

## NONINTERSECTING BROWNIAN MOTIONS ON THE UNIT CIRCLE

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We consider an ensemble of  $n$  nonintersecting Brownian particles on the unit circle with diffusion parameter  $n^{-1/2}$ , which are conditioned to begin at the same point and to return to that point after time  $T$ , but otherwise not to intersect. There is a critical value of  $T$  which separates the subcritical case, in which it is vanishingly unlikely that the particles wrap around the circle, and the supercritical case, in which particles may wrap around the circle. In this paper, we show that in the subcritical and critical cases the probability that the total winding number is zero is almost surely 1 as  $n \rightarrow \infty$ , and in the supercritical case that the distribution of the total winding number converges to the discrete normal distribution. We also give a streamlined approach to identifying the Pearcey and tacnode processes in scaling limits. The formula of the tacnode correlation kernel is new and involves a solution to a Lax system for the Painlevé II equation of size  $2 \times 2$ . The proofs are based on the determinantal structure of the ensemble, asymptotic results for the related system of discrete Gaussian orthogonal polynomials, and a formulation of the correlation kernel in terms of a double contour integral.

**1. Introduction.** The probability models of nonintersecting Brownian motions have been studied extensively in last decade; see Tracy and Widom (2004, 2006), Adler and van Moerbeke (2005), Adler, Orantin and van Moerbeke (2010), Delvaux, Kuijlaars and Zhang (2011), Johansson (2013), Ferrari and Vetó (2012), Katori and Tanemura (2007) and Schehr et al. (2008), for example. These models are closely related to random matrix theory and (multiple) orthogonal polynomials; see Bleher and Kuijlaars (2004, 2007), Aptekarev, Bleher and Kuijlaars (2005) and Kuijlaars (2010), for example. One interesting feature is that as the number of particles  $n \rightarrow \infty$ , under proper scaling the nonintersecting Brownian motions models converge to *universal* processes, like the sine, Airy, Pearcey and tacnode processes. These processes are called universal since they appear in many other probability problems; see Okounkov and Reshetikhin (2003, 2007), Johansson (2005), Baik and Suidan (2007), Adler, van Moerbeke and Wang (2013), Adler, Ferrari and van Moerbeke (2013) and Adler, Johansson and van Moerbeke (2014), for example.

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Usually the models of nonintersecting Brownian motions turn out to be the most convenient ones to use for study of these universal processes. In particular, the Airy process appears ubiquitously in the Kardar–Parisi–Zhang (KPZ) universality class [Corwin (2012)], an important class of interacting particle systems and random growth models. The analysis of nonintersecting Brownian motions greatly improves the understanding of the Airy process and the KPZ universality class; see Corwin and Hammond (2014). Here, we remark that if we consider the nonintersecting Brownian motions on the real line, in the simplest models the Pearcey process does not occur, and the tacnode process only occurs in models with sophisticated parameters. Thus, the analysis of these universal processes becomes increasingly more difficult.

Due to technical difficulties, most studies of the limiting local properties of the nonintersecting Brownian motions concern models defined on the real line. A model of nonintersecting Brownian motions on a circle was considered by Dyson as a dynamical generalization of random matrix models [Dyson (1962)], and physicists and probabilists have been interested in the nonintersecting Brownian motions on a circle and their discrete counterparts for various reasons; see Forrester (1990), Hobson and Werner (1996) and Cardy (2003), for example. The simplest model of nonintersecting Brownian motions on a circle such that the particles start and end at the same common point is shown to be related to Yang–Mills theory on the sphere [Forrester, Majumdar and Schehr (2011), Schehr et al. (2013)] and the partition function (a.k.a. reunion probability) shows an interesting phase transition phenomenon closely related to the Tracy–Widom distributions in random matrix theory.

In this paper, we show that the Pearcey and (symmetric) tacnode processes mentioned above occur as the limits of the simplest model of nonintersecting Brownian motions on a circle, and give a streamlined method to analyze them. We also consider the total winding number of the particles, a quantity that has no counterpart in the models defined on the real line, and show that its limiting distribution in the nontrivial case is the discrete normal distribution [Szabłowski (2001)], a natural through perhaps not well-known discretization of the normal distribution. We also show that in the supercritical case, the Pearcey process occurs if the model is conditioned to have fixed total winding number. Although the sine and Airy processes also naturally occur, we omit the discussion on them to shorten the paper. A detailed discussion can be found in the preprint [Liechty and Wang (2013)].

Technically, the study of nonintersecting Brownian motions has been carried out in two distinct ways: by double contour integral formula, and by the Riemann–Hilbert problem. In the present work, we introduce a mixed approach, using both a double integral formula and the interpolation problem for discrete Gaussian orthogonal polynomials [Liechty (2012)], which are discrete orthogonal polynomials analogous to Hermite polynomials. In this paper, we analyze the dependence of the discrete Gaussian orthogonal polynomials on the translation of the lattice, which encodes the information of the winding number of the Brownian paths.

1.1. *Statement of main results.* Let  $\mathbb{T} = \{e^{i\theta} \in \mathbb{C}\}$  be the unit circle. Suppose  $x_1, x_2, \dots, x_n$  are  $n$  particles in independent Brownian motions on the unit circle with continuous paths and diffusion parameter  $n^{-1/2}$ , that is,

$$(1) \quad x_k(t) = e^{iB_k(t)/\sqrt{n}}, \quad i = 1, 2, \dots, n,$$

where  $B_k(t)$  are independent Brownian motions with diffusion parameter 1 starting from arbitrary places. The *nonintersecting Brownian motions on the circle with  $n$  particles*, henceforth denoted as NIBM in this paper, is defined by the particles  $x_1, \dots, x_n$  conditioned to have nonintersecting paths, that is,  $x_1(t), \dots, x_n(t)$  are distinct for any  $t$  between the starting time and the ending time. In this paper, we concentrate on the simplest model of NIBM, such that the  $n$  particles start from the common point  $e^{i \cdot 0}$  at the starting time  $t = 0$ , and end at the same common point  $e^{i \cdot 0}$  at the ending time  $t = T$ . We denote this model as  $\text{NIBM}_{0 \rightarrow T}$ .

Throughout this paper, we represent a point in  $\mathbb{T}$  by an angular variable  $\theta \in \mathbb{R}$  with  $\theta = \theta + 2\pi k$  ( $k \in \mathbb{Z}$ ) if there is no possibility of confusion, and use  $\theta \in [-\pi, \pi)$  as the principal value of the angle. Let  $P(a; b; t)$  be the transition probability density of one particle in Brownian motion on  $\mathbb{T}$  with diffusion parameter  $n^{-1/2}$ , starting from point  $a \in \mathbb{T}$  and ending at point  $b \in \mathbb{T}$  after time  $t > 0$ , which is

$$(2) \quad P(a; b; t) = \sqrt{\frac{n}{2\pi t}} \sum_{k \in \mathbb{Z}} e^{-n(b-a+2\pi k)^2/(2t)}.$$

Now consider the transition probability density of NIBM. Let  $A_n = \{a_1, \dots, a_n\}$  and  $B_n = \{b_1, \dots, b_n\}$  be two sets of  $n$  distinct points in  $\mathbb{T}$  such that  $-\pi \leq a_1 < a_2 < \dots < a_n < \pi$  and  $-\pi \leq b_1 < b_2 < \dots < b_n < \pi$ , and denote by  $P(A_n; B_n; t)$  the transition probability density of NIBM with the particles starting at the points  $A_n$  and ending at the points  $B_n$  after time  $t$ . Note that we do not require that the particle which started at point  $a_k$  ends at point  $b_k$ , but only that it ends at point  $b_j$  for some  $j = 1, \dots, n$ . For  $\tau \in \mathbb{R}$ , introduce the notation

$$(3) \quad P(a; b; t; \tau) := \sqrt{\frac{n}{2\pi t}} \sum_{k \in \mathbb{Z}} e^{-n(b-a+2\pi k)^2/(2t)} e^{2k\pi\tau i},$$

which reduces to (2) when  $\tau = 0$ . Introduce also the notation

$$(4) \quad \epsilon(n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \frac{1}{2}, & \text{if } n \text{ is even.} \end{cases}$$

A determinantal formula for  $P(A_n; B_n; t)$  is then given in the following proposition.

PROPOSITION 1.1. *The transition probability density function  $P(A_n; B_n; t)$  is given by the determinant of size  $n \times n$ ,*

$$(5) \quad P(A_n; B_n; t) = \det(P(a_i; b_j; t; \epsilon(n)))_{i,j=1}^n.$$

This proposition follows immediately from the Karlin–McGregor formula in the case that  $n$  is odd. If  $n$  is even then more care must be taken to derive the formula, and in the limited knowledge of the current authors it has not appeared before in the literature. The proof is presented in Section 2.1.

Now we consider the model  $\text{NIBM}_{0 \rightarrow T}$ . At a given time  $t \in [0, T]$ , the joint probability density function for the  $n$  particles in  $\text{NIBM}_{0 \rightarrow T}$  at distinct points  $-\pi \leq \theta_1 < \theta_2 < \dots < \theta_n < \pi$  is given by

$$(6) \quad \lim_{\substack{a_1, \dots, a_n \rightarrow 0 \\ b_1, \dots, b_n \rightarrow 0}} \frac{P(A_n; \Theta_n; t)P(\Theta_n; B_n; T - t)}{P(A_n; B_n; T)},$$

where  $A_n = \{a_1, \dots, a_n\}$ ,  $B_n = \{b_1, \dots, b_n\}$ , and  $\Theta_n = \{\theta_1, \dots, \theta_n\}$  describe the locations of the  $n$  particles at time 0,  $T$  and  $t$ , respectively. It is not difficult to see that such a limit exists, and so that our model is well defined (see Section 2.2).

The model  $\text{NIBM}_{0 \rightarrow T}$  is a determinantal process, meaning that the correlation functions of the particles may be described by a determinantal formula [Soshnikov (2000)]. To define the determinantal structure, fix  $m$  times  $0 < t_1 < t_2 < \dots < t_m < T$ , and to each time  $t_i$ , fix  $k_i$  points on  $\mathbb{T}$ ,  $-\pi \leq \theta_1^{(i)} < \theta_2^{(i)} < \dots < \theta_{k_i}^{(i)} < \pi$ . The multi-time correlation function is then defined as

$$(7) \quad \begin{aligned} &R_{0 \rightarrow T}^{(n)}(\theta_1^{(1)}, \dots, \theta_{k_1}^{(1)}; \dots; \theta_1^{(m)}, \dots, \theta_{k_m}^{(m)}; t_1, \dots, t_m) \\ &:= \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^{k_1 + \dots + k_m}} \mathbb{P}(\text{there is a particle in } [\theta_j^{(i)}, \theta_j^{(i)} + \Delta x) \\ &\hspace{15em} \text{for } j = 1, \dots, k_i \text{ at time } t_i). \end{aligned}$$

Then there exists some kernel function  $K_{t_i, t_j}(x, y)$  such that

$$(8) \quad \begin{aligned} &R_{0 \rightarrow T}^{(n)}(\theta_1^{(1)}, \dots, \theta_{k_1}^{(1)}; \dots; \theta_1^{(m)}, \dots, \theta_{k_m}^{(m)}; t_1, \dots, t_m) \\ &= \det(K_{t_i, t_j}(\theta_{l_i}^{(i)}, \theta_{l'_j}^{(j)}))_{i, j=1, \dots, m, l_i=1, \dots, k_i, l'_j=1, \dots, k_j}; \end{aligned}$$

see Section 2.3.

Intuitively, one can imagine the scenario of the model  $\text{NIBM}_{0 \rightarrow T}$  as follows. When the total time  $T$  is small, it is very unlikely that the particles will wrap around the circle before returning to  $e^{i \cdot 0}$ , and so the model is very close to the model of nonintersecting Brownian bridges on the real line. For large  $T$ , the particles which initially move in the positive direction and those which initially move in the negative direction will eventually meet on the far side of the circle, and the behavior of the model is very different. In this paper, this heuristic argument is confirmed, and the critical value of  $T$  which separates these two cases is pinpointed to be

$$(9) \quad T_c = \pi^2.$$

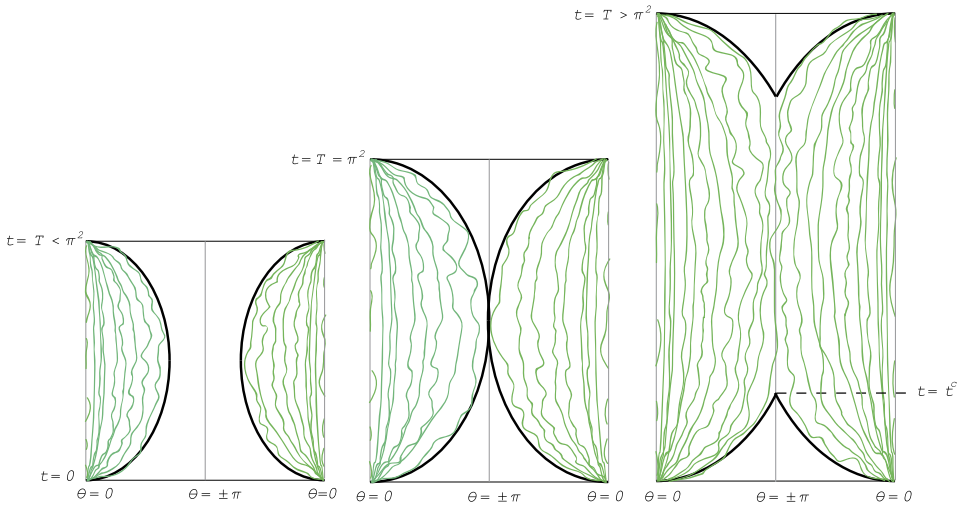


FIG. 1. Typical configurations of nonintersecting paths in the subcritical (left), critical (middle) and supercritical (right) cases. Time is on the vertical axis, and the angular variable  $\theta$  on the horizontal axis. At the initial time  $t = 0$  and the terminal time  $t = T$ , the particles are at  $\theta = 0$ , which is at both the left and right ends of the figures. The far side of the circle,  $\theta = \pm\pi$ , is marked by a light vertical line through the center of the figures. The particles tend to stay within the thick curved lines. In the supercritical case, the critical time  $t^c$  is marked, when the “leftmost” and “rightmost” particles meet on the far side of the circle.

Accordingly, we divide the  $\text{NIBM}_{0 \rightarrow T}$  model into the subcritical, critical and supercritical cases, for  $T < \pi^2$ ,  $T = \pi^2$ , and  $T > \pi^2$ , respectively, as shown in Figure 1.

In the subcritical case  $T < T_c$ , the model is described asymptotically by elementary functions. In the critical case  $T = T_c$  and the supercritical case  $T > T_c$ , the model is described asymptotically by special functions: functions related to the Painlevé II equation for  $T = T_c$ , and elliptic integrals for  $T > T_c$ . Let us define those functions.

*Critical case: The Painlevé II equation, and the related Lax pair.* In the critical case, we consider the model  $\text{NIBM}_{0 \rightarrow T}$  in the scaling limit

$$(10) \quad T = \pi^2(1 - 2^{-2/3}\sigma n^{-2/3}),$$

where  $\sigma \in \mathbb{R}$  is a parameter. In this case, the results of this paper involve a particular solution to the Painlevé II equation, and a solution to a related Lax system. Let us review these objects. The Hastings–McLeod solution [Hastings and McLeod (1980)] to the homogeneous Painlevé II equation (PII) is the solution to the differential equation

$$(11) \quad q''(s) = sq(s) + 2q(s)^3,$$

which satisfies

$$(12) \quad q(s) = \text{Ai}(s)(1 + o(1)) \quad \text{as } s \rightarrow +\infty,$$

where  $\text{Ai}(s)$  is the Airy function. Let  $q(s)$  be this particular solution to PII, and consider the  $2 \times 2$  matrix-valued solutions to the differential equation

$$(13) \quad \frac{d}{d\zeta} \Psi(\zeta; s) = \begin{pmatrix} -4i\zeta^2 - i(s + 2q(s)^2) & 4\zeta q(s) + 2iq'(s) \\ 4\zeta q(s) - 2iq'(s) & 4i\zeta^2 + i(s + 2q(s)^2) \end{pmatrix} \Psi(\zeta; s).$$

This  $2 \times 2$  system was originally studied by Flaschka and Newell (1980). The differential equation (13), together with another one given in (340), form a Lax pair for the PII equation, that is, the compatibility of the two differential equations implies that  $q(s)$  solves PII. We will consider the particular solution to (13) which satisfies

$$(14) \quad \Psi(\zeta; s)e^{i((4/3)\zeta^3 + s\zeta)\sigma_3} = I + O(\zeta^{-1}), \quad \zeta \rightarrow \pm\infty.$$

The asymptotics (14) extend into the sectors  $-\pi/3 < \arg \zeta < \pi/3$ , and  $2\pi/3 < \arg \zeta < 4\pi/3$ . Here, we note that the uniqueness of the boundary value problem (13) and (14) implies

$$(15) \quad \Psi_{i,j}(-\zeta) = \Psi_{3-i,3-j}(\zeta), \quad i, j = 1, 2.$$

*Supercritical case: Elliptic integrals.* In the supercritical case where  $T > T_c = \pi^2$ , we define a  $t^c < T/2$ . To simplify the notation, we parametrize  $T > \pi^2$  by  $k \in (0, 1)$ . For each  $k$ , we have the elliptic integrals

$$(16) \quad \begin{aligned} \mathbf{K} &:= \mathbf{K}(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}, \\ \mathbf{E} &:= \mathbf{E}(k) = \int_0^1 \frac{\sqrt{1-k^2s^2}}{\sqrt{1-s^2}} ds. \end{aligned}$$

We further define

$$(17) \quad \tilde{k} := \frac{2\sqrt{k}}{1+k},$$

and denote

$$(18) \quad \begin{aligned} \tilde{\mathbf{K}} &:= \mathbf{K}(\tilde{k}) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\tilde{k}^2s^2)}}, \\ \tilde{\mathbf{E}} &:= \mathbf{E}(\tilde{k}) = \int_0^1 \frac{\sqrt{1-\tilde{k}^2s^2}}{\sqrt{1-s^2}} ds. \end{aligned}$$

$T$  is then parametrized as

$$(19) \quad T = 4\tilde{\mathbf{K}}\tilde{\mathbf{E}} = 4 \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\tilde{k}^2s^2)}} \int_0^1 \frac{\sqrt{1-\tilde{k}^2s^2}}{\sqrt{1-s^2}} ds,$$

where the well-definedness of the parametrization is given in Lemma 3.2, and  $t^c$  is expressed as

$$(20) \quad t^c = \frac{4}{\tilde{k}^2} \tilde{\mathbf{E}}(\tilde{\mathbf{E}} - (1 - \tilde{k}^2)\tilde{\mathbf{K}}) = 4 \int_0^1 \frac{\sqrt{1 - \tilde{k}^2 s^2}}{\sqrt{1 - s^2}} ds \int_0^1 \frac{\sqrt{1 - s^2}}{\sqrt{1 - \tilde{k}^2 s^2}} ds.$$

The fundamental group of  $\mathbb{T}$  has a canonical identification with  $\mathbb{Z}$ , and so for any closed path on  $\mathbb{T}$  we can define the *winding number* of the path as the integer representative of its homotopy class. For a set of  $n$  particles with continuous paths on  $\mathbb{T}$  that come back the initial position after some time, we can define their *total winding number* as the sum of the winding numbers of the paths of the particles. The following theorem concerns the total winding number of the particles in  $\text{NIBM}_{0 \rightarrow T}$ . Let  $q$  be defined in terms of the complete elliptic integral of the first kind as

$$(21) \quad q := \exp\left(-\frac{\pi \mathbf{K}(\sqrt{1 - k^2})}{2\mathbf{K}(k)}\right) = \exp\left(-\frac{\pi \mathbf{K}(\sqrt{1 - \tilde{k}^2})}{\mathbf{K}(\tilde{k})}\right),$$

where  $k$  and  $\tilde{k}$  are related to  $T$  via (16)–(19).

