressed as $g(x)K_n^{(n)}(x, y)g(y)^{-1}$ for any nonzero function $g(x)$.

Goal of the paper Recently, variations of the Deift-Zhou steepest-descent method applied to vector-valued Riemann-Hilbert problems have been successfully applied to the asymptotic analysis of biorthogonal polynomials p_j, q_j for some specific biorthogonal ensembles, like

- (i) $f(x) = e^{ax}$, $I = (-\infty, +\infty)$, and $w(x) = e^{-nV(x)}$ with V in a rather general class [21],
- (ii) $f(x) = x^\theta$ with $\theta \in (0, \infty)$, $I \subset (0, +\infty)$ compact, and $w(x)$ has n -independent Fisher-Hartwig singularities [18], and
- (iii) $f(x) = x^\theta$ with integer θ , $I = (0, +\infty)$, and $w(x) = x^\alpha e^{-nV(x)}$ with V in a rather general class [42].

These developments support our belief that the method will allow to asymptotically solve more general types of biorthogonal ensembles in the future.

For orthogonal polynomial ensembles defined by (1.7), the asymptotics of the orthogonal polynomials almost immediately yield the asymptotics of the correlation kernel $K_n(x, y) =$

$\sum_{i=0}^{n-1} \frac{1}{h_i} p_i(x)p_i(y) \sqrt{w(x)w(y)}$, and then the “universality” of the ensemble is derived. This is essentially due to the celebrated Christoffel-Darboux formula:

$$\sum_{i=0}^{n-1} \frac{1}{h_i} p_i(x)p_i(y) = \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{h_n}. \quad (1.14)$$

For many other determinantal point processes related to variations of orthogonal polynomials, like multiple orthogonal polynomials, the computation of the limit of the correlation kernel, which is equivalent to solving the universality of the process, is also achieved through an analogue of the Christoffel-Darboux type formula. However, a biorthogonal ensemble does not have a Christoffel-Darboux-like formula in general, so the asymptotics of biorthogonal polynomials do not directly entail the universality of the biorthogonal ensemble. For some determinantal point processes associated to classical (multiple) orthogonal polynomials, the correlation kernel has a double-contour integral formula, like our matrix model with quadratic $V(x)$ and the Muttalib-Borodin ensemble with linear $V(x)$. Of course, most biorthogonal ensembles, including our matrix model with non-quadratic $V(x)$, do not have a double-contour integral kernel formula.

Among all the “universal” correlation kernels occurring in 1-dimensional determinantal point processes, the sine kernel in the bulk and the Airy kernel at the (soft) edge are the simplest ones, and also the most common ones. It is natural to expect that they also occur in the model considered in our paper.

The first successful computation of the limit of the correlation kernel without Christoffel-Darboux type formula or double-contour integral formula in a biorthogonal ensemble was done in [42]. In [42], only the limiting kernel at the “hard” edge of the interval is computed, and it turns out that the hard edge case is technically simpler than the soft edge and bulk cases.

The main technical result of this paper is to prove the universality of the model introduced in (1.1)–(1.5), that is, the sine correlation kernel occurs in the bulk and the Airy correlation kernel occurs at the edges. All of our arguments are based on the asymptotics of $p_j^{(n)}$ and $q_j^{(n)}$ obtained in [21]. Besides confirming the expected universality results for the model, our purpose is more on the methodology: we showcase two methods through which the asymptotics of the correlation kernel, or equivalently, the universality of the model, can be derived from the asymptotics of the biorthogonal polynomials. We hope to convince the reader that the asymptotics of biorthogonal polynomials derived by the Deift-Zhou steepest-descent method for vector-valued Riemann-Hilbert problems are essentially enough for the universality of the biorthogonal ensembles, with the help of arguments analogous to those in our paper. We also expect that arguments similar to ours can be applied to more complicated universal limiting kernels for other biorthogonal ensembles.

Statement of results

Let μ be the unique equilibrium measure which minimizes

$$I(\mu) \equiv \frac{1}{2} \iint \log \frac{1}{|y-x|} d\mu(y)d\mu(x) + \frac{1}{2} \iint \log \frac{1}{|e^y - e^x|} d\mu(y)d\mu(x) + \int V(x)d\mu(x) \quad (1.15)$$

among all Borel probability measures μ supported on \mathbb{R} . If V is strongly convex, it is shown in [21] that μ is supported on a single interval $[a, b]$ and that it has a smooth density $\psi(x)$ there that vanishes like a square root function at the endpoints. To be precise,

$$d\mu(x) = \psi(x)dx, \quad x \in (x, y) \quad \text{with} \quad \begin{aligned} \psi(x) &\sim \alpha(x-a)^{1/2}, & x \searrow a, \\ \psi(x) &\sim \beta(b-x)^{1/2}, & x \nearrow b, \end{aligned} \quad (1.16)$$

and $\alpha, \beta > 0$. The density ψ and the constants α, β have explicit expressions which will be given in Section 2. The measure μ describes the macroscopic behavior of the eigenvalues in (1.11) as the dimension n of the random matrices tends to infinity.

Our result is the following.

Theorem 1.1. *Let V be strongly convex and real analytic on \mathbb{R} , and denote $[a, b]$ for the support of the equilibrium measure μ . Fix $x^* \in (a, b)$. The correlation kernel K_n defined in (1.13) has the following scaling limits:*

$$\lim_{n \rightarrow \infty} \frac{e^{\frac{F'(x^*)(\xi-\eta)}{\pi\psi(x^*)}}}{\pi\psi(x^*)n} K_n^{(n)} \left(x^* + \frac{\xi}{\pi\psi(x^*)n}, x^* + \frac{\eta}{\pi\psi(x^*)n} \right) = \frac{\sin \pi(\xi - \eta)}{\pi(\xi - \eta)}, \quad (1.17)$$

$$\lim_{n \rightarrow \infty} \frac{e^{n(F(u_n) - F(v_n))}}{(\pi\beta n)^{2/3}} K_n^{(n)} \left(b + \frac{\xi}{(\pi\beta n)^{2/3}}, b + \frac{\eta}{(\pi\beta n)^{2/3}} \right) = \frac{\text{Ai}(\xi)\text{Ai}'(\eta) - \text{Ai}'(\xi)\text{Ai}(\eta)}{\xi - \eta}, \quad (1.18)$$

$$\lim_{n \rightarrow \infty} \frac{e^{n(F(\hat{u}_n) - F(\hat{v}_n))}}{(\pi\alpha n)^{2/3}} K_n^{(n)} \left(a - \frac{\xi}{(\pi\alpha n)^{2/3}}, a - \frac{\eta}{(\pi\alpha n)^{2/3}} \right) = \frac{\text{Ai}(\xi)\text{Ai}'(\eta) - \text{Ai}'(\xi)\text{Ai}(\eta)}{\xi - \eta}. \quad (1.19)$$

Here the density function ψ , the constants α and β are defined by (1.16), the variables u_n, v_n and \hat{u}_n, \hat{v}_n depend on ξ, η as

$$u_n = b + \frac{\xi}{(\pi\beta n)^{2/3}}, \quad v_n = b + \frac{\eta}{(\pi\beta n)^{2/3}}, \quad \hat{u}_n = a - \frac{\xi}{(\pi\alpha n)^{2/3}}, \quad \hat{v}_n = a - \frac{\eta}{(\pi\alpha n)^{2/3}}, \quad (1.20)$$

and the function F is defined as

$$F(x) = \frac{1}{2} \int \log \left| \frac{e^x - e^y}{x - y} \right| d\mu(y). \quad (1.21)$$

Outline and heuristics of the proofs

In Section 2, we will collect general results about the equilibrium measure μ and about the biorthogonal polynomials $p_j^{(n)}$ and $q_j^{(n)}$. Most of those results are taken from [21], or are consequences of results in that paper. In Section 3, we will prove the Airy kernel limit (1.18), and in Section 4, we will prove the sine kernel limit (1.17).

In the Airy case, the well-known identity

$$\int_0^\infty \text{Ai}(\xi + y)\text{Ai}(\eta + y)dy = \frac{\text{Ai}(\xi)\text{Ai}'(\eta) - \text{Ai}'(\xi)\text{Ai}(\eta)}{\xi - \eta}, \quad (1.22)$$

which should be seen as a continuous analogue of the Christoffel-Darboux formula for orthogonal polynomials, is of crucial importance. Instead of proving convergence of the correlation kernel to the right hand side of (1.22), we directly prove that the sum in (1.13) converges to the integral on the left hand side of (1.22). This method is similar to the one from [42] for the hard edge case.

For the sine kernel, we will write the correlation kernel as a double sum which can be thought of as an analogue of a Christoffel-Darboux formula. Although no exact cancellations take place in this double sum, it will turn out that only few terms contribute to its large n limit, such that, at least asymptotically, it is reminiscent of the right hand side of (1.14).

2 Equilibrium measure and asymptotics for biorthogonal polynomials

2.1 Equilibrium measure

In this section, we collect some properties about the equilibrium measure μ_t defined as the unique Borel probability measure on \mathbb{R} minimizing

$$I(\mu) \equiv \frac{1}{2} \iint \log \frac{1}{|y-x|} d\mu(y) d\mu(x) + \frac{1}{2} \iint \log \frac{1}{|e^y - e^x|} d\mu(y) d\mu(x) + \frac{1}{t} \int V(x) d\mu(x). \quad (2.1)$$

For $t = 1$, $\mu_1 = \mu$, the equilibrium measure in (1.15) and (1.16). If V is strongly convex, then so is $\frac{1}{t}V(x)$, and this means that we can use the results obtained in [21] about the equilibrium measure for any $t > 0$. Those results can be summarized as follows.

Set

$$\mathbf{J}_{c_1, c_0}(s) = c_1 s + c_0 - \log \frac{s - \frac{1}{2}}{s + \frac{1}{2}}, \quad \mathbb{C} \setminus \left[-\frac{1}{2}, \frac{1}{2} \right], \quad (2.2)$$

with $c_1 > 0$, $c_0 \in \mathbb{R}$, and the logarithm corresponding to arguments between $-\pi$ and π . For any $c_1 > 0$, $c_0 \in \mathbb{R}$, $\mathbf{J}_{c_1, c_0}(s)$ has two critical points $s^-(c_1) < s^+(c_1)$ on the real line, given by (cf. [21, (1.24)])

$$s^\pm(c_1) = \pm \sqrt{\frac{1}{4} + \frac{1}{c_1}}. \quad (2.3)$$

The inverse image of the interval $[\mathbf{J}_{c_1, c_0}(s^-(c_1)), \mathbf{J}_{c_1, c_0}(s^+(c_1))]$ under \mathbf{J} consists of two complex conjugate curves $\gamma_1 = \gamma_1(c_1)$, $\gamma_2 = \gamma_2(c_1)$. We write $\gamma = \gamma(c_1)$ for the counterclockwise union of γ_1 and γ_2 .

For any $t > 0$, there are $c_0(t) \in \mathbb{R}$ and $c_1(t) > 0$ uniquely determined by (see [21, Lemma 2])

$$1 = \frac{c_1(t)}{2\pi i t} \int_{\gamma(c_1(t))} V'(\mathbf{J}_{c_1(t), c_0(t)}(s)) ds, \quad (2.4)$$

$$1 = \frac{1}{2\pi i t} \int_{\gamma(c_1(t))} \frac{V'(\mathbf{J}_{c_1(t), c_0(t)}(s))}{s - \frac{1}{2}} ds. \quad (2.5)$$

In the subsequent part of the paper, we write $\mathbf{J}(s) := \mathbf{J}_{c_1(t), c_0(t)}(s)$, $\gamma := \gamma(c_1(t))$ when there is no possible confusion, hence suppressing the dependence on t in our notations.