REMARK 1.1. Note that we use the notation  $q$  in two different meanings. In the context of the critical asymptotics,  $q$  is the Hastings–McLeod solution to PII and is always written with its argument  $q(\sigma)$ . In the context of the supercritical asymptotics,  $q$  is written with no argument and represents the elliptic nome defined in (21). These are both standard notation, and it should be clear throughout the paper to which object  $q$  refers.

THEOREM 1.2. *In the  $\text{NIBM}_{0 \rightarrow T}$ , as the number of particles  $n \rightarrow \infty$ :*

(a) *In the subcritical case  $T < T_c = \pi^2$ , the winding number is zero with a probability that is exponentially close to 1. That is,*

$$(22) \quad \mathbb{P}(\text{Total winding number equals } 0) = 1 - \mathcal{O}(e^{-cn}),$$

where the constant  $c > 0$  may depend on  $T$ .

(b) *In the critical scaling (10), for any fixed  $\sigma$ ,*

$$(23) \quad \begin{aligned} \mathbb{P}(\text{Total winding number equals } 0) &= 1 - \frac{q(\sigma)}{2^{1/3}n^{1/3}} + \frac{q(\sigma)^2}{2^{2/3}n^{2/3}} + \mathcal{O}(n^{-1}), \\ \mathbb{P}(\text{Total winding number equals } 1) &= \mathbb{P}(\text{Total winding number equals } (-1)) \\ &= \frac{q(\sigma)}{2^{4/3}n^{1/3}} - \frac{q(\sigma)^2}{2^{5/3}n^{2/3}} + \mathcal{O}(n^{-1}), \\ \mathbb{P}(|\text{Total winding number}| > 1) &= \mathcal{O}(n^{-1}). \end{aligned}$$

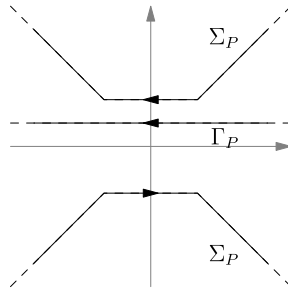


FIG. 2. The shape of contours  $\Sigma_P$  and  $\Gamma_P$ . The upper part of  $\Sigma_P$  consists of the ray from  $2 + 2i$  to  $e^{\pi i/4} \cdot \infty$ , the line segment from  $2 + 2i$  to  $-2 + 2i$ , and the ray from  $-2 + 2i$  to  $e^{3\pi i/4} \cdot \infty$ . The lower part of  $\Sigma_P$  is the reflection of the upper part about the real axis.  $\Gamma_P$  is the horizontal line  $\{z = x + i \mid x \in \mathbb{R}\}$ . Their orientations are shown in the figure.

(c) For  $T > T_c$  and for any  $\omega \in \mathbb{Z}$ ,

$$(24) \quad \mathbb{P}(\text{Total winding number equals } \omega) = q^{\omega^2} \sqrt{\frac{\pi}{2\tilde{\mathbf{K}}}} + \mathcal{O}(n^{-1}).$$

The limiting distribution of the total winding number in the supercritical case is the *discrete normal distribution* defined in Kemp (1997), and the formula in the right-hand side of (24) appears in Szabłowski (2001). See also Johnson, Kemp and Kotz (2005), Section 10.8.3.

The Pearcey process is defined by the extended Pearcey kernel [Tracy and Widom (2006), Section 3],

$$(25) \quad K_{s,t}^{\text{Pearcey}}(\xi, \eta) = \tilde{K}_{s,t}^{\text{Pearcey}}(\xi, \eta) - \phi_{s,t}(\xi, \eta),$$

where

$$(26) \quad \phi_{s,t}(\xi, \eta) = \begin{cases} 0, & \text{if } s \geq t, \\ \frac{1}{\sqrt{2\pi(t-s)}} e^{-(\xi-\eta)^2/(2(t-s))}, & \text{if } s < t, \end{cases}$$

and

$$(27) \quad \tilde{K}_{s,t}^{\text{Pearcey}}(\xi, \eta) = \frac{i}{4\pi^2} \oint_{\Sigma_P} dz \oint_{\Gamma_P} dw \frac{e^{z^4/4 + sz^2/2 + i\xi z}}{e^{w^4/4 + tw^2/2 + i\eta w}} \frac{1}{z - w},$$

where  $\Sigma_P$  and  $\Gamma_P$  are infinite, disjoint contours such that the upper part of  $\Sigma_P$  is from  $e^{\pi i/4} \cdot \infty$  to  $e^{3\pi i/4} \cdot \infty$ , the lower part of  $\Sigma_P$  is from  $e^{5\pi i/4} \cdot \infty$  to  $e^{7\pi i/4} \cdot \infty$ , and  $\Gamma_P$  is the leftward horizontal line. See Figure 2 for the exact description. Our definition of the Pearcey kernel is the same as that in Adler, Orantin and van Moerbeke (2010), Formula 1.2, up to a change of variables.



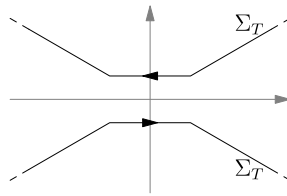


FIG. 3. The shape of contour  $\Sigma_T$ . The upper part of  $\Sigma_T$  consists of the ray from  $\sqrt{3} + i$  to  $e^{\pi i/6} \cdot \infty$ , the line segment from  $\sqrt{3} + i$  to  $-\sqrt{3} + i$ , and the ray from  $-\sqrt{3} + i$  to  $e^{5\pi i/6} \cdot \infty$ . The lower part of  $\Sigma_T$  is the reflection of the upper part about the real axis. The orientation is shown in the figure.

We now define the tacnode kernel. Denote by  $\Psi_{ij}(\zeta; s)$  the  $(i, j)$  entry of the matrix  $\Psi(\zeta; s)$  defined in (13) and (14). It is convenient to also define the functions

$$(28) \quad \begin{aligned} f(u; s) &:= \begin{cases} -\Psi_{12}(u; s), & \text{if } \text{Im } u > 0, \\ \Psi_{11}(u; s), & \text{if } \text{Im } u < 0, \end{cases} \\ g(u, s) &:= \begin{cases} -\Psi_{22}(u; s), & \text{if } \text{Im } u > 0, \\ \Psi_{21}(u; s), & \text{if } \text{Im } u < 0. \end{cases} \end{aligned}$$

We then define the tacnode kernel as

$$(29) \quad K_{s,t}^{\text{tac}}(\xi, \eta; \sigma) = \tilde{K}_{s,t}^{\text{tac}}(\xi, \eta; \sigma) - \phi_{s,t}(\xi, \eta),$$

where  $\phi_{s,t}(\xi, \eta)$  is as in (26), and

$$(30) \quad \begin{aligned} \tilde{K}_{s,t}^{\text{tac}}(\xi, \eta; \sigma) &:= \frac{1}{2\pi} \oint_{\Sigma_T} du \oint_{\Sigma_T} dv e^{su^2/2 - tv^2/2} e^{-i(u\xi - v\eta)} \\ &\quad \times \frac{f(u; \sigma)g(v; \sigma) - g(u; \sigma)f(v; \sigma)}{2\pi i(u - v)}. \end{aligned}$$

Here,  $\Sigma_T$  is a contour consisting of two pieces. One piece of  $\Sigma_T$  lies entirely above the real line, and goes from  $e^{\pi i/6} \cdot \infty$  to  $e^{5\pi i/6} \cdot \infty$ . The other piece lies entirely below the real line and goes from  $e^{7\pi/6} \cdot \infty$  to  $e^{11\pi/6} \cdot \infty$ . See Figure 3 for the exact description. The convergence of the integrals in (30) follows from the asymptotics (14). Let us note that we could deform the two parts of the contour  $\Sigma_T$  to the real line, and write (30) as the sum of four double integrals on  $\mathbb{R}$ . We prefer to write the integral on the contour  $\Sigma_T$  because the integrand of (30) is in fact an  $L^1$  function on  $\Sigma_T$ , whereas convergence of the integral over  $\mathbb{R}$  is the result of rapid oscillations.

The convergence of  $\text{NIBM}_{0 \rightarrow T}$  to the universal processes described above is described in the following theorem.

**THEOREM 1.3.** *In the NIBM<sub>0→T</sub>:*

(a) *Assume  $T > T_c$ . There exists  $d > 0$  defined in (235) such that when we scale  $t_i$  and  $t_j$  close to  $t^c$ , and  $x$  and  $y$  close to  $-\pi$  as*

$$(31) \quad \begin{aligned} t_i &= t^c + \frac{d^2}{n^{1/2}} \tau_i, & t_j &= t^c + \frac{d^2}{n^{1/2}} \tau_j, \\ x &= -\pi - \frac{d}{n^{3/4}} \xi, & y &= -\pi - \frac{d}{n^{3/4}} \eta, \end{aligned}$$

*the correlation kernel  $K_{t_i, t_j}(x, y)$  has the limit*

$$(32) \quad \lim_{n \rightarrow \infty} K_{t_i, t_j}(x, y) \left| \frac{dy}{d\eta} \right| = K_{-\tau_j, -\tau_i}^{\text{Pearcey}}(\eta, \xi).$$

(b) *Let  $T$  be scaled close to  $T_c = \pi^2$  as in (10) with  $\sigma$  fixed, and let*

$$(33) \quad d = 2^{-5/3} \pi.$$

*When we scale  $t_i$  and  $t_j$  close to  $T/2$ , and  $x$  and  $y$  close to  $-\pi$  as*

$$(34) \quad \begin{aligned} t_i &= \frac{T}{2} + \frac{d^2}{n^{1/3}} \tau_i, & t_j &= \frac{T}{2} + \frac{d^2}{n^{1/3}} \tau_j, \\ x &= -\pi - \frac{d}{n^{2/3}} \xi, & y &= -\pi - \frac{d}{n^{2/3}} \eta, \end{aligned}$$

*the correlation kernel  $K_{t_i, t_j}(x, y)$  has the limit*

$$(35) \quad \lim_{n \rightarrow \infty} K_{t_i, t_j}(x, y) \left| \frac{dy}{d\eta} \right| = K_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) = K_{-\tau_j, -\tau_i}^{\text{tac}}(\eta, \xi; \sigma).$$

**REMARK 1.2.** The identity  $K_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) = K_{-\tau_j, -\tau_i}^{\text{tac}}(\eta, \xi; \sigma)$  in (35) is due to the symmetry of the kernel  $K_{s,t}^{\text{tac}}(\xi, \eta)$ , which can be checked by (15).

In the supercritical case, we have finer result for the NIBM<sub>0→T</sub> conditioned to have fixed total winding number. Analogous to (7), we define the multi-time correlation function for the NIBM<sub>0→T</sub> with total winding number  $\omega$  as

$$(36) \quad \begin{aligned} & (R_{0 \rightarrow T}^{(n)})_{\omega}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m) \\ & := \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^{k_1 + \dots + k_m}} \mathbb{P} \left( \begin{array}{l} \text{there is a particle in } [a_j^{(i)}, a_j^{(i)} + \Delta x) \\ \text{for } j = 1, \dots, k_i \text{ at time } t_i, \\ \text{and the total winding number is } \omega \end{array} \right). \end{aligned}$$

If we consider the conditional NIBM<sub>0→T</sub> such that the total winding number is fixed to be  $\omega$ , then the multi-time correlation function of the conditional process

should be

$$(37) \quad \begin{aligned} & (R_{0 \rightarrow T}^{(n)})_{\omega} \sim (a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m) \\ & := \frac{(R_{0 \rightarrow T}^{(n)})_{\omega} (a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m)}{\mathbb{P}(\text{Total winding number equals } \omega)}. \end{aligned}$$

Note that if the total winding number is fixed, then the conditional  $\text{NIBM}_{0 \rightarrow T}$  is no longer a determinantal process. (The reason is as follows: In a determinantal process over time  $[0, T]$ , the movement of particles between two times  $t_1 < t_2 \in (0, T)$  only depends on the positions of the particles at times  $t_1$  and  $t_2$ , but not the trajectories on  $(0, t_1)$  or  $(t_2, T)$ . The conditional  $\text{NIBM}_{0 \rightarrow T}$  with fixed total winding number does not have this property.) Nevertheless, we have results for the limiting  $k$ -correlation functions of the process. The following theorem shows that with the condition of fixed total winding number, the conditional  $\text{NIBM}_{0 \rightarrow T}$  has the same local limiting properties as the  $\text{NIBM}_{0 \rightarrow T}$  with free winding number.

**THEOREM 1.4.** *Assume  $T > T_c = \pi^2$ . Let  $\omega$  be a fixed integer,  $t_1, \dots, t_m \in (0, T)$  be times, and at each time  $t_i$ , let  $x_1^{(i)}, \dots, x_{k_i}^{(i)}$  be locations on  $\mathbb{T}$  such that  $k_1 + \dots + k_m = k$ . We consider the correlation function  $(R_{0 \rightarrow T}^{(n)})_{\omega} \sim (R_{0 \rightarrow T}^{(n)})_{\omega} \sim (x_1^{(1)}, \dots, x_{k_1}^{(1)}; \dots; x_1^{(m)}, \dots, x_{k_m}^{(m)}; t_1, \dots, t_m)$  in the conditional  $\text{NIBM}_{0 \rightarrow T}$  with winding number  $\omega$ . Let*

$$(38) \quad t_i = t^c + \frac{d^2}{n^{1/2}} \tau_i, \quad x_j^{(i)} = -\pi - \frac{d}{n^{3/4}} \xi_j^{(i)},$$

where  $d$  is the same as in Theorem 1.3(a). The multi-time correlation function has the limit

$$(39) \quad \begin{aligned} & \lim_{n \rightarrow \infty} (R_{0 \rightarrow T}^{(n)})_{\omega} \sim \left( \frac{d}{n^{3/4}} \right)^k \\ & = \det(K_{-\tau_j, -\tau_i}^{\text{Pearcey}}(\xi_{l_j}^{(j)}, \xi_{l'_i}^{(i)}))_{i,j=1, \dots, m, l_i=1, \dots, k_i, l'_j=1, \dots, k_j}. \end{aligned}$$

1.2. *Comparison of  $K^{\text{tac}}$  with other tacnode kernels.* The tacnode process was first studied by three different groups [Adler, Ferrari and van Moerbeke (2013), Johansson (2013), Delvaux, Kuijlaars and Zhang (2011)], each using different methods and obtaining different formulas for the tacnode process. The formulas obtained in Adler, Ferrari and van Moerbeke (2013) and Johansson (2013) each involve Airy functions and related operators, whereas the formula of Delvaux, Kuijlaars and Zhang (2011) involves a Lax system for the Painlevé II equation of size  $4 \times 4$ . As it turns out, the various matrix entries of the  $4 \times 4$  Lax system appearing in Delvaux, Kuijlaars and Zhang (2011) can be explicitly expressed in terms of Airy functions and related operators [Delvaux (2013)] [see also Kuijlaars (2014)], and the equivalence of the formulas in Johansson (2013) and Delvaux,

Kuijlaars and Zhang (2011) was recently proven by Delvaux [Delvaux (2013)]. The equivalence of the two different Airy formulas obtained in Johansson (2013) and Adler, Ferrari and van Moerbeke (2013) was proved in Adler, Johansson and van Moerbeke (2014), although the proof is somewhat indirect in that it relies on computing the limiting kernel from a particular model in two different ways.

Indeed the formula for the tacnode kernel obtained in the NIBM<sub>0→T</sub> is equivalent to the existing formulas. In order to state this equivalence precisely, we define the kernel  $\mathcal{L}^{\text{tac}}$  obtained in Johansson (2013), using some notation which was introduced in Delvaux (2013) and Baik, Liechty and Schehr (2012). Let  $\mathbf{B}_s$  be the integral operator defined in Baik, Liechty and Schehr (2012), Formula (3), which is denoted as  $A_\sigma$  in Delvaux (2013), Formula (4.1), acting on  $L^2[0, \infty)$  with kernel

$$(40) \quad \mathbf{B}_s(x, y) = \text{Ai}(x + y + s),$$

and let  $\mathbf{A}_s := \mathbf{B}_s^2$  be the Airy operator, which is defined in Baik, Liechty and Schehr (2012), Formula (17) and is denoted as  $K_{\text{Ai}, \sigma}$  in Delvaux (2013), Formula (4.2). Define the functions  $Q_s$  and  $R_s$  as in Baik, Liechty and Schehr (2012), Formula (18)

$$(41) \quad Q_s := (\mathbf{1} - \mathbf{A}_s)^{-1} \mathbf{B}_s \delta_0, \quad R_s := (\mathbf{1} - \mathbf{A}_s)^{-1} \mathbf{A}_s \delta_0,$$

where the delta function  $\delta_0$  is defined such that

$$(42) \quad \int_{[0, \infty)} f(x) \delta_0(x) dx = f(0),$$

for functions  $f(x)$  which are right-continuous at zero. Define also the function

$$(43) \quad b_{\tau, z, \sigma}(x) := e^{-(2/3)\tau^3 - \tau z - 2^{1/3}\tau x - 2^{-2/3}\tau\sigma} \text{Ai}(2^{1/3}x + z + 2^{-2/3}\sigma + \tau^2),$$

which was introduced in Delvaux (2013), Formula (2.16). Note that our  $b_{\tau, z, \sigma}(x)$  is equivalent to  $b_{\tau, z}(x) = \tilde{b}_{\tau, -z}(x)$  in Delvaux (2013), Formula (2.16) with  $\lambda = 1$ . Then the symmetric tacnode kernel obtained in Johansson (2013) is given by

$$(44) \quad \mathcal{L}_{\text{tac}}(u, v; \sigma, \tau_1, \tau_2) = \tilde{\mathcal{L}}_{\text{tac}}(u, v; \sigma, \tau_1, \tau_2) - \phi_{2\tau_1, 2\tau_2}(u, v),$$

where  $\phi_{s, t}(u, v)$  is defined in (26) and by Delvaux (2013), Formula (2.29),

$$(45) \quad \begin{aligned} &\tilde{\mathcal{L}}_{\text{tac}}(u, v; \sigma, \tau_1, \tau_2) \\ &= \frac{1}{2^{2/3}} \int_{\sigma}^{\infty} (\hat{p}_1(u; s, \tau_1) \hat{p}_1(v; s, -\tau_2) \\ &\quad + \hat{p}_1(-u; s, \tau_1) \hat{p}_1(-v; s, -\tau_2)) ds, \end{aligned}$$

and the function  $\hat{p}_1(z; s, \tau)$  is equivalent to  $\hat{p}_1(z; s, \tau)$  and  $\hat{p}_2(-z; s, \tau)$  defined in Delvaux (2013), Formula (2.26), with  $\lambda = 1$ , and by Delvaux (2013), Lemmas 4.2 and 4.3, it has the expression,

$$(46) \quad \hat{p}_1(z; s, \tau) := \langle b_{\tau, -z, s}, R_s + \delta_0 \rangle_0 - \langle b_{\tau, z, s}, Q_s \rangle_0,$$

where  $\langle \cdot, \cdot \rangle_0$  is the inner product on  $L^2[0, \infty)$ . The kernels  $\mathcal{L}_{\text{tac}}$  and  $K^{\text{tac}}$  are related in the following proposition.

PROPOSITION 1.5.

$$(47) \quad K_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) = 2^{-2/3} \mathcal{L}_{\text{tac}}(2^{-2/3}\xi, 2^{-2/3}\eta; \sigma, 2^{-7/3}\tau_i, 2^{-7/3}\tau_j).$$

The proof of this proposition is given in Appendix B.

1.3. *Organization of the paper.* In Section 2, we derive the exact formulas for the transition probability density of NIBM, the so-called reunion probability of  $\text{NIBM}_{0 \rightarrow T}$ , and the correlation kernel of  $\text{NIBM}_{0 \rightarrow T}$ . We also derive the  $\tau$ -deformed version of the formulas to analyze the conditional  $\text{NIBM}_{0 \rightarrow T}$  with fixed total winding number. In Section 3, we summarize the results about discrete Gaussian orthogonal polynomials that are necessary for the asymptotic analysis in this paper. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorems 1.3 and 1.4. Section 6 is on the interpolation problem and Riemann–Hilbert problem associated to Gaussian discrete orthogonal polynomials, and we prove there the technical results stated in Section 3. Appendix A contains technical results needed in the asymptotic analysis of Section 5, and Appendix B gives a proof of Proposition 1.5.

**2. Nonintersecting Brownian motion on the unit circle and discrete Gaussian orthogonal polynomials.** In this section, we derive the transition probability density of NIBM, and the joint correlation function and the correlation kernel of  $\text{NIBM}_{0 \rightarrow T}$ . For all the probabilistic quantities, we derive the  $\tau$ -deformed versions, which have no direct probabilistic meaning, but are generating functions of the corresponding probabilistic quantities with fixed offset/winding number.

2.1.  *$\tau$ -deformed transition probability density of NIBM.* Let  $P(a; b; t)$  be the transition probability density of one particle in Brownian motion on  $\mathbb{T}$  with diffusion parameter  $n^{-1/2}$ , starting from point  $a$  and ending at point  $b$  after time  $t$  as given in (2). For  $n$  labeled particles in NIBM starting at  $\vec{a} = (a_1, \dots, a_n)$  and ending at  $\vec{b} = (b_1, \dots, b_n)$  after time  $t$ , we denote the transition probability density  $P(\vec{a}; \vec{b}; t)$ . By labeled particles, we mean that the particle beginning at the point  $a_j$  must end at the point  $b_j$  for each  $j = 1, \dots, n$ . Since the Brownian motion on  $\mathbb{T}$  is a stationary strong Markov process with continuous transition probability density, we apply the celebrated Karlin–McGregor formula [Karlin and McGregor (1959), Theorem 1 and assertion D], and have

$$(48) \quad \sum_{\sigma \in S_n} \text{sgn}(\sigma) P(\vec{a}; \vec{b}(\sigma); t) = \det[P(a_i; b_j; t)]_{i,j=1}^n$$

where  $\vec{b}(\sigma) = (b_{\sigma(1)}, \dots, b_{\sigma(n)})$ .

Below we assume that  $-\pi \leq a_1 < a_2 < \dots < a_n < \pi$  and  $-\pi \leq b_1 < b_2 < \dots < b_n < \pi$ . Then  $P(\vec{a}; \vec{b}(\sigma); t)$  is nonzero only if  $\sigma$  is a cyclic permutation. For  $\ell \in$

$\{1, \dots, n\}$ , we use the notation  $[\ell]$  to denote the cyclic permutation which shifts by  $\ell$ . That is,  $[\ell] \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n$  acts on the set  $\{1, \dots, n\}$  as  $[\ell](k) = k + \ell$  or  $k + \ell - n$  in  $\{1, \dots, n\}$ . Hence, (48) becomes

$$(49) \quad \sum_{[\ell] \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n} \text{sgn}([\ell]) P(\vec{a}; \vec{b}([\ell]); t) = \det[P(a_i; b_j; t)]_{i,j=1}^n.$$

Now let  $A_n = \{a_1, \dots, a_n\}$  and  $B_n = \{b_1, \dots, b_n\}$  be two unlabeled sets of points in  $\mathbb{T}$ , and let  $P(A_n; B_n; t)$  be the transition probability for NIBM on  $\mathbb{T}$  with the particles starting at the points  $A_n$  and ending at the points  $B_n$ , as described in the paragraph preceding (3). Then  $P(A_n; B_n; t)$  is obtained from  $P(\vec{a}; \vec{b}(\sigma); t)$  via the relation

$$(50) \quad P(A_n; B_n; t) = \sum_{\sigma \in S_n} P(\vec{a}; \vec{b}(\sigma); t) = \sum_{[\ell] \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n} P(\vec{a}; \vec{b}([\ell]); t).$$

In the case that  $n$  is odd, we have  $\text{sgn}([\ell]) = 1$  for all  $[\ell] \in \mathbb{Z}/n\mathbb{Z}$ , and then (50) and (49) yield

$$(51) \quad P(A_n; B_n; t) = \det[P(a_i; b_j; t)]_{i,j=1}^n.$$

In the case that  $n$  is even, the situation is more complicated. The determinantal formula of  $P(A_n; B_n; t)$  has not appeared before in the literature as far as the current authors can tell, but a discrete analogue was solved by Fulmek (2004/07). We summarize Fulmek’s result below, and take the continuum limit to obtain the result for NIBM.

Consider the cylindrical lattice  $\mathbb{Z}_M \times \mathbb{Z} = \{([m], n) \mid m = -M/2, \dots, M/2 - 1, n \in \mathbb{Z}\}$ , where  $M$  is assumed to be even, and we take the canonical representation for  $\mathbb{Z}_M$  to be the integers between (and including)  $-M/2$  and  $M/2 - 1$ . We define a step to the left as the edge from  $([m], n)$  to  $([m - 1], n + 1)$ , and a step to the right as the edge from  $([m], n)$  to  $([m + 1], n + 1)$ . We assign weight the  $x$  to each step to the left and weight  $y$  to each step to the right, so that

$$(52) \quad w(e) := \begin{cases} x, & \text{if } e = [(m], n) \rightarrow ([m - 1], n + 1) \text{ is a step to the left,} \\ y, & \text{if } e = [(m], n) \rightarrow ([m + 1], n + 1) \text{ is a step to the right.} \end{cases}$$

A path on the lattice is defined as a sequence of adjacent steps, either to the left or to the right. We define the weight of a path as the product of the weights of its edges, so that

$$(53) \quad w(p = (e_1, \dots, e_N)) := \prod_{i=1}^N w(e_i),$$

and for an arbitrary  $n$ -tuple of paths  $(p_1, \dots, p_n)$ , define its weight as  $w((p_1, \dots, p_n)) = \prod_{i=1}^n w(p_i)$ . Furthermore, for a set of objects whose weights are defined,

we define the generating function of these weighted objects as the sum of their weights, so that

$$(54) \quad \mathbf{GF}(A) := \sum_{a \in A} w(a).$$

Let  $-M/2 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n < M/2$  and  $M/2 \leq \beta_1 < \beta_2 < \dots < \beta_n < M/2$  such that  $\alpha_i, \beta_i$  are all even, and  $N$  be an even integer. We denote  $\mathcal{P}(\alpha_i; \beta_j; N)$  as the set of paths connecting  $([\alpha_i], 0)$  and  $([\beta_j], N)$ . For any  $\sigma \in S_n$ , denote  $\mathcal{P}(\vec{\alpha}; \vec{\beta}(\sigma); N)$  as the set of the  $n$ -tuples of nonintersecting paths  $(p_1, \dots, p_n)$  such that  $p_i$  connects  $([\alpha_i], 0)$  and  $([\beta_{\sigma(i)}], N)$ .

The celebrated Lindström–Gessel–Viennot formula [Lindström (1973), Gessel and Viennot (1985)] yields that

$$(55) \quad \begin{aligned} & \sum_{\sigma \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n} \text{sgn}(\sigma) \mathbf{GF}(\mathcal{P}(\vec{\alpha}; \vec{\beta}(\sigma); N)) \\ &= \sum_{[\ell] \in S_n} \text{sgn}([\ell]) \mathbf{GF}(\mathcal{P}(\vec{\alpha}; \vec{\beta}([\ell]); N)) \\ &= \det(\mathbf{GF}(\mathcal{P}(\alpha_i; \beta_j; N)))_{i,j=1}^n, \end{aligned}$$

where in the first identity we have used that there are no nonintersecting paths connecting  $([\alpha_i], 0)$  and  $([\beta_{\sigma(i)}], N)$  for all  $i$  unless  $\sigma$  is a cyclic permutation.

With the weights  $x = y = 1/2$ , we find that  $\mathbf{GF}(\mathcal{P}(\alpha_i; \beta_j; N))$  is the probability that a random walker on  $\mathbb{Z}_M$  that starts at  $[\alpha_i]$  will end at  $[\beta_j]$  after time  $N$ . Similarly  $\mathbf{GF}(\mathcal{P}(\vec{\alpha}; \vec{\beta}(\sigma); N))$  is the probability that  $n$  labeled vicious walkers (i.e., their paths do not intersect) on  $\mathbb{Z}_M$  which start at  $[\alpha_1], \dots, [\alpha_n]$  will end at  $[\beta_{\sigma(1)}], \dots, [\beta_{\sigma(n)}]$ , respectively. By Donsker’s theorem [Durrett (2010)] the path of a random walk converges to the path of Brownian motion in the sense of weak convergence as the step length becomes small and the number of steps becomes large. Similarly, the paths of  $n$  vicious walkers on the circle converge to the paths of NIBM in the weak sense. A rigorous proof of this intuitively clear convergence result, together with a bound of convergence rate, is given by Baik and Suidan (2007) in the setting of nonintersecting Brownian motion on the real line. We do not repeat the proof here. One consequence of the convergence is the following convergence of the transition probability density. Let  $M, N \rightarrow \infty$  such that

$$(56) \quad \frac{\alpha_i}{M} \rightarrow \frac{a_i}{2\pi}, \quad \frac{\beta_i}{M} \rightarrow \frac{b_i}{2\pi}, \quad \frac{N}{M^2} \rightarrow \frac{t}{4\pi^2 n},$$

and the arrays of  $a_i$ ’s and  $b_i$ ’s are distinct, respectively. Then

$$(57) \quad \begin{aligned} & \frac{M}{4\pi} \mathbf{GF}(\mathcal{P}(\alpha_i; \beta_j; N)) \rightarrow P(a_i, b_j; t) \quad \text{and} \\ & \left(\frac{M}{4\pi}\right)^n \mathbf{GF}(\mathcal{P}(\vec{\alpha}; \vec{\beta}(\sigma); N)) \rightarrow P(\vec{a}; \vec{b}(\sigma); t), \end{aligned}$$

and the discrete identity (57) implies (49) as the continuous limit.

We now introduce the phase parameter  $\tau$ , and consider

$$(58) \quad x = \frac{1}{2}e^{-(2\pi i/M)\tau}, \quad y = \frac{1}{2}e^{(2\pi i/M)\tau}.$$

To analyze the information carried by  $\tau$ , we recall the *offset* of the trajectory of a particle moving on  $\mathbb{T}$ . Suppose a particle  $\theta$  moves on  $\mathbb{T}$  such that  $\theta(t_1) = e^{ai}$  and  $\theta(t_2) = e^{bi}$  where  $a, b \in [-\pi, \pi)$ , and the trajectory of  $\theta$  is expressed as  $\theta(t) = e^{ix(t)}$  where  $x(t) : [t_1, t_2] \rightarrow \mathbb{R}$  is continuous for  $t \in [t_1, t_2]$ . Then the offset of the trajectory of  $\theta$  is defined as  $[(x(t_2) - x(t_1)) - (b - a)]/(2\pi)$ . If  $a = b$ , the offset is more commonly called the *winding number*. To consider the path on the lattice  $\mathbb{Z}_M \times \mathbb{Z}$ , we identify the first coordinate  $[m_1] \in \mathbb{Z}_M$  as the discrete point  $e^{2m_1\pi i/M}$  on  $\mathbb{T}$ , and consider the second coordinate  $m_2 \in \mathbb{Z}$  as the discrete time  $4\pi^2nm_2/M^2$ . Then a path on the lattice connecting  $([\alpha_i], 0)$  and  $([\beta_j], N)$  is identified as a trajectory of a particle  $\theta$  on  $\mathbb{T}$  such that  $\theta(0) = e^{2\alpha_i\pi i/M}$ ,  $\theta(4\pi^2nN/M^2) = e^{2\beta_j\pi i/M}$ , and  $\theta(t) = e^{ix(t)}$  where  $x(t)$  is continuous on  $[0, 4\pi^2nN/M^2]$ . Furthermore, we can require  $x(0) = \frac{2\alpha_i\pi}{M}$  and  $x(4\pi^2nN/M^2) = \frac{2\beta_j\pi}{M} + 2\pi o$  where  $o \in \mathbb{Z}$ . Then we say that  $o$  is the offset of the path.

Express

$$(59) \quad \mathcal{P}(\alpha_i; \beta_j; N) = \bigcup_{o \in \mathbb{Z}} \mathcal{P}_o(\alpha_i; \beta_j; N),$$

where

$$(60) \quad \mathcal{P}_o(\alpha_i; \beta_j; N) = \{\text{paths connecting } ([\alpha_i], 0) \text{ and } ([\beta_j], N) \text{ with offset } o\}.$$

Then the paths in  $\mathcal{P}_o(\alpha_i; \beta_j; N)$  on the lattice  $\mathbb{Z}_M \times \mathbb{Z}$  have a canonical 1–1 correspondence with paths on  $\mathbb{Z} \times \mathbb{Z}$  that connect  $(\alpha_i, 0)$  and  $(\beta_j + oM, N)$  and are made of adjacent steps either to the left or to the right. Here, by steps to the left (resp., to the right), we mean edges connecting  $(m_1, m_2)$  and  $(m_1 - 1, m_2 + 1)$  [resp., edges connecting  $(m_1, m_2)$  and  $(m_1 + 1, m_2 + 1)$ ].

Letting

$$(61) \quad \mathbb{P}_o(\alpha_i; \beta_j; N) := \text{transition probability of random walk on } \mathbb{Z} \text{ from } \alpha_i \text{ to } \beta_j + oM \text{ after time } N,$$

we have that

$$(62) \quad \begin{aligned} \mathbf{GF}(\mathcal{P}(\alpha_i; \beta_j; N)) &= \sum_{o \in \mathbb{Z}} \mathbf{GF}(\mathcal{P}_o(\alpha_i; \beta_j; N)) \\ &= \sum_{o \in \mathbb{Z}} \mathbb{P}_o(\alpha_i; \beta_j; N) e^{(\beta_j - \alpha_i)2\tau\pi i/M + 2o\tau\pi i}. \end{aligned}$$

Consider  $n$  nonintersecting paths that connect  $([\alpha_i], 0)$  to  $([\beta_i], N)$ , respectively, for  $i = 1, \dots, n$ . We find that the total offset of these paths has to be  $kn$



( $k \in \mathbb{Z}$ ), since all the paths have the same offset. Similarly, letting  $\sigma = [\ell] \in \mathbb{Z}/n\mathbb{Z}$ , the total offset of  $n$  nonintersecting paths that connect  $([\alpha_i], 0)$  to  $([\beta_{\sigma(i)}], N)$ , respectively, for  $i = 1, \dots, n$  has to be  $kn + \ell$  ( $k \in \mathbb{Z}$ ). Similar to (59), we write for  $\sigma = [\ell]$ ,

$$(63) \quad \mathcal{P}(\vec{\alpha}; \vec{\beta}([\ell]); N) = \bigcup_{o \in n\mathbb{Z} + \ell} \mathcal{P}_o(\vec{\alpha}; \vec{\beta}([\ell]); N),$$

where

$$(64) \quad \mathcal{P}_o(\vec{\alpha}; \vec{\beta}([\ell]); N) := \{n\text{-tuples of nonintersecting paths connecting } ([\alpha_i], 0) \text{ to } ([\beta_{[\ell](i)}], N) \\ (i = 1, \dots, n) \text{ with total offset } o\}.$$

Then, similar to the paths in  $\mathcal{P}_o(\alpha_i; \beta_j; N)$ , the  $n$ -tuples of nonintersecting paths in  $\mathcal{P}_o(\alpha_1, \dots, \alpha_n; \beta_{[\ell](1)}, \dots, \beta_{[\ell](n)}; N)$  on the lattice  $\mathbb{Z}_M \times \mathbb{Z}$  have the canonical 1–1 correspondence with the  $n$ -tuples of paths  $(x_1(t), \dots, x_n(t))$  on  $\mathbb{Z} \times \mathbb{Z}$  such that they connect  $(\alpha_1, 0)$  to  $(\beta_{\ell+1} + knM, N), \dots, (\alpha_{n-\ell}, 0)$  to  $(\beta_n + knM, N), (\alpha_{n-\ell+1}, 0)$  to  $(\beta_1 + k(n+1)M, N), \dots, (\alpha_n, 0)$  to  $(\beta_\ell + k(n+1)M, N)$ , respectively, and satisfy  $x_n(t) - x_1(t) < M$  for all  $t = 0, \dots, N$ . Similar to (61), let us denote

$$(65) \quad \mathbb{P}_o(\vec{\alpha}; \vec{\beta}([\ell]); N) := \text{transition probability of } n \text{ vicious walkers } x_1(t), \dots, x_n(t) \\ \text{on } \mathbb{Z} \text{ such that } x_i(0) = \alpha_i, x_i(N) = \beta_{[\ell](i)} + \left\lfloor \frac{o+i-1}{n} \right\rfloor M \\ \text{and } x_n(t) - x_1(t) < M \text{ for all } t = 0, \dots, N.$$

Then, similar to (62), we have that

$$(66) \quad \mathbf{GF}(\mathcal{P}(\vec{\alpha}; \vec{\beta}([\ell]); N)) = \sum_{o \in n\mathbb{Z} + \ell} \mathbf{GF}(\mathcal{P}_o(\vec{\alpha}; \vec{\beta}([\ell]); N)) \\ = \sum_{o \in n\mathbb{Z} + \ell} \mathbb{P}_o(\vec{\alpha}; \vec{\beta}([\ell]); N) e^{\sum_{k=1}^n (\beta_k - \alpha_k) 2\tau\pi i / M + 2o\tau\pi i}.$$

Note that if  $n$  is even and  $[\ell] \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n$ , then for any  $k \in \mathbb{Z}$ ,  $\text{sgn}([\ell]) = (-1)^{kn+\ell}$ . Thus, by (62) and (66), the determinantal identity (55) implies

$$(67) \quad e^{\sum_{k=1}^n (\beta_k - \alpha_k) (2\tau\pi i / M)} \sum_{o \in \mathbb{Z}} \mathbb{P}_o(\vec{\alpha}; \vec{\beta}([o \bmod n]); N) (-1)^o e^{2o\tau\pi i} \\ = \sum_{o \in \mathbb{Z}} (-1)^o \mathbf{GF}(\mathcal{P}_o(\vec{\alpha}; \vec{\beta}([o \bmod n]); N)) \\ = \det \left( \sum_{o \in \mathbb{Z}} \mathbb{P}_o(\alpha_i; \beta_j; N) e^{(\beta_j - \alpha_i) (2\tau\pi i / M) + 2o\tau\pi i} \right)_{i,j=1}^n.$$

In the scaling limit  $M, N \rightarrow \infty$  given in (56) with distinct arrays of  $a_i$ 's and  $b_i$ 's, respectively, the random walk converges to Brownian motion with diffusion parameter  $n^{-1/2}$ . Therefore, analogous to (57) we obtain

$$(68) \quad \begin{aligned} \frac{M}{4\pi} \mathbb{P}_o(\alpha_i; \beta_j; N) &\rightarrow \frac{\sqrt{n}}{\sqrt{2\pi t}} e^{-n(b_j - a_i + 2o\pi)^2 / (2t)} \quad \text{and} \\ \left(\frac{M}{4\pi}\right)^n \mathbb{P}_o(\vec{\alpha}; \vec{\beta}([o \bmod n]); N) &\rightarrow P_o(A_n; B_n; t), \end{aligned}$$

where  $P_o(A_n; B_n; t)$  is the transition probability of NIBM with fixed offset  $o$ , defined as

$$(69) \quad \begin{aligned} &P_o(A_n; B_n; t) \\ &:= \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^n} \mathbb{P} \left( \begin{array}{c} n \text{ particles in NIBM start at } a_1, \dots, a_n \\ \text{and after time } t \text{ end in} \\ [b_1, b_1 + \Delta x), \dots, [b_n, b_n + \Delta x) \text{ with total offset } o \end{array} \right). \end{aligned}$$

Denote

$$(70) \quad P(A_n; B_n; t; \tau) := \det(P(a_i; b_j; t; \tau))_{i,j=1}^n,$$

where  $P(a; b; t; \tau)$  is defined in (3). We now take (67) in the scaling limit (56), and derive that if  $n$  is even

$$(71) \quad \begin{aligned} &e^{\sum_{k=1}^n (b_k - a_k) \tau i} \sum_{o \in \mathbb{Z}} P_o(A_n; B_n; t) (-1)^o e^{2o\tau\pi i} \\ &= e^{\sum_{k=1}^n (b_k - a_k) \tau i} P(A_n; B_n; t; \tau). \end{aligned}$$

With  $\tau = 1/2$ , (71) implies

$$(72) \quad P(A_n; B_n; t) = \sum_{o \in \mathbb{Z}} P_o(A_n; B_n; t) = P\left(A_n; B_n; t; \frac{1}{2}\right),$$

for  $n$  even. For  $n$  odd, we have a similar formula in (51), which can be written as

$$(73) \quad P(A_n; B_n; t) = \sum_{o \in \mathbb{Z}} P_o(A_n; B_n; t) = P(A_n; B_n; t; 0).$$

The two formulas (72) and (73) are combined to give Proposition 1.1.