The equilibrium measure μ_t is supported on a single interval $[a(t), b(t)]$, and the endpoints are given by

$$a(t) = \mathbf{J}(s_a(t)), \quad b(t) = \mathbf{J}(s_b(t)), \quad s_a(t) := s^-(c_1(t)), \quad s_b(t) := s^+(c_1(t)), \quad (2.6)$$

or equivalently in terms of $c_0(t), c_1(t)$ by (cf. [21, (1.21)–(1.22)])

$$c_0(t) = \frac{b(t) + a(t)}{2}, \quad (2.7)$$

$$c_1(t) s_b(t) - \log \frac{s_b(t) - \frac{1}{2}}{s_b(t) + \frac{1}{2}} = \frac{b(t) - a(t)}{2}. \quad (2.8)$$

By taking derivatives on both sides of (2.7) and (2.8), we obtain

$$a'(t) = c_0'(t) - c_1'(t) s_b(t), \quad b'(t) = c_0'(t) + c_1'(t) s_b(t). \quad (2.9)$$

Let us denote \mathbf{I}_+ (resp. \mathbf{I}_-) for the inverse of \mathbf{J} which maps $[a(t), b(t)]$ to $\gamma_1 \subset \mathbb{C}^+$ (resp. $\gamma_2 \subset \mathbb{C}^-$). The equilibrium measure μ_t is supported on $[a(t), b(t)]$ with a continuous density ψ_t that is given by (see [21, Theorem 3])

$$\psi_t(x) = \frac{1}{2\pi^2} \int_{a(t)}^{b(t)} V''(u) \log \left| \frac{\mathbf{I}_+(u) - \mathbf{I}_-(x)}{\mathbf{I}_+(u) - \mathbf{I}_+(x)} \right| du. \quad (2.10)$$

It follows after a straightforward calculation that

$$\psi_t(x) = \alpha(t)(x - a(t))^{1/2}(1 + \mathcal{O}(x - a(t))), \quad x \searrow a(t), \quad (2.11)$$

$$\psi_t(x) = \beta(t)(b(t) - x)^{1/2}(1 + \mathcal{O}(x - b(t))), \quad x \nearrow b(t), \quad (2.12)$$

with $\alpha(t)$ and $\beta(t)$ given by

$$\alpha(t) = -\frac{1}{\pi^2 c_1(t) |s_a(t)|^{1/2}} \int_{a(t)}^{b(t)} V''(y) \operatorname{Im} \frac{1}{\mathbf{I}_+(y) - s_a(t)} dy > 0, \quad (2.13)$$

$$\beta(t) = -\frac{1}{\pi^2 c_1(t) |s_b(t)|^{1/2}} \int_{a(t)}^{b(t)} V''(y) \operatorname{Im} \frac{1}{\mathbf{I}_+(y) - s_b(t)} dy > 0. \quad (2.14)$$

After the change of variables $y = \mathbf{J}(s)$, noting that $s_a(t) = -s_b(t)$ and that

$$\mathbf{J}'(s) = c_1(t)(s \pm s_b(t)) \left(\frac{\frac{1}{2} \mp s_b(t)}{s - \frac{1}{2}} + \frac{\frac{1}{2} \pm s_b(t)}{s + \frac{1}{2}} \right),$$

and introducing the integrals

$$P_t = \frac{1}{2\pi i} \int_{\gamma} \frac{V''(\mathbf{J}_{c_1(t), c_0(t)}(s))}{s - \frac{1}{2}} ds, \quad Q_t = \frac{1}{2\pi i} \int_{\gamma} \frac{V''(\mathbf{J}_{c_1(t), c_0(t)}(s))}{s + \frac{1}{2}} ds, \quad (2.15)$$

we can rewrite $\alpha(t), \beta(t)$ as

$$\begin{aligned} \alpha(t) &= \frac{1}{\pi s_b(t)^{1/2}} \left(\left(\frac{1}{2} - s_b(t) \right) P_t + \left(\frac{1}{2} + s_b(t) \right) Q_t \right) \\ &= \frac{\frac{1}{2} - s_b(t)}{\pi s_b(t)^{1/2}} \left(P_t - \left(1 + \frac{c_1(t)}{2} + c_1(t) s_b(t) \right) Q_t \right), \end{aligned} \quad (2.16)$$

$$\begin{aligned} \beta(t) &= \frac{1}{\pi s_b(t)^{1/2}} \left(\left(\frac{1}{2} + s_b(t) \right) P_t + \left(\frac{1}{2} - s_b(t) \right) Q_t \right) \\ &= \frac{\frac{1}{2} + s_b(t)}{\pi s_b(t)^{1/2}} \left(P_t - \left(1 + \frac{c_1(t)}{2} - c_1(t) s_b(t) \right) Q_t \right). \end{aligned} \quad (2.17)$$

We first establish the monotonicity of the endpoints $a(t), b(t)$ as a function of t .

Lemma 2.1. *For any $t > 0$, the equilibrium measure μ_t is supported on the interval $[a(t), b(t)]$. Moreover, $a(t)$ and $b(t)$ depend smoothly on t and*

$$a'(t) = \frac{\frac{1}{2} - s_b(t)}{\pi \alpha(t) s_b(t)^{1/2}} < 0, \quad b'(t) = \frac{\frac{1}{2} + s_b(t)}{\pi \beta(t) s_b(t)^{1/2}} > 0. \quad (2.18)$$

Proof. The fact that the support of μ_t consists of a single interval for strongly convex V was proved in [21]. Let the mapping $F : (u, v) \rightarrow (U, V)$ be defined by

$$(U, V) = F(u, v) = \left(\frac{u}{2\pi i} \int_{\gamma} V'(\mathbf{J}_{u,v}(s)) ds, \frac{1}{2\pi i} \int_{\gamma} \frac{V'(\mathbf{J}_{u,v}(s))}{s - \frac{1}{2}} ds \right). \quad (2.19)$$

The Jacobian matrix is given by

$$\frac{\partial(U, V)}{\partial(u, v)} = \begin{bmatrix} \frac{1}{2\pi i} \int_{\gamma} [V'(\mathbf{J}_{u,v}(s)) + V''(\mathbf{J}_{u,v}(s))s] ds & \frac{1}{2\pi i} \int_{\gamma} u V''(\mathbf{J}_{u,v}(s)) ds \\ \frac{1}{2\pi i} \int_{\gamma} V''(\mathbf{J}_{u,v}(s)) \frac{s}{s - \frac{1}{2}} ds & \frac{1}{2\pi i} \int_{\gamma} V''(\mathbf{J}_{u,v}(s)) \frac{1}{s - \frac{1}{2}} ds \end{bmatrix}. \quad (2.20)$$

Noting that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} [V'(\mathbf{J}_{u,v}(s)) + V''(\mathbf{J}_{u,v}(s))s] ds - \frac{1}{2\pi i} \int_{\gamma} \frac{1}{2} V''(\mathbf{J}_{u,v}(s)) \left(\frac{1}{s - \frac{1}{2}} + \frac{1}{s + \frac{1}{2}} \right) ds \\ = \frac{1}{2\pi i} \int_{\gamma} \frac{d}{ds} [V'(\mathbf{J}_{u,v}(s))s] ds = 0, \end{aligned} \quad (2.21)$$

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} V''(\mathbf{J}_{u,v}(s)) ds + \frac{1}{2\pi i} \int_{\gamma} \frac{1}{u} V''(\mathbf{J}_{u,v}(s)) \left(\frac{1}{s + \frac{1}{2}} - \frac{1}{s - \frac{1}{2}} \right) ds \\ = \frac{1}{2\pi i} \int_{\gamma} \frac{d}{ds} [u^{-1} V'(\mathbf{J}_{u,v}(s))] ds = 0, \end{aligned} \quad (2.22)$$

we have

$$\left. \frac{\partial(U, V)}{\partial(u, v)} \right|_{u=c_1(t), v=c_0(t)} = \begin{bmatrix} \frac{1}{2}(P_t + Q_t) & P_t - Q_t \\ \frac{1}{c_1(t)}(P_t - Q_t) + \frac{1}{2}P_t & P_t \end{bmatrix}, \quad (2.23)$$

and, with the help of expressions (2.16) and (2.17) for $\alpha(t)$ and $\beta(t)$, we have

$$\det \left. \frac{\partial(U, V)}{\partial(u, v)} \right|_{u=c_1(t), v=c_0(t)} = P_t Q_t - \frac{1}{c_1(t)} (P_t - Q_t)^2 = \pi s_b(t) \alpha(t) \beta(t) > 0. \quad (2.24)$$

By (2.4) and (2.5), $c_1(t)$ and $c_0(t)$ are defined by the relation $F(c_1(t), c_0(t)) = (t, t)$, and they are smooth functions of t as the Jacobian of F is non-singular at $(c_1(t), c_0(t))$. Moreover, taking derivatives with respect to t in $F(c_1(t), c_0(t)) = (t, t)$, by (2.19), we have

$$c'_1(t) = \frac{Q_t}{P_t Q_t - \frac{1}{c_1(t)} (P_t - Q_t)^2}, \quad c'_0(t) = \frac{\frac{Q_t}{2} - \frac{1}{c_1(t)} (P_t - Q_t)}{P_t Q_t - \frac{1}{c_1(t)} (P_t - Q_t)^2}. \quad (2.25)$$

At last, by (2.9), we have

$$a'(t) = \frac{Q_t \left(\frac{1}{2} - s_b(t) + \frac{1}{c_1(t)} \right) - \frac{1}{c_1(t)} P_t}{P_t Q_t - \frac{1}{c_1(t)} (P_t - Q_t)^2} = \frac{1}{P_t - (1 + \frac{c_1(t)}{2} + c_1(t) s_b(t)) Q_t}, \quad (2.26)$$

$$b'(t) = \frac{Q_t \left(\frac{1}{2} + s_b(t) + \frac{1}{c_1(t)} \right) - \frac{1}{c_1(t)} P_t}{P_t Q_t - \frac{1}{c_1(t)} (P_t - Q_t)^2} = \frac{1}{P_t - (1 + \frac{c_1(t)}{2} - c_1(t) s_b(t)) Q_t}. \quad (2.27)$$

Recalling the expressions (2.16) and (2.17) for $\alpha(t), \beta(t)$, we obtain (2.18). \square

The next lemma is about the position of the endpoints $a(t), b(t)$ of the support of the equilibrium measure μ_t . Since we assume the external field V is strongly convex, it has a unique absolute minimum that we denote as x_{\min} . The external field $V(x) - tx$ has also a unique minimum, which we denote as $\widehat{x}_{\min}(t)$. Observe that $\widehat{x}_{\min}(t)$ is an increasing function of t , that $\widehat{x}_{\min}(t) > x_{\min}$ for $t > 0$, and that $\lim_{t \rightarrow 0} \widehat{x}_{\min}(t) = x_{\min}$.

Lemma 2.2. (a) For any $t > 0$, $x_{\min} < b(t)$, $a(t) < \widehat{x}_{\min}(t)$.

(b) $\lim_{t \searrow 0} a(t) = \lim_{t \searrow 0} b(t) = x_{\min}$.

Proof. (a) Assume first that $b(t) \leq x_{\min}$. Consider the logarithmic energy defined in (2.1) and assume that μ_t is its minimizer supported on $[a(t), b(t)]$. We define the measure μ_t^s by its shifted version $d\mu_t^s(x) = d\mu_t(x - s)$ that is supported on $[a(t) + s, b(t) + s]$. If we substitute μ_t^s into μ in (2.1), then the energy $I(\mu_t^s)$ is a function of s . Since $\mu_t = \mu_t^0$ is the minimizer, it is necessary that $\frac{d}{ds} I(\mu_t^s)|_{s=0} = 0$. However, we observe that the first term on the right side of (2.1) is unchanged as s varies; the second term on the right side of (2.1) decreases as s increases, and the last term there also decreases since $V'(x) < 0$ for $x \in [a(t), b(t)]$. We thus have $\frac{d}{ds} I(\mu_t^s)|_{s=0} < 0$, which is in contradiction with the fact that μ minimizes $I(\mu)$. It follows that $x_{\min} < b(t)$.

If we replace $V(x)$ by $V(-x) + tx$ in (2.1), it is straightforward to verify that the minimizer μ_t with $d\mu_t(x) = \psi_t(x)dx$ supported on $[a(t), b(t)]$ is replaced by μ_t^r with $d\mu_t^r(x) = \psi_t(-x)dx$ supported on $[-b(t), -a(t)]$. It follows from the argument above that $-a(t)$ is bigger than the minimum of $V(-x) + tx$, which is $-\widehat{x}_{\min}(t)$, and then $a(t) < \widehat{x}_{\min}(t)$.

(b) By (2.18), we have that as $t \searrow 0$, $b(t) - a(t) \searrow l^0 \geq 0$, and then by (2.8) $c_1(t) \searrow c_1^0 \geq 0$. If $c_1^0 > 0$, then we can see that (2.4) cannot hold if t is small enough. Hence, as $t \searrow 0$, $a(t)$ and $b(t)$ converge to the same point. Combining this with the results in part (a) and using the fact that $\lim_{t \rightarrow 0} \widehat{x}_{\min}(t) = x_{\min}$, we obtain the result. \square

2.2 Zeros and asymptotics of biorthogonal polynomials

Detailed uniform in $z \in \mathbb{C}$ large n asymptotics for the biorthogonal polynomials $p_{n+k}^{(n,V)}(z)$ and $q_{n+k}^{(n,V)}(z)$ have been obtained in [21] for strongly convex V . In our setting, these asymptotics thus also apply to the external field $\frac{1}{t}V$. Write $t = j/n$. In order to describe the asymptotics, uniform in t , for the polynomials $p_j^{(j,V/t)}(z) = p_j^{(n,V)}(z)$ and $q_j^{(j,V/t)}(z) = q_j^{(n,V)}(z)$ corresponding to the orthogonality weight $e^{-nV(x)} = e^{-\frac{j}{t}V(x)}$, we need to introduce some notations.