In what follows we consider  $P(A_n; B_n; t; \tau)$  for a general  $\tau \in \mathbb{R}$ . To get the transition probability density for NIBM, we simply let  $\tau = 0$  or  $\tau = 1/2$  depending on the parity of the number of particles. One advantage of working with  $P(A_n; B_n; t; \tau)$  with general  $\tau$  is that  $P(A_n; B_n; t; \tau)$  is a generating function for  $P_o(A_n; B_n; t)$ . We call  $P(A_n; B_n; t; \tau)$  the  $\tau$ -deformed transition probability density of NIBM.

2.2.  $\tau$ -deformed reunion probability. Now we consider the limiting case that  $a_1, \dots, a_n$  are close to 0 and/or  $b_1, \dots, b_n$  are close to 0. In the case that  $a_i \rightarrow 0$  and  $b_i$  are fixed and distinct, by l'Hôpital's rule,

$$(74) \quad P(A_n; B_n; t; \tau) = \frac{\prod_{1 \leq j < k \leq n} (a_k - a_j)}{\prod_{j=0}^{n-1} j!} \times \det \left( \frac{d^{j-1}}{dx^{j-1}} P(x; b_k; t; \tau) \Big|_{x=0} \right) (1 + \mathcal{O}(\max(|a_i|))).$$

Similarly, in the case that  $b_i \rightarrow 0$  and  $a_i$  are fixed and distinct,

$$(75) \quad P(A_n; B_n; t; \tau) = \frac{\prod_{1 \leq j < k \leq n} (b_k - b_j)}{\prod_{j=0}^{n-1} j!} \times \det \left( \frac{d^{j-1}}{dx^{j-1}} P(a_k; x; t; \tau) \Big|_{x=0} \right) (1 + \mathcal{O}(\max(|b_i|))).$$

In the case that both  $a_i \rightarrow 0$  and  $b_i \rightarrow 0$ , we define

$$(76) \quad R_n(t; \tau) = \det \left( \frac{d^{j+k-2}}{dx^{j+k-2}} P(0; x; t; \tau) \Big|_{x=0} \right),$$

and have the  $\tau$ -deformed reunion probability

$$(77) \quad \begin{aligned} P(A_n; B_n; t; \tau) &= \frac{\prod_{1 \leq j < k \leq n} (a_j - a_k)(b_k - b_j)}{\prod_{j=0}^{n-1} j!^2} R_n(t; \tau) (1 + \mathcal{O}(\max(|a_i|, |b_i|))). \end{aligned}$$

The transition probability density  $P(A_n; B_n; t; \epsilon(n))$  of the particles in NIBM with starting point  $a_i \rightarrow 0$  and ending point  $b_i \rightarrow 0$  is called the *reunion probability* in Forrester, Majumdar and Schehr (2011). In Forrester, Majumdar and Schehr (2011), the *normalized reunion probability* is defined in the setting of our paper as

$$(78) \quad \tilde{G}_n(L) = \frac{(2\pi/L)^{2n^2} R_n(4\pi^2 n/L^2, \epsilon(n))}{\lim_{t \rightarrow 0} t^{n^2} R_n(nt, \epsilon(n))}.$$

Note that the normalized reunion probability is not real probability since it can exceed 1.

In our paper, we are interested in the  $\tau$ -deformed transition probability  $P(A_n; B_n; t; \tau)$  and  $R_n(t; \tau)$  because they contain information on the total winding number in NIBM with common starting point and the same common ending point. By (77), as  $a_1, \dots, a_n \rightarrow 0$  and  $b_1, \dots, b_n \rightarrow 0$ ,

$$(79) \quad \begin{aligned} P_\omega(A_n; B_n; t) &= \frac{\prod_{1 \leq j < k \leq n} (a_j - a_k)(b_k - b_j)}{\prod_{j=0}^{n-1} j!^2} e^{2\pi\epsilon(n)\omega i} R_{n,\omega}(t) \\ &\times (1 + \mathcal{O}(\max(|a_i|, |b_i|))), \end{aligned}$$

where  $R_{n,\omega}(t)$  is defined as

$$(80) \quad R_{n,\omega}(t) = \int_0^1 R_n(t; \tau) e^{-2\omega\tau\pi i} d\tau.$$

Note that the ratio

$$(81) \quad \frac{e^{2\pi\epsilon(n)\omega i} R_{n,\omega}(t)}{R_n(t; \epsilon(n))} = \lim_{\substack{a_1, \dots, a_n \rightarrow 0 \\ b_1, \dots, b_n \rightarrow 0}} \frac{P_\omega(A_n; B_n; t)}{P(A_n; B_n; t)},$$

is the probability that the total winding number of the  $n$  particles in NIBM starting at a common point and ending at the same common point is  $\omega$ .

To evaluate  $R_n(t; \tau)$  and the determinants on the right-hand sides of (74) and (75), we consider the Fourier series of entries of these determinants. Introduce the lattice

$$(82) \quad L_{n,\tau} := \left\{ \frac{k + \tau}{n} \mid k \in \mathbb{Z} \right\}.$$

By the Poisson resummation formula, we find

$$(83) \quad \begin{aligned} P(a; \theta; t; \tau) &= \frac{\sqrt{n}}{\sqrt{2\pi t}} \sum_{l \in \mathbb{Z}} e^{-n(\theta - a + 2l\pi)^2 / (2t)} e^{2l\pi\tau i} \\ &= \frac{\sqrt{n}}{\sqrt{2\pi t}} \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-n(\theta - a + 2\xi\pi)^2 / (2t)} e^{-2\pi i \xi(k - \tau)} d\xi \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-t(k - \tau)^2 / (2n)} e^{i(\theta - a)(k - \tau)} \\ &= \frac{1}{2\pi} \sum_{x \in L_{n,\tau}} e^{-tnx^2/2} e^{-inx(\theta - a)}. \end{aligned}$$

It follows that

$$(84) \quad \frac{d^j}{d\theta^j} P(a; \theta; t; \tau) = \frac{(-ni)^j}{2\pi} \sum_{x \in L_{n,\tau}} x^j e^{-tnx^2/2} e^{-inx(\theta - a)}.$$

Similarly,

$$(85) \quad P(\theta; b; t; \tau) = \frac{1}{2\pi} \sum_{x \in L_{n,\tau}} e^{-tnx^2/2} e^{inx(\theta - b)},$$

$$(86) \quad \frac{d^j}{d\theta^j} P(\theta; b; t; \tau) = \frac{(ni)^j}{2\pi} \sum_{x \in L_{n,\tau}} x^j e^{-tnx^2/2} e^{inx(\theta - b)},$$

and in particular

$$(87) \quad \frac{d^j}{d\theta^j} P(0; \theta; t; \tau) \Big|_{\theta=0} = \frac{(-ni)^j}{2\pi} \sum_{x \in L_{n,\tau}} x^j e^{-tnx^2/2}.$$

Now setting  $t = T$ , we find that

$$(88) \quad R_n(T; \tau) = (-1)^{n(n-1)/2} \frac{n^{n^2}}{(2\pi)^n} \mathcal{H}_n(T; \tau)$$

where  $\mathcal{H}_n(T; \tau) := \det \left( \frac{1}{n} \sum_{x \in L_{n,\tau}} x^{j+k-2} e^{-Tnx^2/2} \right)_{j,k=1}^n$ .

Note that  $\mathcal{H}_n(t; \tau)$  is the Hankel determinant with respect to the discrete measure on the lattice  $L_{n,\tau}$ ,

$$(89) \quad \frac{1}{n} \sum_{y \in L_{n,\tau}} e^{-Tny^2/2} \delta(x - y).$$

REMARK 2.1. Formula (88) was obtained in Forrester, Majumdar and Schehr (2011) and Schehr et al. (2013) with  $\tau = 0$  and more recently in Castillo and Dupic (2014) with  $\tau = \epsilon(n)$ . We note that the NIBM $_{0 \rightarrow T}$  model is related to Yang–Mills theory on the sphere, as shown in Forrester, Majumdar and Schehr (2011), and a similar formula was derived in the Yang–Mills theory setting in Douglas and Kazakov (1993) with  $\tau = \epsilon(n)$ .

By a standard result for Hankel determinants, we can express  $\mathcal{H}_n(T; \tau)$  using the discrete Gaussian orthogonal polynomials. Let  $p_{n,j}^{(T;\tau)}(x)$  be the monic polynomial of degree  $j$  that satisfies

$$(90) \quad \frac{1}{n} \sum_{x \in L_{n,\tau}} p_{n,j}^{(T;\tau)}(x) p_{n,k}^{(T;\tau)}(x) e^{-Tnx^2/2} = 0 \quad \text{if } j \neq k.$$

We then have [see e.g., Bleher and Liechty (2014), Proposition 2.2.2],

$$(91) \quad \mathcal{H}_n(T; \tau) = \prod_{j=0}^{n-1} h_{n,j}^{(T;\tau)},$$

where

$$(92) \quad h_{n,k}^{(T;\tau)} := \frac{1}{n} \sum_{x \in L_{n,\tau}} p_{n,k}^{(T;\tau)}(x)^2 e^{-Tnx^2/2}.$$

The orthogonal polynomials (90) satisfy the three term recurrence equation [see Szegő (1975)],

$$(93) \quad xp_{n,j}^{(T;\tau)}(x) = p_{n,j+1}^{(T;\tau)}(x) + \beta_{n,j}^{(T;\tau)} p_{n,j}^{(T;\tau)}(x) + (\gamma_{n,j}^{(T;\tau)})^2 p_{n,j-1}^{(T;\tau)}(x),$$

where  $\{\beta_{n,j}^{(T;\tau)}\}_{j=0}^\infty$  is a sequence of real constants, and

$$(94) \quad \gamma_{n,j}^{(T;\tau)} := \left( \frac{h_{n,j}^{(T;\tau)}}{h_{n,j-1}^{(T;\tau)}} \right)^{1/2}.$$

2.3.  $\tau$ -deformed multi-time correlation functions. Next, we consider the joint probability density of  $n$ -particles in NIBM at times  $t_1, \dots, t_m$  such that  $0 < t_1 < \dots < t_m < T$  with the initial condition that they start from the common position  $0 \in [-\pi, \pi) = \mathbb{T}$  at time 0 and end at the same common position at  $T$ . That is, we consider the joint probability density in  $\text{NIBM}_{0 \rightarrow T}$ . We also want to extract the information of joint probability density for each fixed total offset/winding number of the  $n$ -particles. Thus, we consider the  $\tau$ -deformed joint probability density function for the Brownian particles. This density function is the one given in (6) in the physical setting. In order to get the  $\tau$ -deformed version, we start with the discrete model as in Section 2.1.

Let  $N_0 = 0 < N_1 < \dots < N_m < N_{m+1} = N$  be even integers and  $\alpha_i^{(k)}$  be even integers for  $k = 0, \dots, m + 1$  and  $i = 1, \dots, n$  such that for all  $k = 0, \dots, m + 1$ ,

$$(95) \quad -\frac{M}{2} \leq \alpha_1^{(k)} < \alpha_2^{(k)} < \dots < \alpha_n^{(k)} < \frac{M}{2}.$$

Let  $\sigma_1, \dots, \sigma_{m+1} \in S_n$  be permutations. Denote  $\mathcal{P}(\vec{\alpha}^{(0)}; \vec{\alpha}^{(1)}(\sigma_1); \dots; \vec{\alpha}^{(m+1)}(\sigma_{m+1}); N_1; \dots; N_{m+1})$  be the set of  $n$ -tuples of nonintersecting paths  $(p_1, \dots, p_n)$  such that  $p_i$  connects  $([\alpha_i^{(0)}], 0), ([\alpha_{\sigma_1(i)}^{(1)}], N_1), \dots, ([\alpha_{\sigma_{m+1}(i)}^{(m+1)}], N_{m+1})$  successively, and denote  $\mathcal{P}^{(\sigma)}(\vec{\alpha}^{(0)}; \dots; \vec{\alpha}^{(m)}; \vec{\alpha}^{(m+1)}; N_1; \dots; N_{m+1})$  as the union of  $\mathcal{P}(\vec{\alpha}^{(0)}; \vec{\alpha}^{(1)}(\sigma_1); \dots; \vec{\alpha}^{(m)}(\sigma_m); \vec{\alpha}^{(m+1)}(\sigma); N_1; \dots; N_{m+1})$  for all  $\sigma_1, \dots, \sigma_m \in S_n$ . Note that we only need to consider cyclic permutations  $\sigma_k \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n$  due to the nonintersecting assumption. Using the Lindström–Gessel–Viennot formula repeatedly, we have, as a generalization of (55),

$$(96) \quad \begin{aligned} & \sum_{[\ell] \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n} \text{sgn}([\ell]) \mathbf{GF}(\mathcal{P}^{[\ell]}(\vec{\alpha}^{(0)}; \vec{\alpha}^{(1)}; \dots; \vec{\alpha}^{(1)}; \vec{\alpha}^{(m+1)}; N_1; \dots; N_{m+1})) \\ &= \sum_{\sigma_1, \dots, \sigma_m, [\ell] \in \mathbb{Z}/n\mathbb{Z} \subseteq S_n} \text{sgn}([\ell]) \mathbf{GF}(\mathcal{P}(\vec{\alpha}^{(0)}; \vec{\alpha}^{(1)}(\sigma_1); \dots; \vec{\alpha}^{(m)}(\sigma_m); \\ & \quad \vec{\alpha}^{(m+1)}([\ell]); N_1; \dots; N_{m+1})) \\ &= \prod_{k=1}^{m+1} \det(\mathbf{GF}(\mathcal{P}(\alpha_i^{(k-1)}; \alpha_j^{(k)}; N_k - N_{k-1})))_{i,j=1}^n. \end{aligned}$$

Let the weight for each step in (52) be given by  $x = e^{-2\pi\tau i/M}/2$  and  $y = e^{2\pi\tau i/M}/2$  as in (58). Similar to (65), suppose  $o = kn + \ell$  where  $\ell = 0, \dots, n - 1$ ,

we denote

$$\begin{aligned}
 & \mathbb{P}_o(\vec{\alpha}^{(0)}; \dots; \vec{\alpha}^{(m)}; \vec{\alpha}^{(m+1)}; N_1; \dots; N_m; N_{m+1}) \\
 & \quad := \text{transition probability of } n \text{ vicious walkers} \\
 & \quad x_1(t), \dots, x_n(t) \text{ on } \mathbb{Z} \text{ such that } x_i(0) = \alpha_i^{(0)}, \\
 (97) \quad & x_i(N_{m+1}) = \alpha_{[\ell](i)}^{(m+1)} + \left[ \frac{o+i-1}{n} \right] M, \\
 & x_i(N_j) = \alpha_i^{(j)} + c_i^{(j)} M \text{ for some } l = 1, \dots, n \text{ and } c_i^{(j)} \in \mathbb{Z}, \\
 & \text{and } x_n(t) - x_1(t) < M \text{ for all } t = 0, \dots, N.
 \end{aligned}$$

Then, similar to (66), we have

$$\begin{aligned}
 & \mathbf{GF}(\mathcal{P}^{[\ell]}(\vec{\alpha}^{(0)}; \vec{\alpha}^{(1)}; \dots; \vec{\alpha}^{(1)}; \vec{\alpha}^{(m+1)}; N_1; \dots; N_{m+1})) \\
 (98) \quad & = \sum_{o \in n\mathbb{Z} + \ell} \mathbb{P}_o(\vec{\alpha}^{(0)}; \dots; \vec{\alpha}^{(m)}; \vec{\alpha}^{(m+1)}; N_1; \dots; N_{m+1}) \\
 & \quad \times e^{\sum_{k=1}^n (\alpha_k^{(m+1)} - \alpha_k^{(0)}) (2\tau\pi i / M) + 2o\tau\pi i}.
 \end{aligned}$$

In the limit that  $M, N \rightarrow \infty$  such that analogous to (56),

$$(99) \quad \frac{\alpha_i^{(j)}}{M} \rightarrow \frac{a_i^{(j)}}{2\pi}, \quad \frac{N_j}{M^2} \rightarrow \frac{t_j}{4\pi^2 n},$$

where  $0 = t_0 < t_1 < \dots < t_{m+1} = T$ , and  $-\pi \leq a_1^{(j)} < \dots < a_n^{(j)} < \pi$  for each  $j = 0, \dots, m + 1$ , we obtain, similar to (68),

$$\begin{aligned}
 (100) \quad & \left( \frac{M}{4\pi} \right)^{mn} \mathbb{P}_o(\vec{\alpha}^{(0)}; \dots; \vec{\alpha}^{(m)}; \vec{\alpha}^{(m+1)}; N_1; \dots; N_{m+1}) \\
 & \rightarrow P_o(A^{(0)}; \dots; A^{(m+1)}; t_1; \dots; t_{m+1}),
 \end{aligned}$$

where

$$\begin{aligned}
 & P_o(A^{(0)}; \dots; A^{(m+1)}; t_1; \dots; t_{m+1}) \\
 (101) \quad & := \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^{mn}} \\
 & \quad \times \mathbb{P} \left( \begin{array}{l} n \text{ particles in NIBM start at } a_1^{(0)}, \dots, a_n^{(0)} \text{ at time } 0, \\ \text{stay in } [a_1^{(k)}, a_1^{(k)} + \Delta x), \dots, [a_n^{(k)}, a_n^{(k)} + \Delta x) \text{ at time } t_k \\ (k = 1, \dots, m + 1) \text{ with total offset } o \text{ at time } t_{m+1}. \end{array} \right).
 \end{aligned}$$

Thus, similar to (71), equations (98) and (96) imply that the  $\tau$ -deformed joint transition probability density of  $n$  particles in NIBM is [here  $\epsilon(n)$  accommodates both

even and odd  $n$ ]

$$\begin{aligned}
 (102) \quad & \sum_{o \in \mathbb{Z}} P_o(A^{(0)}; \dots; A^{(m+1)}; t_1; \dots; t_{m+1}) e^{2\pi \epsilon(n)oi} e^{2o\tau \pi i} \\
 &= \prod_{j=1}^{m+1} P(A^{(j-1)}; A^{(j)}; t_j - t_{j-1}; \tau),
 \end{aligned}$$

where  $P(A^{(j-1)}; A^{(j)}; t_j - t_{j-1}; \tau)$  is defined by (70) with  $A_n, B_n$  replaced by  $A^{(j-1)}, A^{(j)}$ . Letting  $\tau = \epsilon(n)$ , we have the *joint transition probability density* in NIBM, which is the sum of all  $P_o(A^{(0)}; \dots; A^{(m+1)}; t_1; \dots; t_{m+1})$ , expressed as

$$\begin{aligned}
 (103) \quad & \sum_{o \in \mathbb{Z}} P_o(A^{(0)}; \dots; A^{(m+1)}; t_1; \dots; t_{m+1}) \\
 &= \prod_{j=1}^{m+1} P(A^{(j-1)}; A^{(j)}; t_j - t_{j-1}; \epsilon(n)).
 \end{aligned}$$

In the limiting case  $a_i^{(0)} \rightarrow 0$  and/or  $a_i^{(m+1)} \rightarrow 0$ , we have the result similar to (74), (75) and (77). For  $\text{NIBM}_{0 \rightarrow T}$  we are interested in the ratio between the  $\tau$ -deformed transition probability density of the particles from  $A^{(0)}$  to  $A^{(1)}, \dots, A^{(m+1)}$  successively and the  $\tau$ -deformed transition probability (i.e., the  $\tau$ -deformed reunion probability) of the particles from  $A^{(0)}$  to  $A^{(m+1)}$ , as  $a_i^{(0)} \rightarrow 0, a_i^{(m+1)} \rightarrow 0$ . After changing the notation  $t_{m+1}$  into  $T$ , we have the  $\tau$ -deformed *joint probability density* in  $\text{NIBM}_{0 \rightarrow T}$ ,

$$\begin{aligned}
 (104) \quad & P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m; \tau) \\
 &:= \lim_{\substack{a_i^{(0)} \rightarrow 0, \\ a_i^{(m+1)} \rightarrow 0}} \frac{\prod_{j=1}^{m+1} P(A^{(j-1)}; A^{(j)}; t_j - t_{j-1}; \tau)}{P(A^{(0)}; A^{(m+1)}; t_{m+1}; \tau)} \\
 &= \frac{1}{R_n(T; \tau)} \det \left( \frac{d^{j-1}}{dx^{j-1}} P(x; a_k^{(1)}; t_1; \tau) \Big|_{x=0} \right)_{j,k=1}^n \\
 &\quad \times \det \left( \frac{d^{j-1}}{dx^{j-1}} P(a_k^{(m)}; x; T - t_m; \tau) \Big|_{x=0} \right)_{j,k=1}^n \\
 &\quad \times \prod_{j=2}^m P(A^{(j-1)}; A^{(j)}; t_j - t_{j-1}; \tau).
 \end{aligned}$$

Note that for any  $\tau$ , the denominator  $R_n(T; \tau)$  is a nonzero real number, by (88) and (92). With  $\tau = \epsilon(n)$ ,  $P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m; \epsilon(n))$  gives the *joint transition probability density* of particles in  $\text{NIBM}_{0 \rightarrow T}$ . With the help of Fourier



expansion,  $P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m; \tau)$  yields the conditional joint transition probability density with fixed total winding number. To be precise, we have

$$(105) \quad \frac{R_n(T; \tau)}{R_n(T, \epsilon(n))} P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m; \tau) = \sum_{\omega \in \mathbb{Z}} (P_{0 \rightarrow T})_\omega(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m) e^{2\pi\omega(\tau - \epsilon(n))i},$$

where

$$(106) \quad (P_{0 \rightarrow T})_\omega(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m) = \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^{mn}} \mathbb{P} \left( \begin{array}{l} n \text{ particles in NIBM}_{0 \rightarrow T} \text{ with total winding} \\ \text{number } \omega, \text{ there is a particle in} \\ [a_j^{(i)}, a_j^{(i)} + \Delta x] \text{ at time } t_i \end{array} \right).$$

Note that although  $P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m; \tau)$  may not be nonnegative-valued, it is normalized in the sense that total integral over all possible positions of  $a_j^{(k)}$  is 1.

By (104), we find that  $P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m; \tau)$  has properties similar to the joint probability density function of a determinantal process, and thus is characterized by a reproducing kernel. We apply the Eynard–Mehta theorem [Eynard and Mehta (1998)], to  $P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; t_1, \dots, t_m; \tau)$ , following the notational conventions in Borodin and Rains (2005).

Denote for  $k = 1, \dots, m - 1$  and  $j = 1, \dots, n$ ,

$$(107) \quad W_k(x, y) := P(x; y; t_{k+1} - t_k; \tau),$$

$$(108) \quad \phi_j(x) := \text{linear combination of } \left\{ \frac{d^l}{dy^l} P(y; x; t_1; \tau) \Big|_{y=0} \right\} \text{ for } l = 0, \dots, j - 1,$$

$$(109) \quad \psi_j(x) := \text{linear combination of } \left\{ \frac{d^l}{dy^l} P(x; y; T - t_m; \tau) \Big|_{y=0} \right\} \text{ for } l = 0, \dots, j - 1,$$

where we suppress the dependence on  $\tau$ , and the concrete formulas for  $\phi_j(x)$  and  $\psi_j(x)$  are to be fixed later in (118) and (129). Then we define the operator  $\Phi : L^2(\mathbb{T}) \rightarrow \ell^2(n)$  as

$$(110) \quad \Phi(f(\theta)) = \left( \int_{-\pi}^{\pi} f(\theta) \phi_1(\theta) d\theta, \dots, \int_{-\pi}^{\pi} f(\theta) \phi_n(\theta) d\theta \right)^T,$$

the operator  $\Psi : \ell^2(n) \rightarrow L^2(\mathbb{T})$  as

$$(111) \quad \Psi((v_1, \dots, v_n)^T) = \sum_{k=1}^n v_k \psi_k(\theta),$$

and define the operator  $W_k : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$  by the kernel function  $W_k(x, y)$  in (107). Furthermore, we define the operators

$$(112) \quad \begin{aligned} W_{[i,j]} &:= \begin{cases} W_i \cdots W_{j-1}, & \text{for } i < j, \\ \mathbf{1}, & \text{for } i = j, \\ 0, & \text{for } i > j, \end{cases} & \text{and} \\ \mathring{W}_{[i,j]} &:= \begin{cases} W_i \cdots W_{j-1}, & \text{for } i < j, \\ 0, & \text{for } i \geq j. \end{cases} \end{aligned}$$

We also define the operator  $M : \ell^2(n) \rightarrow \ell^2(n)$  as

$$(113) \quad M := \Phi W_{[1,m]} \Psi,$$

which is represented by the  $n \times n$  matrix

$$(114) \quad \begin{aligned} M_{ij} &= \int \cdots \int_{\mathbb{T}^m} \phi_i(\theta_1) W_1(\theta_1, \theta_2) \cdots \\ &\quad \times W_{m-1}(\theta_{m-1}, \theta_m) \psi_j(\theta_m) d\theta_1 \cdots d\theta_m. \end{aligned}$$

Then for any  $k_1, \dots, k_m \leq n$ , we define the  $\tau$ -deformed joint correlation function as

$$(115) \quad \begin{aligned} &R_{0 \rightarrow T}^{(n)}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m; \tau) \\ &= \prod_{j=1}^m \frac{n!}{(n - k_j)!} \\ &\quad \times \int_{[-\pi, \pi]^{mn - (k_1 + \dots + k_m)}} P_{0 \rightarrow T}(A^{(1)}, \dots, A^{(m)}; \\ &\quad \quad \quad t_1, \dots, t_m; \tau) da_{k_1+1}^{(1)} \cdots da_n^{(1)} da_{k_2+1}^{(2)} \cdots da_n^{(m)}, \end{aligned}$$

and the Eynard–Mehta theorem gives the determinantal formula

$$(116) \quad \begin{aligned} &R_{0 \rightarrow T}^{(n)}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m; \tau) \\ &= \det(K_{t_i, t_j}(a_{l_i}^{(i)}, a_{l'_j}^{(j)}))_{i,j=1, \dots, m, l_i=1, \dots, k_i, l'_j=1, \dots, k_j}, \end{aligned}$$

where the  $\tau$ -deformed correlation kernel is defined as

$$(117) \quad \begin{aligned} K_{t_i, t_j}(x, y) &= \tilde{K}_{t_i, t_j}(x, y) - \mathring{W}_{[i,j]} & \text{and} \\ \tilde{K}_{t_i, t_j}(x, y) &= W_{[i,m]} \Psi M^{-1} \Phi W_{[1,j]}. \end{aligned}$$

REMARK 2.2. The kernel  $K_{t_i, t_j}(x, y)$  depends on  $\tau$ , but we suppress it for notational simplicity. If we let  $\tau = \epsilon(n)$ , we obtain the correlation kernel for  $\text{NIBM}_{0 \rightarrow T}$  in (8).

Our next task is to find an expression for  $\tilde{K}_{t_i, t_j}(x, y)$  which is convenient for analysis. We note that by (83), (84) and (85),

$$(118) \quad \begin{aligned} \phi_j(x) &= \sum_{k \in \mathbb{Z} + \tau} f_{j-1}(k) e^{-t_1 k^2 / (2n)} e^{-ikx}, \\ \psi_j(x) &= \sum_{k \in \mathbb{Z} + \tau} g_{j-1}(k) e^{-(T-t_m)k^2 / (2n)} e^{ikx}, \end{aligned}$$

where  $f_i, g_i$  are polynomials of degree exactly  $i$  (with possibly complex coefficients). Note that  $W_j(x, y)$  depends only on  $x - y$ , and so we can write  $W_j(x, y) = h_j(x - y)$ . Thus, we see that  $W_j$  is a convolution operator,

$$(119) \quad (W_j f)(x) = \int_{-\pi}^{\pi} h_j(x - y) f(y) dy =: (h_j * f)(x),$$

where by (107) and (83),

$$(120) \quad h_j(x) = \sum_{k \in \mathbb{Z} + \tau} \hat{h}_j(k) e^{ikx}, \quad \hat{h}_j(k) = \frac{1}{2\pi} e^{-(t_{j+1} - t_j)k^2 / (2n)}.$$

Here, and in what follows, we use the notation  $\hat{h}(k)$  for the  $k$ th coefficient in the  $\tau$ -shifted Fourier series, defined by the first equation in (120). As with the usual Fourier series, we have that for  $i < j$ ,

$$(121) \quad W_{[i, j]}(x, y) = (h_i * h_{i+1} * \cdots * h_{j-1})(x - y),$$

where  $h_i * h_{i+1} * \cdots * h_{j-1}$  has the  $\tau$ -shifted Fourier series

$$(122) \quad (h_i * \cdots * h_{j-1})^\wedge(k) = (2\pi)^{j-i-1} \prod_{l=i}^{j-1} \hat{h}_l(k) = \frac{1}{2\pi} e^{-(t_j - t_i)k^2 / (2n)}.$$

Furthermore, as  $W_{[i, m]} \Psi$  is an operator from  $\ell^2(n)$  to  $L^2(\mathbb{T})$ , it is represented by an  $n$ -dimensional row vector. Its  $l$ th component is

$$(123) \quad (W_{[i, m]} \Psi)_l(x) = \int_{-\pi}^{\pi} W_{[i, m]}(x, y) \psi_l(y) dy = (h_i * \cdots * h_{m-1}) * \psi_l(x),$$

whose  $\tau$ -shifted Fourier series is

$$(124) \quad \begin{aligned} & ((\widehat{W_{[i, m]} \Psi})_l)(k) \\ &= ((h_i * \cdots * h_{m-1}) * \psi_l)^\wedge(k) = 2\pi (h_i * \cdots * h_{m-1})^\wedge(k) \hat{\psi}_l(k) \\ &= g_{l-1}(k) e^{-(T-t_i)k^2 / (2n)}. \end{aligned}$$

Similarly,  $\Phi W_{[1, j]}$  is an operator from  $L^2(\mathbb{T})$  to  $\ell^2(n)$ , and is then represented by an  $n$  dimensional column vector. Its  $l$ th component,

$$(125) \quad (\Phi W_{[1, j]})_l(x) = \int_{-\pi}^{\pi} \phi_l(y) W_{[1, j]}(y, x) dy,$$

satisfies

$$(126) \quad (\Phi W_{[1,j]})_l(-x) = \tilde{\phi}_l * (h_1 * \dots * h_{j-1})(x)$$

where  $\tilde{\phi}_l(x) = \phi_l(-x)$ ,

and the  $\tau$ -shifted Fourier series is

$$(127) \quad \begin{aligned} & ((\widehat{\Phi W_{[1,j]}})_l)(-k) \\ &= (\tilde{\phi}_l * (h_1 * \dots * h_{j-1}))^\wedge(k) = 2\pi \hat{\phi}_l(-k)(h_1 * \dots * h_{j-1})^\wedge(k) \\ &= f_{l-1}(k)e^{-t_j k^2/(2n)}. \end{aligned}$$

Also for the  $(i, j)$  entry of the matrix  $M$  defined in (113), we have

$$(128) \quad \begin{aligned} M_{ij} &= \int_{-\pi}^\pi \int_{-\pi}^\pi \phi_i(x) W_{[1,m]}(x, y) \psi_j(y) dx dy \\ &= (2\pi)^2 \sum_{k \in \mathbb{Z} + \tau} \hat{\phi}_j(-k)(h_1 * \dots * h_{m-1})^\wedge(k) \hat{\psi}_j(k) \\ &= 2\pi \sum_{k \in \mathbb{Z} + \tau} f_{i-1}(k) g_{j-1}(k) e^{-T k^2/(2n)}. \end{aligned}$$

To simplify the expression of  $\tilde{K}_{t_i, t_j}(x, y)$ , we fix the formula (118) for  $\phi_j(x)$  and  $\psi_j(x)$  as

$$(129) \quad f_j(k) = g_j(k) = p_{n,j}^{(T;\tau)}\left(\frac{k}{n}\right),$$

where  $p_{n,j}^{(T;\tau)}$  is the discrete Gaussian orthogonal polynomial defined in (90). Then (128) yields

$$(130) \quad M_{ij} = \begin{cases} 2\pi n h_{n,j}^{(T;\tau)}, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

where  $h_{n,j}^{(T;\tau)}$  is defined in (92). Thus,

$$(131) \quad \begin{aligned} \tilde{K}_{t_i, t_j}(x, y) &= \sum_{l=0}^{n-1} \left( \sum_{k \in \mathbb{Z} + \tau} g_l(k) e^{-(T-t_i)k^2/(2n)} e^{ikx} \right) \\ &\quad \times \frac{1}{2\pi n h_{n,l}^{(T;\tau)}} \left( \sum_{k \in \mathbb{Z} + \tau} f_l(k) e^{-t_j k^2/(2n)} e^{-iky} \right) \\ &= \frac{n}{2\pi} \sum_{k=0}^{n-1} \frac{1}{h_{n,k}^{(T;\tau)}} S_{k, T-t_i}(x) S_{k, t_j}(-y), \end{aligned}$$

where

$$(132) \quad S_{k,a}(x) = \frac{1}{n} \sum_{s \in L_{n,\tau}} p_{n,k}^{(T;\tau)}(s) e^{-ans^2/2} e^{ixns}.$$

At last, by (112), (120), (121), we have that

$$(133) \quad \mathring{W}_{[i,j]}(x, y) = \frac{1}{2\pi} \sum_{s \in L_{n,\tau}} e^{-(t_j-t_i)ns^2/2 - in(y-x)s}.$$

After arriving at a computable formula of  $R_{0 \rightarrow T}^{(n)}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m; \tau)$  defined in (116), we go back to examine its probabilistic meaning. The special choice that  $\tau = \epsilon(n)$  gives us the correlation function of the  $\text{NIBM}_{0 \rightarrow T}$ , namely

$$(134) \quad \begin{aligned} & R_{0 \rightarrow T}^{(n)}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m; \epsilon(n)) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^{k_1 + \dots + k_m}} \\ & \quad \times \mathbb{P} \left( \begin{array}{l} n \text{ particles in } \text{NIBM}_{0 \rightarrow T}, \text{ there is a particle in} \\ [a_j^{(i)}, a_j^{(i)} + \Delta x] \text{ for } j = 1, \dots, k_i \text{ at time } t_i \end{array} \right). \end{aligned}$$

Letting  $\tau$  vary, the Fourier coefficients of  $R_{0 \rightarrow T}^{(n)}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m; \tau)$  encode the correlation functions of particles in  $\text{NIBM}_{0 \rightarrow T}$  with fixed total winding number, so that

$$(135) \quad \begin{aligned} & \frac{R_n(T; \tau)}{R_n(T; \epsilon(n))} R_{0 \rightarrow T}^{(n)}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m; \tau) \\ &= \sum_{\omega \in \mathbb{Z}} (R_{0 \rightarrow T}^{(n)})_{\omega}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m) \\ & \quad \times e^{2\pi i(\tau + \epsilon(n))\omega}, \end{aligned}$$

where

$$(136) \quad \begin{aligned} & (R_{0 \rightarrow T}^{(n)})_{\omega}(a_1^{(1)}, \dots, a_{k_1}^{(1)}; \dots; a_1^{(m)}, \dots, a_{k_m}^{(m)}; t_1, \dots, t_m) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^{k_1 + \dots + k_m}} \\ & \quad \times \mathbb{P} \left( \begin{array}{l} n \text{ particles in } \text{NIBM}_{0 \rightarrow T} \text{ with total} \\ \text{winding number } \omega, \text{ there is a particle in} \\ [a_j^{(i)}, a_j^{(i)} + \Delta x] \text{ for } j = 1, \dots, k_i \text{ at time } t_i \end{array} \right). \end{aligned}$$

**3. Asymptotic results for discrete Gaussian orthogonal polynomials.** In this section, we state the asymptotic results for the discrete Gaussian orthogonal polynomials (90) which will be used in Sections 4 and 5. The results are derived from the interpolation problem and the corresponding Riemann–Hilbert problem associated with the discrete orthogonal polynomials, and the proofs are outlined in Section 6 unless otherwise stated.

3.1. *The equilibrium measure and the g-function.* A key ingredient in the Riemann–Hilbert analysis of orthogonal polynomials is the equilibrium measure associated with the weight function. The equilibrium measure associated with the weight  $e^{-nTx^2/2}$  for the discrete Gaussian orthogonal polynomials defined on the lattice  $L_{n,\tau}$  is the unique probability measure which minimizes the functional,

$$(137) \quad H(\nu) = \int \int \log \frac{1}{|x - y|} d\nu(x) d\nu(y) + \int \frac{Tx^2}{2} d\nu(x),$$

over the set of probability measures  $\nu$  on  $\mathbb{R}$  satisfying

$$(138) \quad d\nu(x) \leq dx,$$

where  $dx$  denotes the differential with respect to Lebesgue measure. It is well known [Kuijlaars (2000)] that there is a unique solution to (137) satisfying (138), and we call it the *equilibrium measure* for the discrete Gaussian orthogonal polynomials. The upper constraint (138) implies that the equilibrium measure is absolutely continuous with respect to Lebesgue measure and, therefore, has an associated density. Let us denote this density by  $\rho_T(x)$ .

We define the  $g$ -function associated with the discrete Gaussian orthogonal polynomials as the log transform of the equilibrium measure:

$$(139) \quad g(z) := \int \log(z - x) \rho_T(x) dx,$$

where we take the principal branch for the logarithm. Then the Euler–Lagrange variational conditions for the equilibrium problem (137) are

$$(140) \quad g_+(x) + g_-(x) - \frac{Tx^2}{2} - l \begin{cases} = 0, & \text{if } 0 < \rho_T(x) < 1, \\ \leq 0, & \text{if } \rho_T(x) = 0, \\ \geq 0, & \text{if } \rho_T(x) = 1, \end{cases}$$

where  $g_+$  and  $g_-$  refer to the limiting values from the upper and lower half-planes, respectively, and  $l \in \mathbb{R}$  is a constant Lagrange multiplier. Since the external potential  $Tx^2/2$  is convex and even, the equilibrium measure is supported on a single interval  $[-\beta, \beta]$ . We have for all  $x \in (-\infty, \beta)$ ,

$$(141) \quad g_+(x) - g_-(x) = 2\pi i \int_x^\beta \rho_T(x).$$

Without the upper constraint (138), it is well known that the solution  $\nu_T$  to the minimization problem (137) is given by the Wigner semicircle law [Deift (1999), Section 6.7]. That is,  $\nu_T$  is supported on a single interval  $[-\beta, \beta]$  and

$$(142) \quad d\nu_T(x) = \rho_T(x)\chi_{[-\beta, \beta]}(x) dx$$

$$\text{where } \beta = \frac{2}{\sqrt{T}}, \rho_T(x) = \frac{T}{2\pi} \sqrt{\frac{4}{T} - x^2}.$$

Clearly, this  $\rho_T(x)$  has its maximum value at  $x = 0$  and  $\rho_T(0) = \sqrt{T}/\pi$ . It follows that (142) satisfies the variational problem (137) with constraint (138) if and only if  $0 < T \leq \pi^2$ . We therefore denote the critical value  $T_c := \pi^2$  as in (9), and we have the following proposition.