Define (cf. [21, (1.33)])

$$\mathbf{g}_t(z) \equiv \int \log(z - s) d\mu_t(s), \quad (2.28)$$

$$\tilde{\mathbf{g}}_t(z) \equiv \int \log(e^z - e^s) d\mu_t(s), \quad (2.29)$$

for $z \in \mathbb{C} \setminus (-\infty, b(t)]$, with the logarithms taking the principal value, such that

$$\operatorname{Re} \mathbf{g}_{t,+}(x) = \operatorname{Re} \mathbf{g}_{t,-}(x), \quad \operatorname{Re} \tilde{\mathbf{g}}_{t,+}(x) = \operatorname{Re} \tilde{\mathbf{g}}_{t,-}(x), \quad x \in (-\infty, b(t)), \quad (2.30)$$

$$-\operatorname{Im} \mathbf{g}_{t,+}(x) = \operatorname{Im} \mathbf{g}_{t,-}(x) = -\operatorname{Im} \tilde{\mathbf{g}}_{t,+}(x) = \operatorname{Im} \tilde{\mathbf{g}}_{t,-}(x) = \begin{cases} \pi \int_x^{b(t)} d\mu_t(s), & x \in (a(t), b(t)), \\ \pi, & x \in (-\infty, a(t)]. \end{cases} \quad (2.31)$$

Thus the function

$$F_t(z) := \frac{t}{2}(\tilde{\mathbf{g}}_t(z) - \mathbf{g}_t(z)), \quad z \in \mathbb{S} = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \pi\} \quad (2.32)$$

is an analytic function on \mathbb{S} . We note that $F_1(x)$ for $x \in \mathbb{R}$ agrees with $F(x)$ in (1.21). We have the Euler-Lagrange variational conditions (cf. [21, (1.19) and (1.21)])

$$\mathbf{g}_{t,\pm}(x) + \tilde{\mathbf{g}}_{t,\mp}(x) - \frac{1}{t}V(x) - \ell_t = 0, \quad x \in [a(t), b(t)], \quad (2.33)$$

$$\mathbf{g}_{t,\pm}(x) + \tilde{\mathbf{g}}_{t,\mp}(x) - \frac{1}{t}V(x) - \ell_t < 0, \quad x \in \mathbb{R} \setminus [a(t), b(t)], \quad (2.34)$$

where ℓ_t is defined such that for any $y \in [a(t), b(t)]$,

$$\ell_t = \int \log|x - y|d\mu_t(x) + \int \log|e^x - e^y|d\mu_t(x) - \frac{V(y)}{t}. \quad (2.35)$$

In terms of $c_0(t)$, $c_1(t)$ defined by (2.4) and (2.5), we further define

$$G_{t,k}(s) := c_1(t)^k \frac{(s + \frac{1}{2})(s - \frac{1}{2})^k}{\sqrt{s^2 - \frac{1}{4} - \frac{1}{c_1(t)}}}, \quad \hat{G}_{t,k}(s) := i \frac{e^{k(\frac{c_1(t)}{2} + c_0(t))}}{\sqrt{c_1(t)}} \frac{(s - \frac{1}{2})^{-k}}{\sqrt{s^2 - \frac{1}{4} - \frac{1}{c_1(t)}}}, \quad (2.36)$$

where the square root $\sqrt{s^2 - 1/4 - 1/c_1(t)}$ has its branch cut along γ_1 in $G_{t,k}(s)$, along γ_2 in $\hat{G}_{t,k}(s)$, and $\sqrt{s^2 - 1/4 - 1/c_1(t)} \sim s$ as $s \rightarrow \infty$ in both cases. Then we define

$$r_{t,k}(x) := 2|G_{t,k}(\mathbf{I}_+(x))|, \quad \theta_{t,k}(x) := \arg(G_{t,k}(\mathbf{I}_+(x))), \quad (2.37)$$

$$\hat{r}_{t,k}(x) := 2|\hat{G}_{t,k}(\mathbf{I}_-(x))|, \quad \hat{\theta}_{t,k}(x) := \arg(\hat{G}_{t,k}(\mathbf{I}_-(x))). \quad (2.38)$$

Below, we summarize the asymptotic results for $p_j^{(n)}(x)$ and $q_j^{(n)}(e^x)$ that are needed in this paper. We give the reference of each result in [21]. To simplify the notations, we denote

$$\tilde{p}_j^{(n)}(x) = e^{-\frac{j}{2}\ell_j/n} e^{-\frac{nV(x)}{2}} p_j^{(n)}(x), \quad \tilde{q}_j^{(n)}(x) = \frac{e^{\frac{j}{2}\ell_j/n}}{h_j^{(n)}} e^{-\frac{nV(x)}{2}} q_j^{(n)}(e^x). \quad (2.39)$$

Lemma 2.3. (a) ([21, Theorem 3(e)] *Asymptotics for the norming constants*) As $n \rightarrow \infty$, we have

$$h_j^{(n)} = 2\pi c_1(t)^{1/2} e^{j\ell_t} (1 + \mathcal{O}(n^{-1})), \quad (2.40)$$

uniformly for $t = j/n$ in any compact $K \subset (0, +\infty)$.

(b) ([21, Theorem 3(a)] *Outer asymptotics on the real line*) For any $\epsilon > 0$ and for any compact $K \subset (0, +\infty)$, there exist $n_0 \in \mathbb{N}$ and a compact $U_\epsilon \subset \mathbb{R} \setminus \{0\}$ such that for any $n \geq n_0$, $t = j/n \in K$ and $x \in \mathbb{R} \setminus [a(t) - \epsilon, b(t) + \epsilon]$, we have

$$\frac{e^{nF_t(x)} \tilde{p}_j^{(n)}(x)}{e^{\frac{j}{2}(\tilde{\mathbf{g}}_t(x) + \mathbf{g}_t(x) - \frac{n}{j}V(x) - \ell_t)}} \in U_\epsilon, \quad \frac{e^{-nF_t(x)} \tilde{q}_j^{(n)}(x)}{e^{\frac{j}{2}(\tilde{\mathbf{g}}_t(x) + \mathbf{g}_t(x) - \frac{n}{j}V(x) - \ell_t)}} \in U_\epsilon. \quad (2.41)$$

(c) ([21, Theorem 3(c) and (d)] *Microscopic scaling limit at the edge*) Let $x = b(t) + u/(\pi\beta(t)j)^{2/3}$ and $y = a(t) - v/(\pi\alpha(t)j)^{2/3}$. For any compacts $K \subset (0, +\infty)$ and $U \subset \mathbb{R}$, we have as

$n \rightarrow \infty$ that

$$\frac{1}{(\pi\beta(t)j)^{1/6}} e^{nF_t(x)} \tilde{p}_j^{(n)}(x) = \frac{\sqrt{2\pi}}{c_1(t)^{\frac{1}{2}} (s_b(t) - \frac{1}{2}) s_b(t)^{1/4}} \text{Ai}(u) + \mathcal{O}(n^{-\frac{1}{3}}), \quad (2.42)$$

$$\frac{1}{(\pi\beta(t)j)^{1/6}} e^{-nF_t(x)} \tilde{q}_j^{(n)}(x) = \frac{1}{\sqrt{2\pi} c_1(t)^{\frac{1}{2}} s_b(t)^{1/4}} \text{Ai}(u) + \mathcal{O}(n^{-\frac{1}{3}}), \quad (2.43)$$

$$\frac{1}{(\pi\alpha(\frac{j}{n})j)^{1/6}} e^{nF_t(y)} \tilde{p}_j^{(n)}(y) = \frac{(-1)^j \sqrt{2\pi}}{c_1(t)^{\frac{1}{2}} (s_b(t) + \frac{1}{2}) s_b(t)^{1/4}} \text{Ai}(v) + \mathcal{O}(n^{-\frac{1}{3}}), \quad (2.44)$$

$$\frac{1}{(\pi\alpha(t)j)^{1/6}} e^{-nF_t(y)} \tilde{q}_j^{(n)}(y) = \frac{(-1)^j}{\sqrt{2\pi} c_1(t)^{\frac{1}{2}} s_b(t)^{1/4}} \text{Ai}(v) + \mathcal{O}(n^{-\frac{1}{3}}), \quad (2.45)$$

uniformly in $t = j/n \in K$ and $u, v \in U$.

(d) ([21, Theorem 3(c) and (d)] Upper bound at the edge) For any compact $K \subset (0, +\infty)$, there exist constants $\epsilon, c > 0$ such that

$$e^{nF_t(x)} \tilde{p}_j^{(n)}(x) = \mathcal{O}(n^{\frac{1}{6}} e^{-cn^{2/3}(x-b(t))}), \quad e^{-nF_t(x)} \tilde{q}_j^{(n)}(x) = \mathcal{O}(n^{\frac{1}{6}} e^{-cn^{2/3}(x-b(t))}), \quad (2.46)$$

$$e^{nF_t(y)} \tilde{p}_j^{(n)}(y) = \mathcal{O}(n^{\frac{1}{6}} e^{-cn^{2/3}(a(t)-y)}), \quad e^{-nF_t(y)} \tilde{q}_j^{(n)}(y) = \mathcal{O}(n^{\frac{1}{6}} e^{-cn^{2/3}(a(t)-y)}), \quad (2.47)$$

as $n \rightarrow \infty$, uniformly in $t = j/n \in K$ and in $x \in [b(t) - \epsilon, b(t) + \epsilon], y \in [a(t) - \epsilon, a(t) + \epsilon]$. These are not sharp estimates, particularly not for $x < b(t), y > a(t)$.

(e) ([21, Theorem 3(c) and (d)] Edge asymptotics upper bound, improvement for $x < b(t)$ and $y > a(t)$) For any compact $K \subset (0, +\infty)$, there exist constants $\epsilon, c > 0$ such that

$$e^{nF_t(x)} \tilde{p}_j^{(n)}(x) = \mathcal{O}((b(t) - x + n^{-\frac{2}{3}})^{-\frac{1}{4}}), \quad e^{-nF_t(x)} \tilde{q}_j^{(n)}(x) = \mathcal{O}((b(t) - x + n^{-\frac{2}{3}})^{-\frac{1}{4}}), \quad (2.48)$$

$$e^{nF_t(y)} \tilde{p}_j^{(n)}(y) = \mathcal{O}((y - a(t) + n^{-\frac{2}{3}})^{-\frac{1}{4}}), \quad e^{-nF_t(y)} \tilde{q}_j^{(n)}(y) = \mathcal{O}((y - a(t) + n^{-\frac{2}{3}})^{-\frac{1}{4}}), \quad (2.49)$$

as $n \rightarrow \infty$, uniformly in $t = j/n \in K$ and in $x \in [b(t) - \epsilon, b(t)]$ and $y \in [a(t), a(t) + \epsilon]$. These are not sharp estimates.

(f) ([21, Theorem 3(b)] Bulk asymptotics) Let $x = x^* + u/(\pi\varphi_t(x^*)n)$. For any $\epsilon > 0$ and for any compacts $K \subset (0, +\infty), U \subset \mathbb{C}$, we have uniformly in $x^* \in (a(t) + \epsilon, b(t) - \epsilon), u \in U$, and $t = j/N \in K$ that

$$e^{nF_t(x)} \tilde{p}_{j+k}^{(n)}(x) = r_{t,k}(x^*) \left[\cos \left(n\pi \int_{x^*}^{b(t)} d\mu_t(\xi) + \theta_{t,k}(x^*) - u \right) + \mathcal{O}(n^{-1}) \right], \quad (2.50)$$

$$e^{-nF_t(x)} \tilde{q}_{j+k}^{(n)}(x) = \frac{\hat{r}_{t,k}(x^*)}{\sqrt{2\pi} c_1(t)^{k+\frac{1}{2}} e^{k(\frac{c_1(t)}{2} + c_0)}} \left[\cos \left(n\pi \int_{x^*}^{b(t)} d\mu_t(\xi) + \hat{\theta}_{t,k}(x^*) - u \right) + \mathcal{O}(n^{-1}) \right], \quad (2.51)$$

as $n \rightarrow \infty$.

(g) ([21, Theorem 3(a)] Another outer asymptotics) Let $k \in \mathbb{Z}$ be fixed. For any compact $U \subset \mathbb{C} \setminus [a(1), b(1)]$, we have (with $c_1 = c_1(1)$, $\mathbf{g} = \mathbf{g}_1$, $G_k = G_{1,k}$, $\tilde{G}_{-k} = \tilde{G}_{1,-k}$, $\ell = \ell_t$, and for all quantities depending implicitly on t , it is assumed that $t = 1$)

$$p_{n+k}^{(n)}(z) = G_k(\mathbf{I}_1(z))e^{n\mathbf{g}(z)}(1 + \mathcal{O}(n^{-1})) = c_1^k \frac{(s + \frac{1}{2})(s - \frac{1}{2})^k}{\sqrt{s^2 - \frac{1}{4} - \frac{1}{c_1}}} e^{n\mathbf{g}(z)}(1 + \mathcal{O}(n^{-1})), \quad (2.52)$$

$$\begin{aligned} Cq_{n-k}^{(n)}(z) &= \tilde{G}_{-k}(\mathbf{I}_1(z))e^{-n\mathbf{g}(z)+n\ell}(1 + \mathcal{O}(n^{-1})) \\ &= \frac{i}{\sqrt{c_1}} \frac{e^{-k(\frac{c_1}{2}+c_0)}(s - \frac{1}{2})^k}{\sqrt{s^2 - \frac{1}{4} - \frac{1}{c_1}}} e^{-n\mathbf{g}(z)+n\ell}(1 + \mathcal{O}(n^{-1})), \end{aligned} \quad (2.53)$$

$$h_{n-k}^{(n)} = 2\pi c_1^{-k+\frac{1}{2}} e^{-j(\frac{c_1}{2}+c_0)} e^{n\ell}(1 + \mathcal{O}(n^{-1})), \quad (2.54)$$

as $n \rightarrow \infty$, uniformly in $z \in U$, where we denote $s = \mathbf{I}_1(z)$, and \mathbf{I}_1 is an inverse function of $\mathbf{J}_{c_1(1), c_0(1)}$ as defined in [21, (1.26)]. Here $Cq_j^{(n)}$ is the Cauchy transform of $q_j^{(n)}$

$$Cq_j^{(n)}(z) = \frac{-1}{2\pi iz} \int_{-\infty}^{\infty} \frac{q_j^{(n)}(e^s)}{1 - s/z} e^{-nV(s)} ds. \quad (2.55)$$

Lemma 2.4. For any $\epsilon > 0$, there is a $\delta > 0$ such that for all n large enough, the zeros of $p_j^{(n)}(x)$ and $q_j^{(n)}(e^x)$ lie in the interval $(x_{\min} - \epsilon, x_{\min} + \epsilon)$ for all $j \leq \delta n$.

Proof. By part (b) of Lemma 2.2, for any $\epsilon > 0$, there exists $\delta > 0$ such that for $t \in (0, \delta)$, the support of μ_t is strictly included in $(x_{\min} - \epsilon/2, x_{\min} + \epsilon/2)$. From the asymptotics of the biorthogonal polynomials $p_k^{(n)}$ and $q_k^{(n)}$ in Lemma 2.3, especially part (b), it follows that for n, k large and k/n sufficiently close to δ , all the zeros of $p_k^{(n)}(x)$ and of $q_k^{(n)}(e^x)$ lie in $[x_{\min} - \epsilon, x_{\min} + \epsilon]$. Moreover, like the zeros of orthogonal polynomials, the zeros of $p_k^{(n)}(x)$ interlace, as well as those of $q_k^{(n)}(e^x)$, which means that the zeros of $p_j^{(n)}(x)$ and of $q_j^{(n)}(e^x)$ lie in $[x_{\min} - \epsilon, x_{\min} + \epsilon]$ for any $j \leq k$. The interlacing property follows from the fact that the biorthogonal polynomials form an AT system [40, Section 4.4]. \square

Next we give rough bounds for the biorthogonal polynomials with small degrees and the norming constants, obtained by elementary estimates based on the zeros of the polynomials.

Lemma 2.5. Let $\epsilon > 0$, and let $\delta > 0$ be associated to ϵ as in Lemma 2.4. Suppose n is large and $0 \leq j < \delta n$.

(a) (Upper bound for multiple orthogonal polynomials of lower degrees)

$$|p_j^{(n)}(x)| \leq \begin{cases} (x - (x_{\min} - \epsilon))^j, & x \geq x_{\min} + \epsilon, \\ ((x_{\min} + \epsilon) - x)^j, & x \leq x_{\min} - \epsilon, \\ (2\epsilon)^j, & x \in [x_{\min} - \epsilon, x_{\min} + \epsilon], \end{cases} \quad (2.56)$$

$$|q_j^{(n)}(x)| \leq \begin{cases} (e^x - e^{x_{\min} - \epsilon})^j, & x \geq x_{\min} + \epsilon, \\ (e^{x_{\min} + \epsilon} - e^x)^j, & x \leq x_{\min} - \epsilon, \\ (e^{x_{\min} + \epsilon} - e^{x_{\min} - \epsilon})^j, & x \in [x_{\min} - \epsilon, x_{\min} + \epsilon]. \end{cases} \quad (2.57)$$

(b) (Lower bound for the norming constants of lower degrees)

$$h_j^{(n)} \geq e^{-n(V(x_{\min})+C(\epsilon))}, \quad (2.58)$$

where $C(\epsilon) > 0$ depends on V but not on j, n , and $C(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof. Part (a) is a direct consequence of Lemma 2.4. To prove part (b), we note that by the orthogonality (1.13),

$$h_j^{(n)} = \int_{\mathbb{R}} p(x) q_j^{(n)}(e^x) e^{-nV(x)} dx, \quad (2.59)$$

for any monic polynomial $p(x)$ of degree j . Assume that the zeros of $q_j^{(n)}(e^x)$ are $x_1, x_2, \dots, x_j \in (x_{\min} - \epsilon, x_{\min} + \epsilon)$. We let $p(x) = \prod_{i=1}^j (x - x_i)$, such that the integrand in (2.59) is positive. We then have

$$\begin{aligned} h_j^{(n)} &= \int_{\mathbb{R}} \left(\prod_{i=1}^j (x - x_i) \right) \left(\prod_{i=1}^j (e^x - e^{x_i}) \right) e^{-nV(x)} dx \\ &> \int_{x_{\min} + 2\epsilon}^{x_{\min} + 3\epsilon} (x - (x_{\min} + \epsilon))^j (e^x - e^{x_{\min} + \epsilon})^j e^{-nV(x)} dx, \end{aligned} \quad (2.60)$$

Noting that $e^{-nV(x)}$ attains its minimum on the interval $[x_{\min} + 2\epsilon, x_{\min} + 3\epsilon]$ at $x_{\min} + 3\epsilon$ due to the strong convexity, we obtain

$$h_j^{(n)} > \epsilon^j (e^{x_{\min} + 2\epsilon} - e^{x_{\min} + \epsilon})^j e^{-nV(x_{\min} + 3\epsilon)}.$$

The desired result now follows by a standard estimate of the right-hand side in the above equation. \square

2.3 Useful corollaries

First, let F_t be defined by (2.32). Since $F_t(z)$ is analytic with respect to z on \mathbb{S} , the derivative $\frac{\partial}{\partial z} F_t(z)$ exists and is continuous there. On the other hand, for $z \in \mathbb{S} \setminus [a(t), b(t)]$, by the definition of $F_t(z)$ and the formula (2.10) for the density of μ_t , we know that $\frac{\partial}{\partial t} F_t(z)$ also exists and is continuous there. Furthermore, if $w \in [a(t), b(t)]$ or in the vicinity of this interval, we have

$$\frac{\partial}{\partial t} F_t(w) = \frac{1}{2\pi i} \oint \frac{\partial}{\partial t} F_t(z) \frac{dz}{z - w}, \quad (2.61)$$

where the contour integral is over a closed contour within \mathbb{S} that encloses $[a(t), b(t)]$. In this way, we have that the derivative $\frac{\partial}{\partial t} F_t(z)$ exists and is continuous on \mathbb{S} . Therefore, we have the following result:

Corollary 2.6. *Let $K_1 \subset \mathbb{S}$ and $K_2 \subset (0, \infty)$ be two compact sets. Then there exists $C_0 > 0$ such that for all $u, v \in K_1$ and $s, t \in K_2$, we have the estimates*

$$|F_t(u) - F_t(v)| < C_0 |u - v|, \quad |F_s(u) - F_t(u)| < C_0 |s - t| \quad (2.62)$$

and

$$|F_s(u) - F_s(v) - F_t(u) + F_t(v)| < C_0 |u - v| |s - t|. \quad (2.63)$$

Next, we state a corollary of Lemmas 2.3 and 2.5. We use the same notation as in these lemmas.

Corollary 2.7. *Let $C_2 > 0$ be a large enough constant. For any $M > 0$, there exist constants C_1, C_3 (depending on C_2 but not on n) such that the following inequalities hold for n sufficiently large and $0 \leq k/n < M$.*

(a) We have

$$\int_{\mathbb{R} \setminus [-C_2, C_2]} |\tilde{p}_k^{(n)}(x)|^2 (e^x + e^{-x}) dx < e^{-C_1 n}, \quad \int_{\mathbb{R} \setminus [-C_2, C_2]} |\tilde{q}_k^{(n)}(x)|^2 (e^x + e^{-x}) dx < e^{-C_1 n}. \quad (2.64)$$

(b) For any $z = x + iy$ such that $x \in \mathbb{R}$ and $y \in (-n^{-1}C_2, n^{-1}C_2)$, with $F(z) = F_1(z)$ defined in (1.21) and (2.32),

$$|e^{nF(z)} \tilde{p}_k^{(n)}(z)| < e^{\frac{1}{3}C_1 n}, \quad |e^{-nF(z)} \tilde{q}_k^{(n)}(z)| < e^{\frac{1}{3}C_1 n}. \quad (2.65)$$

(c) We have

$$\int_{[-C_2, C_2]} |\tilde{p}_k^{(n)}(x)|^2 dx < e^{C_3 n}, \quad \int_{[-C_2, C_2]} |\tilde{q}_k^{(n)}(x)|^2 dx < e^{C_3 n}. \quad (2.66)$$

(d) For any $\delta > 0$, $\delta < k/n < M$, there exists a constant $C_4 > 0$ depending on C_2 such that

$$\int_{[-C_2, C_2]} |e^{nF_{k/n}(x)} \tilde{p}_k^{(n)}(x)|^2 dx < C_4, \quad \int_{[-C_2, C_2]} |e^{-nF_{k/n}(x)} \tilde{q}_k^{(n)}(x)|^2 dx < C_4. \quad (2.67)$$

Proof. The estimates in parts (a) to (c) are consequences of Lemma 2.3 for $\delta n \leq k < Mn$, and of Lemma 2.5 for $0 \leq k < \delta n$, with $\delta > 0$ sufficiently small. They are all straightforward to prove. Estimate (d) follows from the bulk asymptotics and the two upper bounds for the edge asymptotics in Lemma 2.3. \square

3 Airy kernel

In this section we prove the Airy kernel limit (1.18) near the right endpoint b . The Airy kernel limit (1.19) near the left endpoint can be proved in parallel, or alternatively we can use the symmetry of the model with respect to $V(x) \mapsto V(-x) + \frac{n-1}{n}x$ to map the left edge to the right. We omit the details.

In the rest of the section, we will need the auxiliary quantities $a(t)$, $b(t)$, $c_1(t)$, $c_0(t)$, $\alpha(t)$, $\beta(t)$ related to the equilibrium measure μ_t for $0 < t \leq 1$. Whenever we write a , b , c_1 , c_0 , α , β without indicating the t -dependence, we refer to their values $a(1)$, $b(1)$, $c_1(1)$, $c_0(1)$, $\alpha(1)$, $\beta(1)$ for $t = 1$. This notation is consistent with the one used in the introduction to state the results. We also remind the reader that $F(u)$ defined in (1.21) is equal to the $t = 1$ case of $F_t(u)$ defined in (2.32).

In this section we denote

$$u = b + \frac{\xi}{(\pi\beta n)^{2/3}}, \quad v = b + \frac{\eta}{(\pi\beta n)^{2/3}}, \quad (3.1)$$

where ξ and η are in a compact subset $U \subset \mathbb{R}$.