**PROPOSITION 3.1.** *For  $T \leq T_c = \pi^2$ , the equilibrium measure for the discrete Gaussian orthogonal polynomials is given by the Wigner semicircle law (142).*

For  $T > T_c$ , the probability measure given by the Wigner semicircle law (142) does not satisfy the constraint (138). In this case the equilibrium measure is still supported on a single interval  $[-\beta, \beta]$ , but now there is a *saturated region*  $[-\alpha, \alpha]$ , where  $0 < \alpha < \beta$ , on which the density  $\rho_T(x)$  is identically 1. Since  $\rho_T(x)$  is an even function and has total integral 1, (141) then implies that for  $x \in (-\alpha, \alpha)$  we have

$$(143) \quad g_+(x) - g_-(x) = i\pi - 2\pi ix.$$

To present the solution of the minimization problem (137) and (138), we introduce a parameter  $k \in (0, 1)$  and use elliptic integrals with parameter  $k$ , defined as

$$(144) \quad F(z; k) = \int_0^z \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}, \quad E(z; k) = \int_0^z \frac{\sqrt{1-k^2s^2}}{\sqrt{1-s^2}} ds.$$

In the definitions of  $F(z; k)$  and  $E(z; k)$  we assume  $z \in \mathbb{C} \setminus \{(-\infty, 1) \cup (1, \infty)\}$ . We also use the complete elliptic integrals  $\mathbf{K}$  and  $\mathbf{E}$  defined in (16). Given any  $k \in (0, 1)$ , we express the endpoints of the support and saturated region of the equilibrium measure  $\alpha$  and  $\beta$  as

$$(145) \quad \beta = \beta(k) = (2\mathbf{E} - (1 - k^2)\mathbf{K})^{-1}, \quad \alpha = \alpha(k) = k\beta(k).$$

Note that by Erdélyi et al. (1981), Table 4 on page 319, and notation defined in (18),

$$(146) \quad \tilde{\mathbf{K}} = \mathbf{K}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\mathbf{K}(k),$$

$$\tilde{\mathbf{E}} = \mathbf{E}\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{2\mathbf{E}(k) - (1 - k^2)\mathbf{K}(k)}{1+k},$$

and so we have

$$(147) \quad \beta = \frac{1}{(1+k)\tilde{\mathbf{E}}}.$$

Using (146), we parametrize  $T$  by  $k$  as in (19),

$$(148) \quad T = T(k) = 4\mathbf{K}\beta^{-1} = 4\tilde{\mathbf{K}}\tilde{\mathbf{E}}.$$

By the following lemma, the parametrization is well defined.

LEMMA 3.2.  $\mathbf{K}(k)\mathbf{E}(k)$  is a strictly increasing function of  $k \in [0, 1)$  and

$$(149) \quad \lim_{k \rightarrow 0^+} \mathbf{K}(k)\mathbf{E}(k) = T_c = \pi^2, \quad \lim_{k \rightarrow 1} \mathbf{K}(k)\mathbf{E}(k) = +\infty.$$

Now we can state the result of the equilibrium measure for  $T > T_c$ .

PROPOSITION 3.3. For  $T > T_c = \pi^2$ ,  $T = T(k)$  is parametrized by  $k \in (0, 1)$  as in (148), and the equilibrium measure for the discrete Gaussian orthogonal polynomials is supported on a single interval  $[-\beta, \beta]$  with a saturated region  $[-\alpha, \alpha]$  where  $\beta = \beta(k)$  and  $\alpha = \alpha(k)$  are defined in (145). The density  $\rho_T(x)$  for the equilibrium measure is given by the formula

$$(150) \quad \rho_T(x) = \begin{cases} 1, & \text{if } x \in [-\alpha, \alpha], \\ \frac{2}{\pi\alpha} \left[ \mathbf{E} \int_x^\beta \frac{ds}{\sqrt{(\alpha^{-2}s^2 - 1)(1 - \beta^{-2}s^2)}} \right. \\ \quad \left. - \mathbf{K} \int_x^\beta \frac{\sqrt{1 - \beta^{-2}s^2}}{\sqrt{\alpha^{-2}s^2 - 1}} ds \right], & \text{if } x \in (\alpha, \beta), \\ \rho_T(-x), & \text{if } x \in (-\beta, -\alpha), \\ 0, & \text{otherwise.} \end{cases}$$

Note that for  $x \in (\alpha, \beta)$ , using formulas Gradshteyn and Ryzhik (2007), 3.152-10, page 280 and 3.169-17, page 309 and Byrd and Friedman (1971), 413.01, page 228,  $\rho_T(x)$  can be expressed in a more compact form

$$(151) \quad \begin{aligned} \rho_T(x) &= \frac{2}{\pi} \left[ (\mathbf{E} - \mathbf{K}) F \left( \sqrt{\frac{1 - x^2/\beta^2}{1 - k^2}}; k' \right) + \mathbf{K} E \left( \sqrt{\frac{1 - x^2/\beta^2}{1 - k^2}}; k' \right) \right] \\ &= \Lambda_0 \left( \sqrt{\frac{1 - x^2/\beta^2}{1 - k^2}}; k \right) \\ &= \frac{2}{\pi\beta x} \sqrt{(\beta^2 - x^2)(x^2 - \alpha^2)} \Pi_1 \left( -\frac{\alpha^2}{x^2}, k \right), \end{aligned}$$

where  $k' = \sqrt{1 - k^2}$ ,  $\Lambda_0(x; k)$  is the Heuman's Lambda function [see Byrd and Friedman (1971), 150.03, page 36, and note that our  $x$  corresponds to  $\sin \beta$  in



Byrd and Friedman (1971), 150.03, page 36], and the  $\Pi_1$  denotes the complete elliptic integral of the third kind [in the notational conventions of Erdélyi et al. (1981), Section 13.8 (3), page 317],

$$(152) \quad \Pi_1(v, k) = \int_0^1 \frac{dx}{(1 + vx^2)\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

The formulas (151) have appeared several times in the physics literature in the context of Yang–Mills theory [Douglas and Kazakov (1993), Gross and Matytsin (1995)].

In our asymptotic analysis of  $\text{NIBM}_{0 \rightarrow T}$ , the function  $g(z)$  defined in (139) plays an important role. In particular, we must use the derivative of this function to locate critical points. The following proposition gives an explicit formula of  $g'(z)$ .

PROPOSITION 3.4. For  $T \leq T_c$ ,

$$(153) \quad g'(z) = \frac{T}{2} \left( z - \sqrt{z^2 - \frac{4}{T}} \right),$$

and for  $T > T_c$ ,

$$(154) \quad \begin{aligned} g'(z) &= 2 \left[ \frac{\mathbf{K}}{\beta} z - \frac{\mathbf{K}}{\alpha} \int_0^z \frac{\sqrt{1 - \beta^{-2}s^2}}{\sqrt{1 - \alpha^{-2}s^2}} ds \right. \\ &\quad \left. + \frac{\mathbf{E}}{\alpha} \int_0^z \frac{ds}{\sqrt{(1 - \alpha^{-2}s^2)(1 - \beta^{-2}s^2)}} \mp \frac{\pi i}{2} \right] \\ &= 2 \left[ \frac{\mathbf{K}}{\beta} z - \mathbf{KE} \left( \frac{z}{\alpha}; k \right) + \mathbf{EF} \left( \frac{z}{\alpha}; k \right) \mp \frac{\pi i}{2} \right], \end{aligned}$$

for  $\pm \text{Im } z > 0$ .

Note that  $g(z)$  is single valued on  $(\beta, +\infty)$ . This is clear in (153), and we may write (154) in the form

$$(155) \quad g'(z) = \frac{2\mathbf{K}z}{\beta} - 2\mathbf{E}\beta \int_\beta^z \frac{ds}{\sqrt{(s^2 - \alpha^2)(s^2 - \beta^2)}} - \frac{2\mathbf{K}}{\beta} \int_\beta^z \frac{\sqrt{s^2 - \beta^2}}{\sqrt{s^2 - \alpha^2}} ds,$$

where the square roots are positive for  $s > \beta$  and have cuts on  $(-\beta, -\alpha) \cup (\alpha, \beta)$ .

With the notation defined in this section, we rewrite  $t^c$  defined in (20) for the supercritical case of  $\text{NIBM}_{0 \rightarrow T}$  as [by (143),  $g''(z)$  is well defined in a neighborhood of 0]

$$(156) \quad \begin{aligned} t^c := g''(0) &= \frac{T}{2} - \frac{2}{\alpha}(\mathbf{K} - \mathbf{E}) = \frac{2}{\alpha}(\mathbf{E} - (1 - k)\mathbf{K}) \\ &= \frac{(1 + k)^2}{k} \mathbf{E} \left( \frac{2\sqrt{k}}{1 + k} \right) \left( \mathbf{E} \left( \frac{2\sqrt{k}}{1 + k} \right) - \left( \frac{1 - k}{1 + k} \right)^2 \mathbf{K} \left( \frac{2\sqrt{k}}{1 + k} \right) \right). \end{aligned}$$

The formulas (153) and (154) can be integrated to obtain expressions for  $g(z)$ , where the constant of integration is determined by the condition  $g(z) \sim \log(z)$  as  $z \rightarrow \infty$ . Then the Lagrange multiplier  $l$  in (140) can be determined from the equality in (140). Although they are not indispensable in this paper, for completeness we present the formulas for  $g(z)$  and  $l$  below. In the subcritical case  $0 < T < T_c = \pi^2$ , explicit calculations give that

$$\begin{aligned}
 (157) \quad g(z) &= \frac{T}{4}z \left( z - \sqrt{z^2 - \frac{4}{T}} \right) - \log \left( z - \sqrt{z^2 - \frac{4}{T}} \right) \\
 &\quad - \frac{1}{2} + \log 2 - \log T \quad \text{and} \\
 e^l &= \frac{1}{Te}.
 \end{aligned}$$

In the supercritical case  $T > T_c$ , we present the formula for  $g(z)$  and the Lagrange multiplier in the following proposition.

PROPOSITION 3.5. *For  $T > T_c = \pi^2$  the function  $g(z)$  is given by*

$$\begin{aligned}
 (158) \quad g(z) &= zg'(z) - \frac{\mathbf{K}z^2}{\beta} + \frac{\mathbf{K}}{\beta} \sqrt{(z^2 - \beta^2)(z^2 - \alpha^2)} \\
 &\quad + \log(\sqrt{z^2 - \beta^2} + \sqrt{z^2 - \alpha^2}) + \frac{\mathbf{K}\beta}{2}(1 + k^2) - 1 - \log 2,
 \end{aligned}$$

where  $g'(z)$  is as in (155) and the principal branches are taken for the square roots and logarithms. The Lagrange multiplier  $l$  in the Euler–Lagrange variational conditions (140) is given by

$$(159) \quad l = \log(\beta^2 - \alpha^2) + \mathbf{K}\beta(1 + k^2) - 2(1 + \log 2).$$

PROOF. Using integration by parts, we have

$$(160) \quad g(z) = zg'(z) - \int zg''(z) dz + \text{const.}$$

The second term in this formula can be integrated directly using (155), and this determines  $g(z)$  up to the constant term, which is obtained by the condition  $g(z) \sim \log(z)$  as  $z \rightarrow \infty$ . This proves (158). To obtain (159), we use (140) at  $x = \beta$ , which implies

$$(161) \quad l = 2g(\beta) - \frac{T\beta^2}{2},$$

which we evaluate using (158).  $\square$

3.2. *Asymptotics of the discrete Gaussian orthogonal polynomials.* We now summarize the asymptotics of the discrete Gaussian orthogonal polynomials (90) and their discrete Cauchy transforms used in this paper. For a real function  $f(x)$ , define its discrete Cauchy transform  $Cf$  on the weighted lattice  $L_{n,\tau}$  as

$$(162) \quad Cf(z) := \frac{1}{n} \sum_{x \in L_{n,\tau}} \frac{f(x)e^{-(nT/2)x^2}}{z-x}.$$

In the subcritical case  $T < T_c$ , the discrete Gaussian orthogonal polynomials are exponentially close, as  $n \rightarrow \infty$ , to the rescaled Hermite polynomials, for which there are exact formulas. To present the asymptotics in the supercritical case, we first fix some notation. Define the function

$$(163) \quad \gamma(z) := \left( \frac{(z+\beta)(z-\alpha)}{(z-\beta)(z+\alpha)} \right)^{1/4},$$

with a cut on  $[-\beta, -\alpha] \cup [\alpha, \beta]$ , taking the branch such that  $\gamma(z) \sim 1$  as  $z \rightarrow \infty$ . Recall the elliptic nome  $q$  defined in (21) for  $T > T_c$ . We will use the Jacobi theta functions with elliptic nome  $q$ ,

$$(164) \quad \begin{aligned} \vartheta_3(z) &:= \vartheta_3(z; q) = 1 + 2 \sum_{j=1}^{\infty} q^{j^2} \cos(2jz), \\ \vartheta_4(z) &:= \vartheta_4(z; q) = 1 + 2 \sum_{j=1}^{\infty} (-1)^j q^{j^2} \cos(2jz). \end{aligned}$$

We will also use the notation  $\tilde{k}$ ,  $\tilde{\mathbf{K}}$  and  $\tilde{\mathbf{E}}$  defined in (17) and (18), as well as the function

$$(165) \quad u(z) := \frac{\pi(\alpha + \beta)}{4\tilde{\mathbf{K}}} \int_{\beta}^z \frac{dx}{\sqrt{(x^2 - \alpha^2)(x^2 - \beta^2)}}.$$

Fix some  $0 \leq \delta < 1$  and  $\varepsilon > 0$ . Define the domain  $D(\delta, \varepsilon, n)$  as

$$(166) \quad D(\delta, \varepsilon, n) = \{z \mid |z \pm \alpha| > \varepsilon, |z \pm \beta| > \varepsilon, |\operatorname{Im} z| > \varepsilon n^{-\delta}\}.$$

We then have the following proposition which describes the asymptotics of the discrete Gaussian orthogonal polynomials on the domain  $D(\delta, \varepsilon, n)$ .

**PROPOSITION 3.6.** *For any  $T > T_c$ , as  $n \rightarrow \infty$ , the discrete Gaussian orthogonal polynomials (90) satisfy*

$$(167) \quad \begin{aligned} p_{n,n}^{(T;\tau)}(z) &= e^{ng(z)} M_{11}(z)(1 + \operatorname{Er}_{11}(n, z)), \\ \frac{p_{n,n-1}^{(T;\tau)}(z)}{h_{n,n-1}^{(T;\tau)}} &= e^{n(g(z)-l)} M_{21}(z)(1 + \operatorname{Er}_{21}(n, z)), \end{aligned}$$

$$(168) \quad \begin{aligned} (Cp_{n,n}^{(T;\tau)})(z) &= e^{-n(g(z)-l)} M_{12}(z)(1 + \text{Er}_{12}(n, z)), \\ \frac{(Cp_{n,n}^{(T;\tau)})(z)}{h_{n,n-1}^{(T;\tau)}} &= e^{-ng(z)} M_{22}(z)(1 + \text{Er}_{22}(n, z)), \end{aligned}$$

where

$$(169) \quad M_{11}(z) = \frac{1}{2} \left( \gamma(z) + \frac{1}{\gamma(z)} \right) \frac{\vartheta_3(0)\vartheta_3(u(z) - \pi/4 - \pi(\tau - \epsilon(n)))}{\vartheta_3(\pi(\tau - \epsilon(n)))\vartheta_3(u(z) - \pi/4)},$$

$$(170) \quad M_{21}(z) = \frac{1}{4\pi} \left( \gamma(z) - \frac{1}{\gamma(z)} \right) \frac{\vartheta_3(0)\vartheta_3(u(z) + \pi/4 - \pi(\tau - \epsilon(n)))}{\vartheta_3(\pi(\tau - \epsilon(n)))\vartheta_3(u(z) + \pi/4)},$$

$$(171) \quad M_{12}(z) = \pi \left( \gamma(z) - \frac{1}{\gamma(z)} \right) \frac{\vartheta_3(0)\vartheta_3(u(z) + \pi/4 + \pi(\tau - \epsilon(n)))}{\vartheta_3(\pi(\tau - \epsilon(n)))\vartheta_3(u(z) + \pi/4)},$$

$$(172) \quad M_{22}(z) = \frac{1}{2} \left( \gamma(z) + \frac{1}{\gamma(z)} \right) \frac{\vartheta_3(0)\vartheta_3(u(z) - \pi/4 + \pi(\tau - \epsilon(n)))}{\vartheta_3(\pi(\tau - \epsilon(n)))\vartheta_3(u(z) - \pi/4)}.$$

These asymptotics are uniform in  $\tau$  and for  $z \in D(\delta, \epsilon, n)$  in the following sense. There exists a constant  $C(\epsilon) > 0$  such that for each  $0 < \delta < 1$ , the errors in (167) and (168) satisfy

$$(173) \quad \sup_{z \in D(\delta, \epsilon, n)} |\text{Er}_*(n, z)| < C(\epsilon)n^{-(1-\delta)} \quad \text{where } * = 11, 21, 12, 22.$$

A similar result with a weaker error holds in the critical case  $T = T_c + \mathcal{O}(n^{-2/3})$ . We have the following proposition.

PROPOSITION 3.7. Fix  $\epsilon > 0$  and  $0 \leq \delta < 1/3$ . For  $T = T_c(1 - 2^{-2/3}\sigma n^{-2/3})$ , the discrete Gaussian orthogonal polynomials (90) satisfy the asymptotics (167) in the domain  $\{z \mid |z \pm \beta| > \epsilon, |\text{Im } z| > \epsilon n^{-\delta}\}$ , where the function  $g(z)$  is defined in (157), the functions  $M_{11}(z)$  and  $M_{21}(z)$  are given by

$$(174) \quad \begin{aligned} M_{11}(z) &= \frac{1}{2} \left( \gamma(z) + \frac{1}{\gamma(z)} \right), & M_{21}(z) &= \frac{1}{4\pi} \left( \gamma(z) - \frac{1}{\gamma(z)} \right) \\ & & & \text{where } \gamma(z) = \left( \frac{z + \beta}{z - \beta} \right)^{1/4}, \end{aligned}$$

such that  $\beta$  is defined as in (142) and  $\gamma$  is defined with a cut  $[-\beta, \beta]$  and the branch  $\gamma(z) \sim 1$  as  $z \rightarrow \infty$ . The errors  $\text{Er}_{11}(n, z), \text{Er}_{21}(n, z)$  are of the order  $n^{-(1/3-\delta)}$ .

In the critical case, the asymptotic formulas for the discrete Gaussian orthogonal polynomials close to the origin are described in terms of the matrix function  $\Psi(\zeta, s)$  defined in (13) and (14). We do not describe these asymptotics in full generality, but do give the following formula for the Christoffel–Darboux kernel in a small neighborhood of the origin and a rough estimate of the orthogonal polynomials.

PROPOSITION 3.8. Fix  $\varepsilon > 0$  and  $0 < \delta < 1/3$ , and let  $T = T_c(1 - 2^{-2/3}\sigma n^{-2/3})$ . For all  $z, w \in \{z \in \mathbb{C} \mid |z| < \varepsilon n^{-\delta}\}$  the following asymptotic formula holds:

$$\begin{aligned}
 & e^{-(nT/4)(z^2+w^2)} \frac{P_{n,n}^{(T;\tau)}(z)P_{n,n-1}^{(T;\tau)}(w) - P_{n,n-1}^{(T;\tau)}(z)P_{n,n}^{(T;\tau)}(w)}{h_{n,n-1}^{(T;\tau)}(z-w)} \\
 (175) \quad &= \frac{1}{2\pi i(z-w)} \left( \frac{-e^{-i\pi(nz-\tau)}}{e^{i\pi(nz-\tau)}} \right)^T \Psi(dn^{1/3}z; \sigma)^{-1} \Psi(dn^{1/3}w; \sigma) \\
 & \quad \times \left( \frac{e^{i\pi(nw-\tau)}}{e^{-i\pi(nw-\tau)}} \right) (1 + \mathcal{O}(n^{-(1/3-\delta)})),
 \end{aligned}$$

where  $d = 2^{-5/3}\pi$  is defined in (33). Also the following estimate holds uniformly in  $\{z \in \mathbb{C} \mid |z| < \varepsilon n^{-\delta}\}$ :

$$(176) \quad p_{n,n}^{(T;\tau)}(z) = \mathcal{O}(e^{ng(z)}), \quad \frac{P_{n,n-1}^{(T;\tau)}(z)}{h_{n,n-1}^{(T;\tau)}} = \mathcal{O}(e^{n(g(z)-l)}).$$

Proposition 3.8 follows from the Riemann–Hilbert analysis of Liechty (2012). The fomula (175) appears in a slightly different form in Liechty (2012), equation (6.12).