In order to study large n asymptotics, we split the correlation kernel $K_n(u, v)$ defined in (1.13) in 4 parts:

$$K_n = K_n^{(1)} + K_n^{(2)} + K_n^{(3)} + K_n^{(4)}, \quad (3.2)$$

where, with \tilde{p}_j and \tilde{q}_j defined in (2.39),

$$\begin{aligned}
K_n^{(1)}(u, v) &= \sum_{j=0}^{\lfloor \delta n \rfloor} \tilde{p}_j^{(n)}(u) \tilde{q}_j^{(n)}(v), & K_n^{(2)}(u, v) &= \sum_{j=\lfloor \delta n \rfloor + 1}^{\lfloor (1-\delta')n \rfloor} \tilde{p}_j^{(n)}(u) \tilde{q}_j^{(n)}(v), \\
K_n^{(3)}(u, v) &= \sum_{j=\lfloor (1-\delta')n \rfloor + 1}^{\lfloor (1-Mn^{-2/3})n \rfloor} \tilde{p}_j^{(n)}(u) \tilde{q}_j^{(n)}(v), & K_n^{(4)}(u, v) &= \sum_{j=\lfloor (1-Mn^{-2/3})n \rfloor + 1}^{n-1} \tilde{p}_j^{(n)}(u) \tilde{q}_j^{(n)}(v).
\end{aligned} \tag{3.3}$$

$$\tag{3.4}$$

Here we choose δ and δ' to be sufficiently small positive numbers whose values will be fixed in the proof of Lemma 3.1. M can be any sufficiently large number independent of n . The proof of (1.18) is based on the following technical lemmas, which we will prove later in this section.

Lemma 3.1. *There exists a constant $c > 0$ such that for n large enough*

$$|e^{nF(u)} K_n^{(1)}(u, v) e^{-nF(v)}| < e^{-cn}, \tag{3.5}$$

and

$$|e^{nF(u)} K_n^{(2)}(u, v) e^{-nF(v)}| < e^{-cn}. \tag{3.6}$$

Lemma 3.2. *There exist constants $c', c'' > 0$ independent of M, n such that for M, n large enough,*

$$\left| e^{nF(u)} K_n^{(3)}(u, v) e^{-nF(v)} \right| \leq c' n^{2/3} e^{-c''M}. \tag{3.7}$$

Lemma 3.3. *For any $M > 0$,*

$$\lim_{n \rightarrow \infty} \frac{1}{(\pi\beta n)^{2/3}} e^{nF(u)} K_n^{(4)}(u, v) e^{-nF(v)} = \int_0^{(\pi\beta)^{2/3} b'(1)M} \text{Ai}(\xi + y) \text{Ai}(\eta + y) dy. \tag{3.8}$$

The Airy limit (1.18) follows from the lemmas above. By the identity (1.22), we have that for any $\epsilon > 0$, if M is large enough, then

$$\left| \frac{\text{Ai}(\xi) \text{Ai}'(\eta) - \text{Ai}'(\xi) \text{Ai}(\eta)}{\xi - \eta} - \int_0^{(\pi\beta)^{2/3} b'(1)M} \text{Ai}(\xi + y) \text{Ai}(\eta + y) dy \right| < \frac{\epsilon}{3}. \tag{3.9}$$

By Lemma 3.2, if M is large enough, then for all large enough n ,

$$\left| \frac{1}{(\pi\beta n)^{2/3}} e^{nF(u)} K_n^{(3)}(u, v) e^{-nF(v)} \right| \leq \frac{\epsilon}{3}. \tag{3.10}$$

At last by Lemmas 3.1 and 3.3, for n large enough,

$$\left| \frac{1}{(\pi\beta n)^{2/3}} e^{nF(u)} \left(K_n^{(1)}(u, v) + K_n^{(2)}(u, v) \right) e^{-nF(v)} \right| < \frac{\epsilon}{3}. \tag{3.11}$$

The limit identity (3.8), the inequalities (3.9), (3.10), (3.11) together with the arbitrariness of ϵ yield (1.18).

Proof of Lemma 3.1. First we prove (3.5). Assume that $\epsilon > 0$ is small enough such that $b > x_{\min} + 2\epsilon$ and $V(x_{\min}) + C(\epsilon) < V(b - \epsilon)$, where $C(\epsilon)$ is the constant in Lemma 2.5(b). Then we take $\tilde{\delta}$ to be a small enough positive constant such that $a(\tilde{\delta})$ and $b(\tilde{\delta})$ are inside $(x_{\min} - \epsilon, x_{\min} + \epsilon)$.

It is straightforward to check that we can choose $\delta < \tilde{\delta}$ positive and small enough such that we have the following inequalities for all $t \in [0, \delta]$ and $x, y \in [b - \epsilon, b + \epsilon]$:

$$\begin{aligned} & \frac{e^{-\frac{1}{2}V(x)}e^{-\frac{1}{2}V(y)}}{e^{-(V(x_{\min})+C(\epsilon))}}(x - (x_{\min} - \epsilon))^t(e^y - e^{x_{\min}-\epsilon})^t \\ & \leq \frac{e^{-V(b-\epsilon)}}{e^{-(V(x_{\min})+C(\epsilon))}}((b - \epsilon) - (x_{\min} - \epsilon))^\delta(e^{b-\epsilon} - e^{x_{\min}-\epsilon})^\delta \\ & < e^{-C'}, \end{aligned} \quad (3.12)$$

where C' is a positive constant. Using the estimates (2.56), (2.57) and (2.58), we have that (3.12) implies that for all $j < \delta n$,

$$|\tilde{p}_j^{(n)}(u)\tilde{q}_j^{(n)}(v)| = |(h_j^{(n)})^{-1}p_j^{(n)}(u)q_j^{(n)}(v)| < e^{-C'n}. \quad (3.13)$$

Finally using the estimate (2.62) for the term $e^{n(F(u)-F(v))}$ and the fact that $|u - v| \leq C''n^{-2/3}$, we obtain

$$|\tilde{p}_j^{(n)}(u)\tilde{q}_j^{(n)}(v)e^{n(F(u)-F(v))}| < e^{-(C' - \frac{C''}{n^{2/3}}C_0)n}. \quad (3.14)$$

Taking the sum for j from 0 to $\lfloor \delta n \rfloor$, the inequality (3.5) follows with any $c < C'$ for n sufficiently large.

Next we prove (3.6). For $\delta < j/n \leq 1 - \delta'$, there exists $\epsilon' > 0$ such that both u and v are greater than $b(j/n) + \epsilon'$, if n is large enough. We can thus use the outer asymptotics in Lemma 2.3(b). We obtain, uniformly in $t = j/n$,

$$|e^{n(F_t(u)-F_t(v))}\tilde{p}_j^{(n)}(u)\tilde{q}_j^{(n)}(v)| = \mathcal{O}\left(e^{\frac{j}{2}(\tilde{\mathbf{g}}_t(u)+\mathbf{g}_t(u)-\frac{1}{t}V(u)-\ell_t)}e^{\frac{j}{2}(\tilde{\mathbf{g}}_t(v)+\mathbf{g}_t(v)-\frac{1}{t}V(v)-\ell_t)}\right), \quad (3.15)$$

as $j, n \rightarrow \infty$. Using the inequality (2.34), we conclude that for all large enough n and $\delta n < j \leq (1 - \delta')n$, there is $C''' > 0$ such that

$$|e^{n(F_t(u)-F_t(v))}\tilde{p}_j^{(n)}(u)\tilde{p}_j^{(n)}(v)| < e^{-C'''n}. \quad (3.16)$$

Finally, by (2.63), for n large enough,

$$e^{n(F_1(u)-F_1(v))-n(F_{j/n}(u)-F_{j/n}(v))} < e^{nC_0(1-\frac{j}{n})\frac{|\xi-\eta|}{(\pi\beta n)^{2/3}}} < e^{\frac{C'''}{2}n}, \quad (3.17)$$

where C_0 is the same as in (2.63), and then

$$\begin{aligned} & |e^{nF(u)}K_n^{(2)}(u, v)e^{-nF(v)}| \\ & \leq \sum_{j=\lfloor \delta n \rfloor+1}^{\lfloor (1-\delta')n \rfloor} e^{n(F_1(u)-F_1(v))-n(F_{j/n}(u)-F_{j/n}(v))}|e^{n(F_{j/n}(u)-F_{j/n}(v))}\tilde{p}_j^{(n)}(u)\tilde{q}_j^{(n)}(v)| \\ & < ne^{-\frac{C'''}{2}n}. \end{aligned} \quad (3.18)$$

Thus we prove (3.6) for any $c < C'''/2$ if n is sufficiently large. \square

Proof of Lemma 3.2. For all $t = j/n \in [1 - \delta', 1]$, we use the upper bound at the edge in Lemma 2.3(d). Together with (2.63), we obtain

$$\begin{aligned} & |e^{nF(u)}\tilde{p}_j(u)\tilde{q}_j(v)e^{-nF(v)}| = e^{n(F_1(u)-F_1(v)-F_t(u)+F_t(v))}|e^{nF_t(u)}\tilde{p}_j(u)\tilde{q}_j(v)e^{-nF_t(v)}| \\ & \leq C'n^{1/3} \exp\left(\frac{C_0n^{\frac{1}{3}}}{(\pi\beta)^{\frac{2}{3}}}|\xi - \eta||1 - t|\right) e^{-cn^{2/3}(u-b(j/n))} e^{-cn^{2/3}(v-b(j/n))}. \end{aligned} \quad (3.19)$$

where C_0 is the constant in (2.63) and c is the constant in (2.46). By Lemma 2.1, $b(t)$ is differentiable at $t = 1$ with $b'(1) > 0$, and this implies the existence of a constant $C'' < 0$ such that

$$|b(1) - b(t)| \geq C''(1 - t) \quad (3.20)$$

for $t \in [1 - \delta', 1]$, where we assume that δ' is small enough. Hence we have the inequality

$$\begin{aligned} & |e^{nF(u)} \tilde{p}_j(u) \tilde{q}_j(v) e^{-nF(v)}| \\ & \leq C' n^{1/3} \exp\left(\frac{C_0 n^{1/3}}{(\pi\beta)^{2/3}} |\xi - \eta| \left|1 - \frac{j}{n}\right|\right) e^{-c\left(C'' n^{2/3} (1 - \frac{j}{n}) + \frac{\xi}{(\pi\beta)^{2/3}}\right)} e^{-c\left(C'' n^{2/3} (1 - \frac{j}{n}) + \frac{\eta}{(\pi\beta)^{2/3}}\right)}. \end{aligned} \quad (3.21)$$

Taking the sum for all j from $\lfloor (1 - \epsilon_2)n + 1 \rfloor$ to $\lfloor (1 - Mn^{-2/3})n \rfloor$, we prove the lemma with c', c'' in (3.7) determined by c, C_0, C', C'' in (3.21). \square

Proof of Lemma 3.3. By Taylor expansion, for all $j/n \in [1 - Mn^{-2/3}, 1]$, $n \rightarrow \infty$,

$$b(j/n) = b - b'(1) \frac{n-j}{n} + \mathcal{O}(n^{-4/3}), \quad (3.22)$$

$$\beta(j/n) = \beta + \mathcal{O}(n^{-2/3}), \quad c_1(j/n) = c_1 + \mathcal{O}(n^{-2/3}), \quad s_b(j/n) = s_b + \mathcal{O}(n^{-2/3}). \quad (3.23)$$

The expressions (3.1) for u and v can be rewritten as

$$u = b(j/n) + \frac{\xi + \frac{n-j}{n} b'(1) (\pi\beta)^{2/3}}{(\pi\beta j)^{2/3}} + \mathcal{O}(n^{-4/3}), \quad v = b(j/n) + \frac{\eta + \frac{n-j}{n} b'(1) (\pi\beta)^{2/3}}{(\pi\beta j)^{2/3}} + \mathcal{O}(n^{-4/3}). \quad (3.24)$$

By Lemma 2.3(c), we have

$$\begin{aligned} & e^{nF(u)} \tilde{p}_j(u) \tilde{q}_j(v) e^{-nF(v)} \\ & = e^{n(F_1(u) - F_1(v)) - n(F_{j/n}(u) - F_{j/n}(v))} \left(e^{nF_{j/n}(u)} \tilde{p}_j(u) \right) \left(e^{-nF_{j/n}(v)} \tilde{q}_j(v) \right) \\ & = e^{n(F_1(u) - F_1(v)) - n(F_{j/n}(u) - F_{j/n}(v))} \frac{(\pi\beta(\frac{j}{n})j)^{1/3}}{c_1(\frac{j}{n})(s_b(\frac{j}{n}) - \frac{1}{2})s_b(\frac{j}{n})^{1/2}} \\ & \quad \times \text{Ai}\left(\xi + \frac{n-j}{n^{1/3}} b'(1) (\pi\beta)^{2/3} + \mathcal{O}(n^{-2/3})\right) \text{Ai}\left(\eta + \frac{n-j}{n^{1/3}} b'(1) (\pi\beta)^{2/3} + \mathcal{O}(n^{-2/3})\right) (1 + \mathcal{O}(n^{-1/3})). \end{aligned}$$

By (2.63), we have $e^{n(F_1(u) - F_1(v)) - n(F_{j/n}(u) - F_{j/n}(v))} = 1 + \mathcal{O}(n^{-1/3})$, and this implies

$$\begin{aligned} & e^{nF(u)} \tilde{p}_j(u) \tilde{q}_j(v) e^{-nF(v)} \\ & = \frac{(\pi\beta n)^{1/3}}{c_1(s_b - \frac{1}{2})s_b^{1/2}} \text{Ai}\left(\xi + \frac{n-j}{n^{1/3}} b'(1) (\pi\beta)^{2/3}\right) \text{Ai}\left(\eta + \frac{n-j}{n^{1/3}} b'(1) (\pi\beta)^{2/3}\right) (1 + \mathcal{O}(n^{-1/3})). \end{aligned} \quad (3.25)$$

Taking the sum of the left-hand side of (3.25) with respect to j from $\lfloor (1 - Mn^{-2/3})n \rfloor + 1$ to $n - 1$ and replacing the sum by an integral with small correction, we have that

$$\begin{aligned} & e^{nF(u)} K_n^{(4)}(u, v) e^{-nF(v)} = \\ & \frac{n^{2/3}}{c_1(s_b - \frac{1}{2})s_b^{1/2} (\pi\beta)^{1/3} b'(1)} \int_0^{(\pi\beta)^{2/3} b'(1) M} \text{Ai}(u+y) \text{Ai}(v+y) dy (1 + \mathcal{O}(n^{-1/3})). \end{aligned} \quad (3.26)$$

Using expression (2.18) for $b'(1)$ in terms of s_b and (2.3) and (2.6) to obtain an expression for s_b in terms of c_1 , we obtain the result. \square

4 Sine kernel

4.1 Preliminaries

In this section we denote

$$u = x^* + \frac{\xi}{\pi\psi(x^*)n}, \quad v = x^* + \frac{\eta}{\pi\psi(x^*)n}, \quad (4.1)$$

where $x^* \in (a, b)$, and ξ, η are in a compact subset $U \subset \mathbb{C}$. Although the kernel function $K_n^{(n)}(u, v)$ has a probabilistic meaning only if u, v are real, for technical reasons to be explained below, we need to consider complex ξ and η .

The goal of this section is to compute $e^{nF(u)}K_n^{(n)}(u, v)e^{-nF(v)}$ to prove (1.17). Instead of computing it directly, we consider

$$\begin{aligned} & (\exp_{[\delta n]}(u) - \exp(v))e^{nF(u)}K_n^{(n)}(u, v)e^{-nF(v)} \\ &= e^{n(F(u)-F(v))}(\exp_{[\delta n]}(u) - \exp(v)) \sum_{k=0}^{n-1} \tilde{p}_k^{(n)}(u)\tilde{q}_k^{(n)}(v), \end{aligned} \quad (4.2)$$

where δ is a small enough positive number and \exp_k is the truncated exponential sum

$$\exp_k(x) = \sum_{i=0}^k \frac{x^i}{i!}. \quad (4.3)$$

We need several estimates of $\exp_k(z)$. The first ones are very rudimentary: For $k, l \in \mathbb{Z}_+$ and $z = x + iy$,

$$|\exp_k(z)| \leq e^x + e^{-x}, \quad |\exp(z) - \exp_k(z)| \leq e^x + e^{-x}, \quad |\exp_k(z) - \exp_l(z)| \leq e^x + e^{-x}. \quad (4.4)$$

The next one can be shown using Stirling's formula. For any $C_2 > 0$, there exists a constant C_5 , depending only on C_2 , such that for all $k < l \in \mathbb{Z}_+$

$$\max_{|z| < C_2} |\exp(z) - \exp_k(z)| < C_5 e^{-\frac{k}{2} \log(k)}, \quad \max_{|z| < C_2} |\exp_l(z) - \exp_k(z)| < C_5 e^{-\frac{k}{2} \log(k)}. \quad (4.5)$$

Later in Section 4.2, we will prove that for ξ, η in a compact $K \subset \mathbb{C}$, uniformly

$$\lim_{n \rightarrow \infty} (\exp_{[\delta n]}(u) - \exp(v))e^{nF(u)}K_n^{(n)}(u, v)e^{-nF(v)} = \frac{e^{x^*}}{\pi} \sin \pi(\xi - \eta). \quad (4.6)$$

With the help of (4.5), when $\xi \neq \eta$, this is equivalent to

$$\lim_{n \rightarrow \infty} \frac{e^{n(F(u)-F(v))}}{\pi\psi(x^*)n} K_n^{(n)}(u, v) = \lim_{n \rightarrow \infty} \frac{\sin \pi(\xi - \eta)}{\pi(e^{u-x^*} - e^{v-x^*})\pi\psi(x^*)n}, \quad (4.7)$$

which is, by Taylor approximations of $F(u)$, $F(v)$, e^{u-x^*} , e^{v-x^*} at x^* , equivalent to the sine kernel limit (1.17). Hence the $\xi \neq \eta$ part of (1.17) is proved. To prove the $\xi = \eta$ case of (1.17), we note that functions on both sides of (4.6) are analytic in ξ and η . With ξ fixed, for all η on a small circle centred at ξ , the limit identity (4.7) holds uniformly, so it extends to $\eta = \xi$. By this argument we complete the proof of (1.17).

4.2 Proof of (4.6)

We can expand $e^y \tilde{q}_j^{(n)}(y)$ as a linear combination of $\tilde{q}_0^{(n)}, \tilde{q}_1^{(n)}, \dots$,

$$\exp(y) \tilde{q}_j^{(n)}(y) = \sum_{k=0}^{j+1} a_{j,k} \tilde{q}_k^{(n)}(y), \quad a_{j,k} = \int \tilde{p}_k^{(n)}(x) \tilde{q}_j^{(n)}(x) e^x dx. \quad (4.8)$$

On the other hand, we have

$$\exp_{[\delta n]}(x) \tilde{p}_j^{(n)}(x) = \sum_{k=0}^{j+[\delta n]} b_{j,k} \tilde{p}_k^{(n)}(x), \quad b_{j,k} = \int \tilde{p}_j^{(n)}(x) \tilde{q}_k^{(n)}(x) \exp_{[\delta n]}(x) dx. \quad (4.9)$$

Since the left hand side is a polynomial of degree $j + [\delta n]$, we see that $b_{j,k} = 0$ if $k > j + [\delta n]$; similarly $a_{j,k} = 0$ for $k > j + 1$.

To compute the right-hand side of (4.2), we write

$$(\exp_{[\delta n]}(u) - \exp(v)) \sum_{k=0}^{n-1} \tilde{p}_k^{(n)}(u) \tilde{q}_k^{(n)}(v) = J_1(u, v) + J_2(u, v) - a_{n-1, n} \tilde{p}_{n-1}^{(n)}(u) \tilde{q}_n^{(n)}(v), \quad (4.10)$$

where

$$J_1(u, v) = \sum_{j,k=0}^{n-1} (b_{k,j} - a_{j,k}) \tilde{p}_j^{(n)}(u) \tilde{q}_k^{(n)}(v), \quad J_2(u, v) = \sum_{j=n-[\delta n]}^{n-1} \sum_{k=n}^{j+[\delta n]} b_{j,k} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v). \quad (4.11)$$

Substituting (4.11) in (4.10), we get a double sum at the right hand side of (4.10). This should be seen as the analogue of the Christoffel-Darboux formula (1.14) valid in the case where $f(\lambda) = \lambda$. In that case, the counterpart of the prefactor $(\exp_{[\delta n]}(u) - \exp(v))$ would be $(u - v)$, and only two terms survive on the right, by (1.14). In our situation where $f(\lambda) = e^\lambda$, no exact cancellations take place on the right, but fortunately, it will turn out that many terms in the double sum on the right will be small as $n \rightarrow \infty$. Indeed, we will show that J_1 tends to 0 as $n \rightarrow \infty$, and that the terms in J_2 for which $j - k$ is large and the ones for which $n - j$ or $k - n$ is large, have only a small contribution. The main contribution will come from the terms near $j = n - 1$ and $k = n - 1$ in the (j, k) -plane.

Lemma 4.1. *Let J_1 be defined by (4.11), with $a_{j,k}, b_{j,k}$ defined by (4.8) and (4.9), and with u, v as in (4.1). We have*

$$\lim_{n \rightarrow \infty} e^{n(F(u) - F(v))} J_1(u, v) = 0. \quad (4.12)$$

Lemma 4.2. *Let J_2 be defined by (4.11), with $a_{j,k}, b_{j,k}$ defined by (4.8) and (4.9), and with u, v as in (4.1). For any $\epsilon > 0$, there exist M_0, n_0 such that for $M \geq M_0, n \geq n_0$,*

$$\left| e^{n(F(u) - F(v))} \left(J_2(u, v) - \sum_{j=n-M}^{n-1} \sum_{k=n}^{j+M} a_{k,j} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v) \right) \right| < \epsilon. \quad (4.13)$$

Lemma 4.3. *Let $a_{n+j, n-k}$ be defined by (4.8). As $n \rightarrow \infty$, we have*

$$a_{n+j, n-k} = \begin{cases} \alpha_{j+k} + \mathcal{O}(n^{-1}), & j+k \geq -1, \\ 0, & j+k \leq -2, \end{cases} \quad \text{where} \quad \alpha_l = \begin{cases} \left(\frac{c_1}{(1+l)!} + \frac{1}{l} \right) e^{\frac{1}{2}c_1 + c_0}, & l \geq 0, \\ c_1 e^{\frac{1}{2}c_1 + c_0}, & l = -1. \end{cases} \quad (4.14)$$

Here $c_0 = c_0(1)$, $c_1 = c_1(1)$ are defined by (2.7) and (2.8).

Corollary 4.4. *Let $a_{k,j}$ be defined by (4.8). For any $\epsilon > 0$, there exist M_0, n_0 such that for $M \geq M_0, n \geq n_0$, we have*

$$\left| e^{n(F(u)-F(v))} \left(\sum_{j=n-M}^{n-1} \sum_{k=n}^{j+M} a_{k,j} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v) - a_{n-1,n} \tilde{p}_{n-1}^{(n)}(u) \tilde{q}_n^{(n)}(v) \right) - \frac{e^{x^*}}{\pi} \sin \pi(\xi - \eta) \right| < \epsilon. \quad (4.15)$$

Summing up (4.12), (4.13), and (4.15), we obtain that (4.2) holds. Hence it suffices to prove Lemma 4.1, Lemma 4.2, and Corollary 4.4, as well as Lemma 4.3 on which Corollary 4.4 relies. We note that in the proofs below, C_2 is a large enough positive constant, and other constant terms are defined where they are used.