We will also need asymptotic results for the discrete Gaussian orthogonal polynomials on  $\mathbb{R}$  outside of the support of the equilibrium measure. The following proposition extends the asymptotics of Proposition 3.6 to this region. The Cauchy transforms in (168) have poles on  $L_{n,\tau}$ , so we must exclude the points in this lattice from the formulation of the asymptotic result. Define the regions

$$\begin{aligned}
 (177) \quad & E(\varepsilon) = \{(-\infty, -\beta - \varepsilon] \cup [\beta + \varepsilon, \infty)\} \times [-i\varepsilon, i\varepsilon], \\
 & E(\varepsilon; n, \tau) = E(\varepsilon) \setminus \bigcup_{x \in L_{n,\tau}} \left\{ z \mid |z - x| < \frac{\varepsilon}{n} \right\}.
 \end{aligned}$$

Then we have a result parallel to Proposition 3.6.

PROPOSITION 3.9. Fix  $\varepsilon > 0$ . Then the asymptotics (167) are valid on  $E(\varepsilon)$ , and the asymptotics (168) are valid on  $E(\varepsilon; n, \tau)$ . In both cases, the errors are of the order  $n^{-1}$ .

The functions  $M_{11}(z), M_{21}(z), M_{12}(z)$ , and  $M_{22}(z)$  in Proposition 3.6 are entries of the  $2 \times 2$  matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix}^{-1} \mathbf{M}(z) \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix}$  as in (316), where  $\mathbf{M}(z)$  is defined in Section 6.2.1; see formula (314). By the Riemann–Hilbert problem satisfied by  $\mathbf{M}(z)$ , we have that  $\det \mathbf{M}(z) = 1$ , and so

$$(178) \quad M_{11}(z)M_{22}(z) - M_{12}(z)M_{21}(z) = 1,$$

for all  $z$  where they are defined. The jump condition for the  $2 \times 2$  matrix Riemann–Hilbert problem for  $\mathbf{M}(z)$  given in Section 6.2.1 implies that for  $x \in (-\alpha, \alpha)$ ,

$$(179) \quad (M_{11})_+(x) = (M_{11})_-(x)e^{2\pi i(\tau + \epsilon(n))},$$

$$(M_{21})_+(x) = (M_{21})_-(x)e^{2\pi i(\tau + \epsilon(n))},$$

$$(180) \quad (M_{12})_+(x) = (M_{12})_-(x)e^{-2\pi i(\tau + \epsilon(n))},$$

$$(M_{22})_+(x) = (M_{22})_-(x)e^{-2\pi i(\tau + \epsilon(n))}.$$

We now summarize the asymptotic formulas for the recurrence coefficients and the normalizing constants. In (183), we use the Jacobi elliptic function  $\operatorname{dn}(u, \tilde{k})$ ; see, for example, Whittaker and Watson (1996).

PROPOSITION 3.10. *As  $n \rightarrow \infty$  the recurrence coefficients  $(\gamma_{n,n}^{(T;\tau)})^2$  in (93) satisfy the following asymptotic formulas:*

(a) *In the subcritical case  $T < T_c = \pi^2$ ,*

$$(181) \quad (\gamma_{n,n}^{(T;\tau)})^2 = \frac{1}{T} + \mathcal{O}(e^{-cn}),$$

where  $c > 0$  is a constant which depends on  $T$ .

(b) *In the critical case  $T = T_c(1 - 2^{-2/3}\sigma n^{-2/3})$ , as  $n \rightarrow \infty$ ,*

$$(182) \quad (\gamma_{n,n}^{(T;\tau)})^2 = \frac{1}{T} \left( 1 - \frac{2^{5/3}}{n^{1/3}} q(\sigma) \cos(2\pi(\tau + \epsilon(n))) \right. \\ \left. + \frac{2^{4/3}}{n^{2/3}} q(\sigma)^2 \cos(4\pi\tau) + \mathcal{O}(n^{-1}) \right).$$

(c) *In the supercritical case  $T > T_c = \pi^2$ ,*

$$(183) \quad (\gamma_{n,n}^{(T;\tau)})^2 = \frac{\operatorname{dn}^2(2\tilde{\mathbf{K}}(\tau + 1/2 + \epsilon(n)), \tilde{k})}{4\tilde{\mathbf{E}}^2} + \mathcal{O}(n^{-1}).$$

The formula (181) states that in the subcritical case, the recurrence coefficients are exponentially close as  $n \rightarrow \infty$  to the recurrence coefficients for the rescaled Hermite polynomials; see, for example, Liechty (2012), Appendix B. The asymptotic formula (182) was proved in Liechty (2012), and formula (183) follows from the Riemann–Hilbert analysis presented in Section 6.

**4. Distribution of winding numbers.** In this section, we prove Theorem 1.2.

For the proof of this theorem, we will use the formulas (80) and (81). They state that the total winding number for  $n$  particles in  $\text{NIBM}_{0 \rightarrow T}$  is given by the formula

$$(184) \quad \mathbb{P}(\text{Total winding number equals } \omega) = e^{2\pi i \omega \epsilon(n)} \int_0^1 \frac{R_n(T; \tau) e^{-2\pi i \omega \tau}}{R_n(T; \epsilon(n))} d\tau,$$

which according to (88) is

$$\begin{aligned}
 \mathbb{P}(\text{Total winding number equals } \omega) &= e^{2\pi i \omega \epsilon(n)} \int_0^1 \frac{\mathcal{H}_n(T; \tau) e^{-2\pi i \omega \tau}}{\mathcal{H}_n(T; \epsilon(n))} d\tau \\
 (185) \qquad \qquad \qquad &= \int_0^1 \frac{\mathcal{H}_n(T; \tau - \epsilon(n)) e^{-2\pi i \omega \tau}}{\mathcal{H}_n(T; \epsilon(n))} d\tau.
 \end{aligned}$$

In order to evaluate this integral, we will use the following deformation equation for  $\mathcal{H}_n(T; \tau)$  with respect to  $\tau$ .

PROPOSITION 4.1. *The Hankel determinant  $\mathcal{H}_n(T; \tau)$  satisfies the differential equation*

$$(186) \qquad \frac{\partial^2}{\partial \tau^2} \log \mathcal{H}_n(T; \tau) = T^2 (\gamma_{n,n}^{(T;\tau)})^2 - T,$$

where the recurrence coefficient  $\gamma_{n,n}^{(T;\tau)}$  is defined in (93).

PROOF. Introducing a linear term into the exponent of the symbol for the Hankel determinant, we define

$$(187) \qquad \mathcal{H}_n(T; \tau; t) := \det \left( \frac{1}{n} \sum_{x \in L_{n,\tau}} x^{j+k-2} e^{-(nT/2)(x^2+2tx/n)} \right)_{j,k=1}^n,$$

and the monic orthogonal polynomials

$$(188) \qquad \frac{1}{n} \sum_{x \in L_{n,\tau}} p_{n,j}^{(T;\tau;t)}(x) p_{n,l}^{(T;\tau;t)}(x) e^{-(nT/2)(x^2+2tx/n)} = h_{n,j}^{(T;\tau;t)} \delta_{jl}.$$

It is well known then [see, e.g., Bleher and Liechty (2014), Theorem 2.4.3] that this Hankel determinant satisfies

$$(189) \qquad \frac{\partial^2}{\partial t^2} \log \mathcal{H}_n(T; \tau; t) = \frac{T^2 h_{n,n}^{(T;\tau;t)}}{h_{n,n-1}^{(T;\tau;t)}} = T^2 (\gamma_{n,n}^{(T;\tau;t)})^2,$$

$$\text{where } \gamma_{n,j}^{(T;\tau;t)} := \left( \frac{h_{n,j}^{(T;\tau;t)}}{h_{n,j-1}^{(T;\tau;t)}} \right)^{1/2}.$$

Completing the square in (187), we find that

$$\begin{aligned}
 \mathcal{H}_n(T; \tau; t) &= \det \left( \frac{e^{Tt^2/(2n)}}{n} \sum_{x \in L_{n,\tau}} x^{j+k-2} e^{-(nT/2)(x+t/n)^2} \right)_{j,k=1}^n \\
 (190) \qquad \qquad &= e^{Tt^2/2} \mathcal{H}_n(T; \tau + t; 0).
 \end{aligned}$$

Taking the logarithm and differentiating twice with respect to  $t$ , we obtain

$$(191) \quad \frac{\partial^2}{\partial t^2} \log \mathcal{H}_n(T; \tau; t) = T + \frac{\partial^2}{\partial t^2} \log \mathcal{H}_n(T; \tau + t; 0),$$

and combining (189) with (191) gives

$$(192) \quad \frac{\partial^2}{\partial t^2} \log \mathcal{H}_n(T; \tau + t; 0) = T^2(\gamma_{n,n}^{(T;\tau;t)})^2 - T.$$

Now replacing  $\partial^2/\partial t^2$  with  $\partial^2/\partial \tau^2$  on the left-hand side of (192) and plugging in  $t = 0$  gives (186), and the proposition is proved.  $\square$

We can now use this proposition to write an integral equation for the ratio in equation (185). For  $\epsilon(n) = 0$  or  $\epsilon(n) = 1/2$ , it is clear that  $\mathcal{H}_n(T, \tau)$  satisfies the symmetries

$$(193) \quad \mathcal{H}_n(T; \epsilon(n) + \tau) = \mathcal{H}_n(T; \epsilon(n) - \tau) = \mathcal{H}_n(T; \tau - \epsilon(n)).$$

Therefore, we have

$$(194) \quad \left. \frac{\partial}{\partial \tau} \log \mathcal{H}_n(T; \tau) \right|_{\tau=\epsilon(n)} = 0,$$

and then Proposition 4.1 implies the integral formula

$$(195) \quad \begin{aligned} \log \frac{\mathcal{H}_n(T; \tau - \epsilon(n))}{\mathcal{H}_n(T; \epsilon(n))} &= \log \frac{\mathcal{H}_n(T; \epsilon(n) + \tau)}{\mathcal{H}_n(T; \epsilon(n))} \\ &= \int_{\epsilon(n)}^{\epsilon(n)+\tau} \int_{\epsilon(n)}^u (T^2(\gamma_{n,n}^{(T;v)})^2 - T) dv du. \end{aligned}$$

*Subcritical case.* In the subcritical case  $T < T_c$  we can apply the asymptotic formula (181) for  $(\gamma_{n,n}^{(T;v)})^2$ . Then combining (185) and (195) gives (22).

*Supercritical case.* In the supercritical case  $T > T_c$ , We will use the notation  $\tilde{k}$ ,  $\tilde{\mathbf{K}}$  and  $\tilde{\mathbf{E}}$  introduced in (17) and (18), as well as the elliptic nome  $q$  introduced in (21). We apply the asymptotic formula (183) to the integral equation (195), giving

$$(196) \quad \begin{aligned} &\log \frac{\mathcal{H}_n(T, \tau - \epsilon(n))}{\mathcal{H}_n(T, \epsilon(n))} \\ &= \int_{\epsilon(n)}^{\epsilon(n)+\tau} \int_{\epsilon(n)}^u \left( \frac{T^2}{4\tilde{\mathbf{E}}^2} \operatorname{dn}^2 \left( 2\tilde{\mathbf{K}} \left( v + \frac{1}{2} + \epsilon(n) \right), \tilde{k} \right) - T \right) dv du + \mathcal{O}(n^{-1}) \\ &= \int_0^\tau \int_0^u \left( \frac{T^2}{4\tilde{\mathbf{E}}^2} \operatorname{dn}^2 \left( 2\tilde{\mathbf{K}} \left( v + \frac{1}{2} + 2\epsilon(n) \right), \tilde{k} \right) - T \right) dv du + \mathcal{O}(n^{-1}) \\ &= \int_0^\tau \int_0^u \left( \frac{T^2}{4\tilde{\mathbf{E}}^2} \operatorname{dn}^2 \left( 2\tilde{\mathbf{K}} \left( v + \frac{1}{2} \right), \tilde{k} \right) - T \right) dv du + \mathcal{O}(n^{-1}), \end{aligned}$$



where we use that  $\operatorname{dn}(u, \tilde{k})$  has period  $2\tilde{\mathbf{K}}$  as a function of  $u$  [Erdélyi et al. (1981), Table 5 on page 341]. Let us discuss how to compute the integral

$$(197) \quad \int_0^\tau \int_0^u \operatorname{dn}^2\left(2\tilde{\mathbf{K}}\left(v + \frac{1}{2}\right), \tilde{k}\right) dv du.$$

The inner integral can be written as

$$(198) \quad \frac{1}{2\tilde{\mathbf{K}}} \left[ \int_0^{2\tilde{\mathbf{K}}u + \tilde{\mathbf{K}}} \operatorname{dn}^2(t, \tilde{k}) dt - \int_0^{\tilde{\mathbf{K}}} \operatorname{dn}^2(t, \tilde{k}) dt \right].$$

The above integrals can be written in terms of the Jacobi Zeta function  $Z(u, \tilde{k})$  [Erdélyi et al. (1981), Section 13.16], which can be expressed by the Jacobi theta function as [Whittaker and Watson (1996), Sections 22.731, 21.11, 21.62],

$$(199) \quad Z(t, \tilde{k}) = \frac{\partial}{\partial t} \log \Theta(t) \quad \text{where } \Theta(t) = \vartheta_4\left(\frac{\pi t}{2\tilde{\mathbf{K}}}\right).$$

Using Erdélyi et al. (1981), Section 13.16, Formulas (12) and (14), we have

$$(200) \quad \int_0^u \operatorname{dn}^2(t, \tilde{k}) dt = Z(u, \tilde{k}) + \frac{\tilde{\mathbf{E}}}{\tilde{\mathbf{K}}} u,$$

and then

$$(201) \quad \int_0^u \operatorname{dn}^2(2\tilde{\mathbf{K}}(v + 1/2), \tilde{k}) dv = \frac{1}{2\tilde{\mathbf{K}}} [Z(2\tilde{\mathbf{K}}u + \tilde{\mathbf{K}}, \tilde{k}) + 2\tilde{\mathbf{E}}u],$$

where we have used that  $Z(\tilde{\mathbf{K}}, \tilde{k}) = 0$  by (199) and that  $\vartheta_4'(\pi/2) = 0$  [see Whittaker and Watson (1996), Section 21.11]. The integral (197) is thus

$$(202) \quad \begin{aligned} & \int_0^\tau \int_0^u \operatorname{dn}^2(2\tilde{\mathbf{K}}(v + 1/2), \tilde{k}) dv du \\ &= \frac{1}{2\tilde{\mathbf{K}}} \left[ \frac{1}{2\tilde{\mathbf{K}}} \int_{\tilde{\mathbf{K}}}^{2\tilde{\mathbf{K}}\tau + \tilde{\mathbf{K}}} Z(t, \tilde{k}) dt + \int_0^\tau 2\tilde{\mathbf{E}}u du \right]. \end{aligned}$$

Integrating the right-hand side of (202), we obtain

$$(203) \quad \begin{aligned} & \int_0^\tau \int_0^u \operatorname{dn}^2(2\tilde{\mathbf{K}}(v + 1/2), \tilde{k}) dv du \\ &= \frac{1}{2\tilde{\mathbf{K}}} \left[ \frac{1}{2\tilde{\mathbf{K}}} \log\left(\frac{\Theta(2\tilde{\mathbf{K}}\tau + \tilde{\mathbf{K}})}{\Theta(\tilde{\mathbf{K}})}\right) + \tilde{\mathbf{E}}\tau^2 \right]. \end{aligned}$$

Combining with (195) and (196), we obtain

$$(204) \quad \begin{aligned} & \log \frac{\mathcal{H}_n(T, \tau - \epsilon(n))}{\mathcal{H}_n(T, \epsilon(n))} \\ &= \frac{T^2}{8\tilde{\mathbf{K}}\tilde{\mathbf{E}}^2} \left[ \frac{1}{2\tilde{\mathbf{K}}} \log\left(\frac{\Theta(2\tilde{\mathbf{K}}\tau + \tilde{\mathbf{K}})}{\Theta(\tilde{\mathbf{K}})}\right) + \tilde{\mathbf{E}}\tau^2 \right] - \frac{T}{2}\tau^2 + \mathcal{O}(n^{-1}). \end{aligned}$$

The parametrization  $T = 4\tilde{\mathbf{K}}\tilde{\mathbf{E}}$  in (148) then implies [Whittaker and Watson (1996), Section 21.11],

$$\begin{aligned}
 (205) \quad \log \frac{\mathcal{H}_n(T, \tau - \epsilon(n))}{\mathcal{H}_n(T, \epsilon(n))} &= \log \left( \frac{\Theta(2\tilde{\mathbf{K}}\tau + \tilde{\mathbf{K}})}{\Theta(\tilde{\mathbf{K}})} \right) + \mathcal{O}(n^{-1}) \\
 &= \log \left( \frac{\vartheta_3(\pi\tau)}{\vartheta_3(0)} \right) + \mathcal{O}(n^{-1}).
 \end{aligned}$$

Then the Fourier series (164) for the function  $\vartheta_3$  and the identity [Whittaker and Watson (1996), Section 21.8],

$$(206) \quad \vartheta_3(0)^2 = \frac{2\tilde{\mathbf{K}}}{\pi},$$

imply (24).

*Critical case.* We now consider the critical case  $T = T_c(1 - 2^{-2/3}\sigma n^{-2/3})$ . In this part of the proof, we use the notation  $q(s)$  for the Hastings–McLeod solution to the Painlevé equation (11) and (12). Inserting the asymptotic formula (182) into this integral equation (195) yields

$$\begin{aligned}
 (207) \quad \log \frac{\mathcal{H}_n(T; \tau - \epsilon(n))}{\mathcal{H}_n(T; \epsilon(n))} &= T2^{4/3} \int_{\epsilon(n)}^{\epsilon(n)+\tau} \int_{\epsilon(n)}^u \left( \frac{2^{1/3}}{n^{1/3}} q(\sigma) \cos(2\pi(v + \epsilon(n))) \right. \\
 &\quad \left. + \frac{1}{n^{2/3}} q(\sigma)^2 \cos(4\pi v) + \mathcal{O}(n^{-1}) \right) dv du,
 \end{aligned}$$

which is integrated to obtain

$$\begin{aligned}
 (208) \quad \log \frac{\mathcal{H}_n(T; \tau - \epsilon(n))}{\mathcal{H}_n(T; \epsilon(n))} &= T2^{4/3} \left( -\frac{2^{1/3}q(\sigma)}{4\pi^2 n^{1/3}} (1 - \cos(2\pi\tau)) \right. \\
 &\quad \left. + \frac{q(\sigma)^2}{16\pi^2 n^{2/3}} (1 - \cos(4\pi\tau)) \right) + \mathcal{O}(n^{-1}).
 \end{aligned}$$

Using the scaling (10) for  $T$ , we find

$$\begin{aligned}
 (209) \quad \log \frac{\mathcal{H}_n(T; \tau - \epsilon(n))}{\mathcal{H}_n(T; \epsilon(n))} &= -\frac{q(\sigma)}{2^{1/3}n^{1/3}} (1 - \cos(2\pi\tau)) + \frac{2^{1/3}q(\sigma)^2}{8n^{2/3}} (1 - \cos(4\pi\tau)) + \mathcal{O}(n^{-1}),
 \end{aligned}$$

which we exponentiate to obtain

$$\begin{aligned}
 & \frac{\mathcal{H}_n(T; \tau - \epsilon(n))}{\mathcal{H}_n(T; \epsilon(n))} \\
 (210) \quad &= 1 - \frac{q(\sigma)}{2^{1/3}n^{1/3}}(1 - \cos(2\pi\tau)) + \frac{q(\sigma)^2}{2^{2/3}n^{2/3}}(1 - \cos(2\pi\tau)) + \mathcal{O}(n^{-1}),
 \end{aligned}$$

and the formulas (23) follow immediately from (185).