Proof of Lemma 4.1. Using (4.8), (4.9), and the orthogonality of $\tilde{p}_j^{(n)}$ and $\tilde{q}_j^{(n)}$, we obtain for any $0 \leq j, k \leq n$,

$$|b_{k,j} - a_{j,k}| = \left| \int_{\mathbb{R}} (\exp(x) - \exp_{[\delta n]}(x)) \tilde{p}_k^{(n)}(x) \tilde{q}_j^{(n)}(x) dx \right|. \quad (4.16)$$

Using (4.4), the Cauchy-Schwarz inequality, and (2.64), we obtain

$$\begin{aligned} I_1 &:= \left| \int_{\mathbb{R} \setminus [-C_2, C_2]} (\exp(x) - \exp_{[\delta n]}(x)) \tilde{p}_k^{(n)}(x) \tilde{q}_j^{(n)}(x) dx \right| \\ &\leq \left| \int_{\mathbb{R} \setminus [-C_2, C_2]} (e^x + e^{-x}) \tilde{p}_k^{(n)}(x) \tilde{q}_j^{(n)}(x) dx \right| \\ &\leq e^{-C_1 n}, \end{aligned} \quad (4.17)$$

and by (4.5), the Cauchy-Schwarz inequality, and (2.66),

$$\begin{aligned} I_2 &:= \left| \int_{[-C_2, C_2]} (\exp(x) - \exp_{[\delta n]}(x)) \tilde{p}_k^{(n)}(x) \tilde{q}_j^{(n)}(x) dx \right| \\ &\leq C_5 e^{-\frac{[\delta n]}{2} \log([\delta n])} \left| \int_{[-C_2, C_2]} \tilde{p}_k^{(n)}(x) \tilde{q}_j^{(n)}(x) dx \right| \\ &\leq C_5 e^{-\frac{\delta n}{2} \log(\delta n) + C_3 n}. \end{aligned} \quad (4.18)$$

Using the estimate (2.65) for $e^{n(F(u)-F(v))} \tilde{p}_j^{(n)}(u) \tilde{q}_k^{(n)}(v)$, we have that for each pair j, k ,

$$\begin{aligned} \left| e^{n(F(u)-F(v))} (b_{k,j} - a_{j,k}) \tilde{p}_j^{(n)}(u) \tilde{q}_k^{(n)}(v) \right| &= (I_1 + I_2) \left| e^{nF(u)} \tilde{p}_j^{(n)}(u) e^{-nF(v)} \tilde{q}_k^{(n)}(v) \right| \\ &< \left(e^{-C_1 n} + C_5 e^{-\frac{\delta n}{2} \log(\delta n) + C_3 n} \right) e^{\frac{2C_1}{3} n}, \end{aligned} \quad (4.19)$$

and then finish the proof by

$$|e^{n(F(u)-F(v))} J_1(u, v)| < n^2 (e^{-C_1 n} + C_5 e^{-\frac{\delta n}{2} \log(\delta n) + C_3 n}) e^{\frac{2C_1}{3} n} = o(1), \quad \text{as } n \rightarrow \infty. \quad (4.20)$$

□

Proof of Lemma 4.2. When we consider J_2 defined in (4.11), the indices j, k satisfy $n - [\delta n] \leq j \leq n - 1$ and $n \leq k \leq j + [\delta n]$. If $m < k - j$, then by the biorthogonality,

$$\int_{\mathbb{R}} \exp_m(x) \tilde{p}_j^{(n)}(x) \tilde{q}_k^{(n)}(x) dx = 0. \quad (4.21)$$

Hence we have

$$|b_{j,k}| = \left| \int_{\mathbb{R}} [\exp_{[\delta n]}(x) - \exp_{k-j-1}(x)] \tilde{p}_j^{(n)}(x) \tilde{q}_k^{(n)}(x) dx \right| \leq I_3 + I_4, \quad (4.22)$$

where

$$I_3 := \left| \int_{\mathbb{R} \setminus [-C_2, C_2]} [\exp_{[\delta n]}(x) - \exp_{k-j-1}(x)] \tilde{p}_j^{(n)}(x) \tilde{q}_k^{(n)}(x) dx \right|, \quad (4.23)$$

$$I_4 := \left| \int_{[-C_2, C_2]} [\exp_{[\delta n]}(x) - \exp_{k-j-1}(x)] \tilde{p}_k^{(n)}(x) \tilde{q}_j^{(n)}(x) dx \right|. \quad (4.24)$$

Similar to (4.17), we have $I_3 < e^{-C_1 n}$. On the other hand,

$$I_4 = \left| \int_{[-C_2, C_2]} [\exp_{[\delta n]}(x) - \exp_{k-j-1}(x)] e^{-n(F_{k/n}(x) - F_{j/n}(x))} \times e^{nF_{k/n}(x)} \tilde{p}_k^{(n)}(x) e^{-nF_{j/n}(x)} \tilde{q}_j^{(n)}(x) dx \right|. \quad (4.25)$$

Using the estimate (4.5) to estimate $\exp_{[\delta n]}(x) - \exp_{k-j-1}(x)$, and then using (2.62) to estimate $F_{k/n}(x) - F_{j/n}(x)$ and (2.67) to estimate $e^{nF_{k/n}(x)} \tilde{p}_k(x) e^{-nF_{j/n}(x)} \tilde{q}_j(x)$, we have

$$I_4 \leq C_4 C_5 e^{-\frac{k-j-1}{2} \log(k-j-1) + C_0(k-j)}. \quad (4.26)$$

It follows that

$$|b_{j,k}| \leq e^{-C_1 n} + C_4 C_5 e^{-\frac{k-j-1}{2} \log(k-j-1) + C_0(k-j)}. \quad (4.27)$$

Now we estimate $e^{n(F(u) - F(v))} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v)$. Since in the setting of Lemma 4.2, $|j - n|, |k - n| < [\delta n]$ and δ small, we have that x^* lies in $(a(j/n), b(j/n))$ and $(a(k/n), b(k/n))$, where $[a(t), b(t)]$ is the support of the equilibrium measure μ_t , and it follows from the bulk asymptotics in Lemma 2.3(f) that $e^{nF_{k/n}(u)} \tilde{p}_k^{(n)}(u)$ and $e^{-nF_{j/n}(v)} \tilde{q}_j^{(n)}(v)$ are bounded in n . Combining this with (2.62), there is a constant C' such that

$$\begin{aligned} e^{n(F(u) - F(v))} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v) &= e^{n(F_1(u) - F_{k/n}(u))} e^{-n(F_1(v) - F_{j/n}(v))} e^{nF_{k/n}(u)} \tilde{p}_k^{(n)}(u) e^{-nF_{j/n}(v)} \tilde{q}_j^{(n)}(v) \\ &\leq e^{C_0(k-n)} e^{C_0(n-j)} C' \\ &= C' e^{C_0(k-j)}. \end{aligned} \quad (4.28)$$

Combining (4.27) with the estimate (4.28) for $e^{n(F(u) - F(v))} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v)$, we have that if n is large enough, then the value of $|b_{j,k} e^{n(F(u) - F(v))} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v)|$

- decays exponentially fast as $k - j$ increases, as long as $k - j \leq \sqrt{n}$, and
- is bounded by $e^{-\sqrt{n}}$ for $k - j \in (\sqrt{n}, \delta n]$, given that δ is small enough.

Thus, in the double sum defining J_2 in (4.11), the terms for $k - j$ large will give a small contribution in $e^{n(F(u) - F(v))} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v)$: for any $\epsilon_1 > 0$, we have that

$$\left| e^{n(F(u) - F(v))} \left(J_2(u, v) - \sum_{j=n-M}^{n-1} \sum_{k=n}^{j+M} b_{j,k} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v) \right) \right| < \epsilon_1, \quad (4.29)$$

if M and n are large enough.

We note that inequality (4.19) also holds when k, j are around n . Hence for any finite M , there exists $\epsilon > 0$, such that

$$\left| e^{n(F(u)-F(v))} \sum_{j=n-M}^{n-1} \sum_{k=n}^{j+M} (b_{j,k} - a_{k,j}) \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v) \right| = \mathcal{O}(e^{-\epsilon n}), \quad n \rightarrow \infty. \quad (4.30)$$

This, together with (4.29), completes the proof. \square

Proof of Lemma 4.3. When $j + k \leq -2$ or $(n - k + 1) < (n + j)$, the result follows from the biorthogonality. Below we prove the other cases. Let $C_2 > 0$ be a sufficiently large constant such that the support of the equilibrium measure μ is contained in the interval $(-C_2, C_2)$. Using the exponential decay of the polynomials $\tilde{p}_j^{(n)}$ and $\tilde{q}_j^{(n)}$, see (2.64), it is straightforward to approximate $a_{n+j, n-k}$ for large n as follows,

$$\begin{aligned} a_{n+j, n-k} &= \int_{-\infty}^{\infty} \tilde{p}_{n-k}^{(n)}(x) \exp(x) \tilde{q}_{n+j}^{(n)}(x) dx \\ &= \frac{1}{h_{n+j}} \int_{[-C_2, C_2]} p_{n-k}^{(n)}(x) \exp(x) q_{n+j}^{(n)}(e^x) e^{-nV(x)} dx + \mathcal{O}(e^{-C_1 n}) \\ &= -\frac{1}{h_{n+j}} \oint_{\Gamma} p_{n-k}^{(n)}(z) \exp(z) C q_{n+j}^{(n)}(z) dz + \mathcal{O}(e^{-C_1 n}), \end{aligned} \quad (4.31)$$

where the Cauchy transform $C q_{n+j}^{(n)}(z)$ is defined in (2.55), and Γ is a counterclockwise oriented contour surrounding the support of the equilibrium measure $[a, b]$ and intersecting with the real line at the points $-C_2, C_2$. See Figure 1.

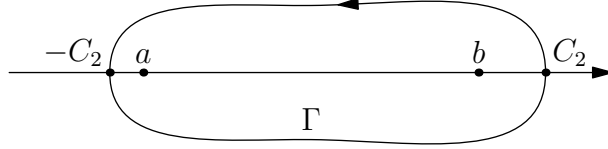


Figure 1: Contour Γ .

Below we compute the coefficients $a_{n+j, n-k}$. We need to use the mapping $\mathbf{J}_{c_1(1), c_0(1)}(s)$ defined in (2.2), and abbreviate it as $\mathbf{J}(s)$ here. We also need \mathbf{I}_1 , an inverse functions of \mathbf{J} , as defined in [21, (1.26)]. Recall (2.52)–(2.54) describing the outer asymptotics in Lemma 2.3(b). Substituting those in (4.31) and writing $s = \mathbf{I}_1(z)$, we obtain as $n \rightarrow \infty$,

$$a_{n+j, n-k} = \frac{c_1^{-j-k-1}}{2\pi i} \oint_{\Gamma} \frac{(s + \frac{1}{2})(s - \frac{1}{2})^{-j-k}}{s^2 - \frac{1}{4} - \frac{1}{c_1}} e^z (1 + \mathcal{O}(n^{-1})) dz + \mathcal{O}(e^{-C_1 n}). \quad (4.32)$$

Using \mathbf{J} , we get

$$\begin{aligned} a_{n+j, n-k} &= \frac{c_1^{-j-k-1}}{2\pi i} \oint_{\Gamma'} \frac{(s + \frac{1}{2})(s - \frac{1}{2})^{-j-k}}{s^2 - \frac{1}{4} - \frac{1}{c_1}} e^{\mathbf{J}(s)} \frac{d\mathbf{J}(s)}{ds} (1 + \mathcal{O}(n^{-1})) ds + \mathcal{O}(e^{-C_1 n}) \\ &= \frac{c_1^{-j-k} e^{c_0}}{2\pi i} \oint_{\Gamma'} (s + \frac{1}{2})(s - \frac{1}{2})^{-j-k-2} e^{c_1 s} (1 + \mathcal{O}(n^{-1})) ds + \mathcal{O}(e^{-C_1 n}), \end{aligned} \quad (4.33)$$

with $\Gamma' = \mathbf{I}_1(\Gamma)$, and (4.14) follows from a simple residue calculus. \square

Proof of Corollary 4.4. We need to estimate

$$F_{n,M} := e^{n(F(u)-F(v))} \left(\sum_{j=n-M}^{n-1} \sum_{k=n}^{j+M} a_{k,j} \tilde{p}_k^{(n)}(u) \tilde{q}_j^{(n)}(v) - a_{n-1,n} \tilde{p}_{n-1}^{(n)}(u) \tilde{q}_n^{(n)}(v) \right). \quad (4.34)$$

Recall that, for u, v near the point x^* which lies in (a, b) , the interior of the support of the equilibrium measure $\mu = \mu_1$, the asymptotics of $e^{nF(u)} \tilde{p}_{n+k}^{(n)}(u)$ and $e^{-nF(v)} \tilde{q}_{n+k}^{(n)}(v)$ are given by (2.50) and (2.51), with $j = n$ and $t = j/n = 1$ (here $F = F_1$). Then

$$e^{nF(u)} \tilde{p}_{n+k}^{(n)}(u) \tilde{q}_{n+j}^{(n)}(v) e^{-nF(v)} = A_{k,j}(u, v) + \mathcal{O}(n^{-1}), \quad (4.35)$$

with $(\mu = \mu_1, r_k = r_{1,k}, \hat{r}_k = \hat{r}_{1,k}, \theta_k = \theta_{1,k} \text{ and } \hat{\theta}_k = \hat{\theta}_{1,k} \text{ in (2.50) and (2.51))$

$$A_{k,j}(u, v) = \frac{1}{2\pi} c_1^{-j-\frac{1}{2}} e^{-j(\frac{c_1}{2}+c_0)} r_k(x^*) \left[\cos \left(n\pi \int_{x^*}^b d\mu(t) + \theta_k(x^*) - \xi \right) \right] \\ \times \hat{r}_j(x^*) \left[\cos \left(n\pi \int_{x^*}^b d\mu(t) + \hat{\theta}_j(x^*) - \eta \right) \right]. \quad (4.36)$$

Now we denote $G_k = G_{1,k}$ and $\hat{G}_k = \hat{G}_{1,k}$ where $G_{t,k}$ and $\hat{G}_{t,k}$ are defined in (2.36). Using the relations $\mathbf{I}_-(\cdot) = \overline{\mathbf{I}_+(\cdot)}$, $G_k(\mathbf{I}_-(\cdot)) = \overline{G_k(\mathbf{I}_+(\cdot))}$, this can be further computed as

$$A_{k,j}(u, v) = \frac{1}{2\pi} c_1^{-j-\frac{1}{2}} e^{-j(\frac{c_1}{2}+c_0)} 4 \operatorname{Re} \left(G_k(\mathbf{I}_+(x^*)) e^{in\pi \int_{x^*}^b d\mu-i\xi} \right) \operatorname{Re} \left(\hat{G}_j(\mathbf{I}_-(x^*)) e^{in\pi \int_{x^*}^b d\mu-i\eta} \right) \\ = \frac{1}{2\pi} c_1^{-j-\frac{1}{2}} e^{-j(\frac{c_1}{2}+c_0)} \left(G_k(\mathbf{I}_+(x^*)) e^{in\pi \int_{x^*}^b d\mu-i\xi} + G_k(\mathbf{I}_-(x^*)) e^{-in\pi \int_{x^*}^b d\mu+i\xi} \right) \\ \times \left(\hat{G}_j(\mathbf{I}_+(x^*)) e^{-in\pi \int_{x^*}^b d\mu+i\eta} + \hat{G}_j(\mathbf{I}_-(x^*)) e^{in\pi \int_{x^*}^b d\mu-i\eta} \right) \\ = \frac{1}{2\pi} c_1^{-j-\frac{1}{2}} e^{-j(\frac{c_1}{2}+c_0)} \\ \times \left[G_k(\mathbf{I}_+(x^*)) \hat{G}_j(\mathbf{I}_-(x^*)) e^{2in\pi \int_{x^*}^b d\mu-i(\xi+\eta)} + G_k(\mathbf{I}_-(x^*)) \hat{G}_j(\mathbf{I}_+(x^*)) e^{-2in\pi \int_{x^*}^b d\mu+i(\xi+\eta)} \right. \\ \left. + G_k(\mathbf{I}_+(x^*)) \hat{G}_j(\mathbf{I}_+(x^*)) e^{-i(\xi-\eta)} + G_k(\mathbf{I}_-(x^*)) \hat{G}_j(\mathbf{I}_-(x^*)) e^{i(\xi-\eta)} \right] \\ = \frac{1}{\pi} c_1^{-j-\frac{1}{2}} e^{-j(\frac{c_1}{2}+c_0)} \left[\operatorname{Re} (G_k(\mathbf{I}_+(x^*)) \hat{G}_j(\mathbf{I}_-(x^*)) e^{2in\pi \int_{x^*}^b d\mu-i(\xi+\eta)}) \right. \\ \left. + \operatorname{Re} (G_k(\mathbf{I}_+(x^*)) \hat{G}_j(\mathbf{I}_+(x^*)) e^{-i(\xi-\eta)}) \right]. \quad (4.37)$$

We want to find asymptotics for $F_{n,M}$ in (4.34) by substituting the formula for $A_{k,-l}$. We have for any fixed M , as $n \rightarrow \infty$,

$$F_{n,M} = \left(\sum_{l=1}^M \sum_{k=0}^{M-l} a_{n+k,n-l} A_{k,-l}(u, v) - a_{n-1,n} A_{-1,0}(u, v) \right) + \mathcal{O}(n^{-1}) \\ = \left(\sum_{l=1}^M \sum_{k=0}^{M-l} \alpha_{k+l} \frac{1}{\pi} c_1^{l-\frac{1}{2}} e^{l(\frac{c_1}{2}+c_0)} \left[\operatorname{Re} (G_k(\mathbf{I}_+(x^*)) \hat{G}_{-l}(\mathbf{I}_-(x^*)) e^{2in\pi \int_{x^*}^b d\mu-i(\xi+\eta)}) \right. \right. \\ \left. \left. - \operatorname{Re} (G_k(\mathbf{I}_+(x^*)) \hat{G}_{-l}(\mathbf{I}_+(x^*)) e^{-i(\xi-\eta)}) \right] \right. \\ \left. - \alpha_{-1} \frac{1}{\pi} c_1^{-\frac{1}{2}} \left[\operatorname{Re} (G_{-1}(\mathbf{I}_+(x^*)) \hat{G}_0(\mathbf{I}_-(x^*)) e^{2in\pi \int_{x^*}^b d\mu-i(\xi+\eta)}) \right. \right. \\ \left. \left. \operatorname{Re} (G_{-1}(\mathbf{I}_+(x^*)) - \hat{G}_0(\mathbf{I}_+(x^*)) e^{-i(\xi-\eta)}) \right] \right) + \mathcal{O}(n^{-1}). \quad (4.38)$$

Next, we define

$$I_5 := -c_1 G_{-1}(\mathbf{I}_+(x^*)) \hat{G}_0(\mathbf{I}_+(x^*)) \\ + \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{c_1}{(1+k+l)!} + \frac{1}{(k+l)!} \right) c_1^l e^{l(\frac{c_1}{2}+c_0)} G_k(\mathbf{I}_+(x^*)) \hat{G}_{-l}(\mathbf{I}_+(x^*)), \quad (4.39)$$

$$I_6 := -c_1 G_{-1}(\mathbf{I}_+(x^*)) \hat{G}_0(\mathbf{I}_-(x^*)) \\ + \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{c_1}{(1+k+l)!} + \frac{1}{(k+l)!} \right) c_1^l e^{l(\frac{c_1}{2}+c_0)} G_k(\mathbf{I}_+(x^*)) \hat{G}_{-l}(\mathbf{I}_-(x^*)), \quad (4.40)$$

and by (4.14), we have the identity between absolute convergent series

$$\begin{aligned} & \frac{1}{\pi} c_1^{-\frac{1}{2}} e^{\frac{c_1}{2}+c_0} \left(\operatorname{Re} [I_6 e^{2in\pi \int_{x^*}^b d\mu - i(\xi+\eta)}] - \operatorname{Re} [I_5 e^{-i(\xi-\eta)}] \right) \\ = & \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \alpha_{k+l} \frac{1}{\pi} c_1^{l-\frac{1}{2}} e^{l(\frac{c_1}{2}+c_0)} \left[\operatorname{Re} (G_k(\mathbf{I}_+(x^*)) \hat{G}_{-l}(\mathbf{I}_-(x^*)) e^{2in\pi \int_{x^*}^b d\mu - i(\xi+\eta)}) \right. \\ & \left. - \operatorname{Re} (G_k(\mathbf{I}_+(x)) \hat{G}_{-l}(\mathbf{I}_+(y)) e^{-i(\xi-\eta)}) \right] \\ & - \alpha_{-1} \frac{1}{\pi} c_1^{-\frac{1}{2}} \left[\operatorname{Re} (G_{-1}(\mathbf{I}_+(x^*)) \hat{G}_0(\mathbf{I}_-(x^*)) e^{2in\pi \int_{x^*}^b d\mu - i(\xi+\eta)}) \right. \\ & \left. \operatorname{Re} (G_{-1}(\mathbf{I}_+(x^*)) - \hat{G}_0(\mathbf{I}_+(x^*)) e^{-i(\xi-\eta)}) \right]. \end{aligned} \quad (4.41)$$

If we can prove

$$I_5 = i\sqrt{c_1} e^{-(\frac{c_1}{2}+c_0)} e^{x^*}, \quad I_6 = 0, \quad (4.42)$$

then the corollary is proved.

To finish the proof, we first consider I_5 in (4.39). By the definition (2.36) of G_k and \hat{G}_{-l} , we get

$$I_5 = \frac{i}{\sqrt{c_1}} \frac{\mathbf{I}_+(x^*) + \frac{1}{2}}{\mathbf{I}_+(x^*)^2 - \frac{1}{4} - \frac{1}{c_1}} \\ \times \left[-\frac{1}{\mathbf{I}_+(x^*) - \frac{1}{2}} + \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{c_1}{(1+k+l)!} + \frac{1}{(k+l)!} \right) c_1^{k+l} \left(\mathbf{I}_+(x^*) - \frac{1}{2} \right)^{k+l} \right]. \quad (4.43)$$

By the identities

$$\sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(k+l)!} x^{k+l} = x e^x, \quad \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(1+k+l)!} x^{k+l} = \left(1 - \frac{1}{x}\right) e^x + \frac{1}{x}, \quad (4.44)$$

we have

$$\begin{aligned} & \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{c_1}{(1+k+l)!} + \frac{1}{(k+l)!} \right) c_1^{k+l} \left(\mathbf{I}_+(x^*) - \frac{1}{2} \right)^{k+l} = \\ & \left(c_1 \left(\mathbf{I}_+(x^*) + \frac{1}{2} \right) - \frac{1}{\mathbf{I}_+(x^*) - \frac{1}{2}} \right) e^{c_1(\mathbf{I}_+(x^*) - \frac{1}{2})} + \frac{1}{\mathbf{I}_+(x^*) - \frac{1}{2}}. \end{aligned} \quad (4.45)$$

Using the identity $\mathbf{J}(\mathbf{I}_+(x)) = x$, we have

$$(\mathbf{I}_+(x^*) + \frac{1}{2})e^{c_1(\mathbf{I}_+(x^*) - \frac{1}{2})} = e^{-(\frac{c_1}{2} + c_0)}e^{x^*}(\mathbf{I}_+(x^*) - \frac{1}{2}), \quad \frac{e^{c_1(\mathbf{I}_+(x^*) - \frac{1}{2})}}{\mathbf{I}_+(x^*) - \frac{1}{2}} = e^{-(\frac{c_1}{2} + c_0)}\frac{e^{x^*}}{\mathbf{I}_+(x^*) + \frac{1}{2}}. \quad (4.46)$$

Substituting (4.46) into (4.45), and then substituting the simplified form of (4.45) into (4.43), we simplify (4.43), or equivalently (4.39), and obtain the expression for I_5 from (4.42).

Similarly we simplify (4.40) as

$$\frac{i}{\sqrt{c_1}} \frac{\mathbf{I}_+(x^*) + \frac{1}{2}}{\sqrt{\mathbf{I}_+(x^*)^2 - \frac{1}{4} - \frac{1}{c_1}} \sqrt{\mathbf{I}_-(x^*)^2 - \frac{1}{4} - \frac{1}{c_1}}} \times \left[-\frac{1}{\mathbf{I}_+(x^*) - \frac{1}{2}} + \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{c_1}{(1+k+l)!} + \frac{1}{(k+l)!} \right) c_1^{k+l} (\mathbf{I}_+(x^*) - \frac{1}{2})^k (\mathbf{I}_-(x^*) - \frac{1}{2})^l \right]. \quad (4.47)$$

Then, using the identities

$$\sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(k+l)!} x^k y^l = \frac{y}{y-x} (e^y - e^x), \quad \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(1+k+l)!} x^k y^l = \frac{y}{y-x} \left(\frac{e^y}{y} - \frac{e^x}{x} \right) + \frac{1}{x}, \quad (4.48)$$

we have analogous to (4.45),

$$\sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{c_1}{(1+k+l)!} + \frac{1}{(k+l)!} \right) c_1^{k+l} (\mathbf{I}_+(x^*) - \frac{1}{2})^k (\mathbf{I}_-(x^*) - \frac{1}{2})^l = \frac{\mathbf{I}_-(x^*) - \frac{1}{2}}{\mathbf{I}_-(x^*) - \mathbf{I}_+(x^*)} \left(\frac{\mathbf{I}_-(x^*) + \frac{1}{2}}{\mathbf{I}_-(x^*) - \frac{1}{2}} e^{c_1(\mathbf{I}_-(x^*) - \frac{1}{2})} - \frac{\mathbf{I}_+(x^*) + \frac{1}{2}}{\mathbf{I}_+(x^*) - \frac{1}{2}} e^{c_1(\mathbf{I}_+(x^*) - \frac{1}{2})} \right) + \frac{1}{\mathbf{I}_+(x) - \frac{1}{2}}. \quad (4.49)$$

Using the relation that $\mathbf{J}(\mathbf{I}_+(x^*)) = \mathbf{J}(\mathbf{I}_-(x^*))$, we find that the two terms between the parentheses in (4.49) cancel each other. Substituting (4.49) into (4.47), we find that $I_6 = 0$. Thus the proof is complete. \square

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