Theorem 1.2 is thus proved.

**5. Correlation function of particles.** In this section, we do asymptotic analysis to the  $\tau$ -deformed correlation kernel  $K_{t_i, t_j}(x, y)$  in (131), and prove Theorems 1.3 and 1.4 for the limiting behavior of  $\text{NIBM}_{0 \rightarrow T}$  in the critical and supercritical cases. In the critical case, we simply let  $\tau = \epsilon(n)$  and the asymptotics of  $K_{t_i, t_j}(x, y)$  gives Theorem 1.3(b); see Remark 2.2. In the supercritical case, we need the following technical result.

**THEOREM 5.1.** *Assume  $T > T_c$ . There exists  $d > 0$  defined in (235) such that when we scale  $t_i$  and  $t_j$  close to  $t^c$ , and  $x$  and  $y$  close to  $-\pi$  as in (31), the  $\tau$ -deformed correlation kernel  $K_{t_i, t_j}(x, y)$  has the limit independent of the parameter  $\tau$*

$$(211) \quad \lim_{n \rightarrow \infty} K_{t_i, t_j}(x, y) \Big| \frac{dy}{d\eta} \Big| = K_{-\tau_j, -\tau_i}^{\text{Pearcey}}(\eta, \xi).$$

Theorem 5.1 yields Theorem 1.3(a) as  $\tau = \epsilon(n)$ , while in Section 5.4 it is shown that Theorem 1.4 also follows from Theorem 5.1.

In Section 5.1, we lay out the contour integral formulas to do asymptotic analysis, and the supercritical and critical cases are undertaken in Sections 5.2 and 5.3, respectively. Throughout this section, we simplify the notation for the orthogonal polynomials (90) a bit, writing  $p_k(x)$  for  $p_{n,k}^{(T; \tau)}(x)$  when there is no possibility of confusion.

**5.1. Contour integral formula of the  $\tau$ -deformed correlation kernel.** First, we express the function  $S_{k,a}(x)$  defined in (132) in contour integral formulas that are convenient for asymptotic analysis. Under some circumstances, it is convenient to express  $S_{k,a}(x)$  by an integral over an infinite contour. Consider the function

$$(212) \quad P_{k,a}(z; x) = \pi p_k(z) e^{-anz^2/2} \frac{e^{i(x-\pi)nz+i\tau\pi}}{\sin(\pi nz - \tau\pi)}.$$

It is straightforward to check that  $P_{k,a}(z; x)$  has poles only at lattice points of  $L_{n,\tau}$ , and

$$(213) \quad \text{Res}_{z=s \in L_{n,\tau}} P_{k,a}(z; x) = \frac{1}{n} p_k(s) e^{-ans^2/2} e^{ixns}.$$

Since  $a$  is assumed to be positive,  $P_{k,a}(z; x)$  vanishes exponentially fast as  $z \rightarrow \infty$  in the direction 0 or direction  $\pi$ . Thus, if  $\Sigma^+$  is a contour in the upper half-plane

and from  $e^0 \cdot \infty$  to  $e^{\pi i} \cdot \infty$ , and  $\Sigma^-$  is a contour in the lower half-plane and from  $e^{\pi i} \cdot \infty$  to  $e^0 \cdot \infty$ , we have

$$(214) \quad S_{k,a}(x) = \frac{1}{2\pi i} \oint_{\Sigma} P_{k,a}(z; x) dz = \oint_{\Sigma} p_k(z) e^{-anz^2/2} \frac{e^{ixnz}}{e^{2\pi inz - 2\tau\pi i} - 1} dz$$

where  $\Sigma = \Sigma^+ \cup \Sigma^-$ .

Under some other circumstances, it is convenient to express  $S_{k,a}(x)$  as the sum of a contour integral and a remainder that is negligible in the asymptotic analysis. For any  $M > 0$  such that  $\pm M$  are not lattice points in  $L_{n,\tau}$ , we write

$$(215) \quad S_{k,a}(x) = \frac{1}{n} \sum_{\substack{s \in L_{n,\tau} \\ |s| \leq M}} p_k(s) e^{-ans^2/2} e^{ixns} + s_{k,a}^{(M)}(x)$$

where  $s_{k,a}^{(M)}(x) = \frac{1}{n} \sum_{\substack{s \in L_{n,\tau} \\ |s| > M}} p_k(s) e^{-ans^2/2} e^{ixns}$ .

Recall the discrete Cauchy transform  $Cp_k(z)$  defined in (162). Let  $\Gamma$  be a closed contour such that the part of  $L_{n,\tau}$ ,  $\{s \in L_{n,\tau} \mid |s| \leq M\}$  is enclosed in  $\Gamma$  while the rest of  $L_{n,\tau}$  is outside of  $\Gamma$ . By the calculation of residues,

$$(216) \quad S_{k,a}(x) = \frac{1}{2\pi i} \oint_{\Gamma} Cp_k(z) e^{(T-a)nz^2/2} e^{ixnz} dz + s_{k,a}^{(M)}(x).$$

Therefore, by (214) and (216), we can write (131) as

$$(217) \quad \begin{aligned} \tilde{K}_{t_i,t_j}(x, y) &= K_{t_i,t_j}^{\text{major}}(x, y) + K_{t_i,t_j}^{\text{minor}}(x, y) \\ &= K_{t_i,t_j}^{\text{major}}(x, y; M) + K_{t_i,t_j}^{\text{minor}}(x, y; M), \end{aligned}$$

where  $K_{t_i,t_j}^{\text{major}}(x, y; M)$  and  $K_{t_i,t_j}^{\text{minor}}(x, y; M)$  depend on the positive constant  $M$  which we suppress if there is no possibility of confusion. They are defined as

$$(218) \quad \begin{aligned} K_{t_i,t_j}^{\text{major}}(x, y) &= \frac{n}{4\pi^2 i} \oint_{\Gamma} dz \oint_{\Sigma} dw \left( \sum_{k=0}^{n-1} \frac{1}{h_{n,k}^{(T;\tau)}} Cp_k(z) p_k(w) \right) e^{t_i n z^2 / 2 - t_j n w^2 / 2} \\ &\quad \times \frac{-e^{ixnz - iynw}}{1 - e^{2\pi inw - 2\tau\pi i}}, \end{aligned}$$

$$(219) \quad \begin{aligned} K_{t_i,t_j}^{\text{minor}}(x, y) &= \frac{n}{2\pi} \oint_{\Sigma} dw \left( \sum_{k=0}^{n-1} \frac{1}{h_{n,k}^{(T;\tau)}} s_{k,T-t_i}^{(M)}(x) p_k(w) \right) e^{-t_j n w^2 / 2} \\ &\quad \times \frac{-e^{-iynw}}{1 - e^{2\pi inw - 2\tau\pi i}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \oint_{\Sigma} dw \sum_{\substack{s \in L_{n,\tau} \\ |s| > M}} \left( \sum_{k=0}^{n-1} \frac{1}{h_{n,k}^{(T;\tau)}} p_k(s) p_k(w) \right) e^{-(T-t_i)ns^2/2 - t_jnw^2/2} \\
 &\quad \times \frac{-e^{ixns - iynw}}{1 - e^{2\pi inw - 2\tau\pi i}}.
 \end{aligned}$$

In (218) and (219), we assume that the contour  $\Gamma$  is the same as in (216),  $\Sigma$  is the same as in (214), and  $\Gamma$  and  $\Sigma$  are disjoint.

Recall the well-known Christoffel–Darboux formula [Szegő (1975), Chapter 3.2]

$$\begin{aligned}
 (220) \quad &\sum_{k=0}^{n-1} \frac{1}{h_{n,k}^{(T;\tau)}} p_k(z) p_k(w) \\
 &= \frac{1}{h_{n,n-1}^{(T;\tau)}} \frac{p_n(z) p_{n-1}(w) - p_{n-1}(z) p_n(w)}{z - w}.
 \end{aligned}$$

We derive its straightforward variation

$$\begin{aligned}
 &\sum_{k=0}^{n-1} \frac{1}{h_{n,k}^{(T;\tau)}} C p_k(z) p_k(w) \\
 &= \sum_{k=0}^{n-1} \frac{1}{nh_{n,k}^{(T;\tau)}} \sum_{s \in L_{n,\tau}} \frac{p_k(s) e^{-Tns^2/2}}{z - s} p_k(w) \\
 &= \sum_{s \in L_{n,\tau}} \frac{1}{n} \frac{e^{-Tns^2/2}}{z - s} \sum_{k=0}^{n-1} \frac{1}{h_{n,k}^{(T;\tau)}} p_k(s) p_k(w) \\
 &= \sum_{s \in L_{n,\tau}} \frac{1}{h_{n,n-1}^{(T;\tau)}} \frac{p_n(s) p_{n-1}(w) - p_{n-1}(s) p_n(w)}{n(z - s)(s - w)} e^{-Tns^2/2} \\
 &= \frac{1}{h_{n,n-1}^{(T;\tau)}} \frac{1}{z - w} \sum_{s \in L_{n,\tau}} \frac{1}{n} \frac{p_n(s) e^{-Tns^2/2}}{z - s} p_{n-1}(w) \\
 &\quad - \frac{1}{n} \frac{p_n(s) e^{-Tns^2/2}}{w - s} p_{n-1}(w) \\
 &\quad - \frac{1}{n} \frac{p_{n-1}(s) e^{-Tns^2/2}}{z - s} p_n(w) \\
 &\quad + \frac{1}{n} \frac{p_{n-1}(s) e^{-Tns^2/2}}{w - s} p_n(w)
 \end{aligned}$$

(221)

$$= \frac{1}{h_{n,n-1}^{(T;\tau)}} \left( \frac{Cp_n(z)p_{n-1}(w) - Cp_{n-1}(z)p_n(w)}{z-w} - \frac{Cp_n(w)p_{n-1}(w) - Cp_{n-1}(w)p_n(w)}{z-w} \right).$$

Using (221) and (220) and noting that  $\oint_{\Gamma} \frac{dz}{z-w} (Cp_n(w)p_{n-1}(w) - Cp_{n-1}(w)p_n(w)) = 0$  for  $w \in \Sigma$ , we simplify (218) and (219) as

$$(222) \quad K_{t_i,t_j}^{\text{major}}(x,y) = \frac{n}{4\pi^2 i} \oint_{\Gamma} dz \oint_{\Sigma} dw \frac{1}{h_{n,n-1}^{(T;\tau)}} \frac{Cp_n(z)p_{n-1}(w) - Cp_{n-1}(z)p_n(w)}{z-w} \times e^{t_i n z^2 / 2 - t_j n w^2 / 2} \frac{-e^{ixnz - iynw}}{1 - e^{2\pi i n w - 2\tau \pi i}},$$

$$(223) \quad K_{t_i,t_j}^{\text{minor}}(x,y) = \frac{1}{2\pi} \oint_{\Sigma} dw \sum_{\substack{s \in L_{n,\tau} \\ |s| > M}} \frac{1}{h_{n,n-1}^{(T;\tau)}} \frac{p_n(s)p_{n-1}(w) - p_{n-1}(s)p_n(w)}{s-w} \times e^{-(T-t_i)ns^2 / 2 - t_j n w^2 / 2} \frac{-e^{ixns - iynw}}{1 - e^{2\pi i n w - 2\tau \pi i}}.$$

These formulas are convenient in the derivation of the Pearcey kernel. For the tacnode kernel, however, it is more convenient to write (131) in the form

$$(224) \quad \frac{n}{2\pi} \oint_{\Sigma} dz \oint_{\Sigma} dw e^{-(n/2)[t_j w^2 + (T-t_i)z^2]} e^{in(xz-yw)} \left( \sum_{k=0}^{n-1} \frac{p_k(z)p_k(w)}{h_{n,k}^{(T;\tau)}} \right) \times \frac{e^{2\pi i(nz-\tau)}}{(e^{2\pi i(nz-\tau)} - 1)(e^{2\pi i(nw-\tau)} - 1)} = \frac{n}{2\pi} \oint_{\Sigma} dz \oint_{\Sigma} dw e^{-(n/2)[t_j w^2 + (T-t_i)z^2]} e^{in(xz-yw)} \times \left( \frac{p_n(z)p_{n-1}(w) - p_{n-1}(z)p_n(w)}{h_{n,n-1}^{(T;\tau)}(z-w)} \right) \times \frac{e^{2\pi i(nz-\tau)}}{(e^{2\pi i(nz-\tau)} - 1)(e^{2\pi i(nw-\tau)} - 1)},$$

by (214) and (220), noting that the term  $e^{ixnz}$  can be replaced by  $e^{ixnz+2\pi i(nz-\tau)}$  in (214).

5.2. *Limiting Pearcey process.* In this subsection, we assume that  $t_i, t_j, x, y$  are defined by (31), and the parameter  $d$  in (31) is to be determined later in (239).

To evaluate  $K_{t_i, t_j}^{\text{major}}(x, y)$  in (222), we define some notation. We denote for any  $z \in \mathbb{C} \setminus (-\infty, \beta)$

$$(225) \quad \begin{aligned} I(z) &= g(z) - \frac{t_c z^2}{2} + i\pi z, \\ \tilde{I}(z) &= \begin{cases} g(z) - \frac{t_c z^2}{2} + i\pi z = I(z), & \text{if } \text{Im } z > 0, \\ g(z) - \frac{t_c z^2}{2} - i\pi z, & \text{if } \text{Im } z < 0. \end{cases} \end{aligned}$$

Although  $I(z)$  and  $\tilde{I}(z)$  are generally not well defined on the real line, we define

$$(226) \quad I(x) = \lim_{y \rightarrow 0^+} I(x + iy), \quad \tilde{I}(x) = \lim_{y \rightarrow 0^+} \tilde{I}(x + iy) \quad \text{for } x \in \mathbb{R}.$$

Note that by the relation (143) of  $g_+(x)$  and  $g_-(x)$  for  $x \in (-\alpha, \alpha)$ , the  $g$ -function defined on  $\mathbb{C}_+$  can be analytically continued to  $\mathbb{C}_-$  through the interval  $(-\alpha, \alpha)$ . This analytic continuation is well defined on  $\mathbb{C} \setminus ((-\infty, -\alpha) \cup (\alpha, \infty))$ , and we denote it as  $\tilde{g}(z)$ . By (143), we have

$$(227) \quad \tilde{g}(z) = \begin{cases} g(z), & \text{if } \text{Im } z > 0, \\ g(z) + i\pi - 2\pi iz, & \text{if } \text{Im } z < 0. \end{cases}$$

Thus, we can express  $\tilde{I}(z)$  as

$$(228) \quad \tilde{I}(z) = \begin{cases} \tilde{g}(z) - \frac{t_c z^2}{2} + i\pi z, & \text{if } \text{Im } z > 0, \\ \tilde{g}(z) - \frac{t_c z^2}{2} + i\pi z - i\pi, & \text{if } \text{Im } z < 0. \end{cases}$$

We also define the function  $F(z, w)$  for  $z, w \in \mathbb{C} \setminus \mathbb{R}$  as

$$(229) \quad \begin{aligned} F(z, w) &= \frac{e^{n(g(z)-g(w))}}{h_{n, n-1}^{(T; \tau)}} (C p_n(z) p_{n-1}(w) - C p_{n-1}(z) p_n(w)) \\ &\quad \times \frac{-1}{1 - e^{2\pi i n w - 2\tau \pi i}}. \end{aligned}$$

Then we write (222) as

$$(230) \quad \begin{aligned} K_{t_i, t_j}^{\text{major}}(x, y) &= \frac{n}{4\pi^2 i} \oint_{\Gamma} dz \oint_{\Sigma} dwe^{-nI(z)+n\tilde{I}(w)} \\ &\quad \times \frac{e^{n^{1/2}(d^2/2)(\tau_i z^2 - \tau_j w^2) - in^{1/4} d(\xi z - \eta w)}}{z - w} F(z, w). \end{aligned}$$

In Appendix A, we construct the contour  $\tilde{\Sigma}$  that intersects the real axis at 0 and lies above the real axis elsewhere, such that  $\text{Re } I(z)$  attains its global maximum

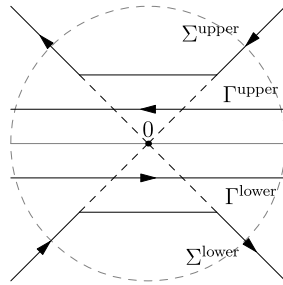


FIG. 4.  $\Sigma^{\text{upper}}$  and  $\Gamma^{\text{upper}}$  are deformed from  $\tilde{\Sigma}$  and  $\tilde{\Gamma}$ , respectively. The circled region is the neighborhood  $\{z \in \mathbb{C} \mid |z| < 4d^{-1}n^{-1/4}\}$ .

on  $\tilde{\Sigma}$  uniquely at 0, and construct the contour  $\tilde{\Gamma}$  that lies above or on the real axis, passes through 0, overlaps with the real axis in the vicinity of 0, starts at  $M$  and ends at  $-M$ , where  $M > \beta$  such that  $\text{Re } I(z)$  attains its global minimum on  $\tilde{\Gamma}$  uniquely at 0. We define  $\Sigma^{\text{upper}}$  by a deformation of  $\tilde{\Sigma}$  such that  $\Sigma^{\text{upper}}$  is identical to  $\tilde{\Sigma}$  outside of the region  $\{z \in \mathbb{C} \mid |z| < 4d^{-1}n^{-1/4}\}$ , and in this region the corner of  $\tilde{\Sigma}$  is leveled to be a horizontal base that is above 0 by  $2d^{-1}n^{-1/4}$ . We also define  $\Gamma^{\text{upper}}$  by a deformation of  $\tilde{\Gamma}$  as follows. First, we shift  $\tilde{\Gamma}$  upward by  $d^{-1}n^{-1/4}$ , and then connect the two end points of the shifted  $\tilde{\Gamma}$ , namely  $\pm M + id^{-1}n^{-1/4}$ , to  $\pm M$ , respectively, by vertical bars of length  $d^{-1}n^{-1/4}$ .  $\Gamma^{\text{upper}}$  is the result of the deformation. At last, we construct  $\Sigma^{\text{lower}}$  and  $\Gamma^{\text{lower}}$  by a reflection of  $\Sigma^{\text{upper}}$  and  $\Gamma^{\text{upper}}$ , respectively, about the real axis, and define

$$(231) \quad \Sigma = \Sigma^{\text{upper}} \cup \Sigma^{\text{lower}}, \quad \Gamma = \Gamma^{\text{upper}} \cup \Gamma^{\text{lower}},$$

with the orientation prescribed for  $\Sigma$  and  $\Gamma$ . See Figures 4 and 5. We assume, without loss of generality, that  $\pm M$  defined in Appendix A are not lattice points of  $L_{n,\tau}$ , otherwise we deform the contour around  $\pm M$  by  $\mathcal{O}(n^{-1})$ .

Then we denote

$$(232) \quad \begin{aligned} \Gamma_{\text{local}} &= \Gamma \cap N_{n^{-2/9}}(0), & \Sigma_{\text{local}} &= \Sigma \cap N_{n^{-2/9}}(0) \\ & & & \text{where } N_{n^{-2/9}}(0) = \{z \in \mathbb{C} \mid |z| < n^{-2/9}\}, \end{aligned}$$

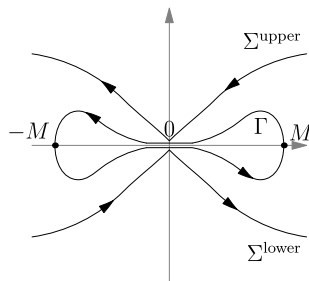


FIG. 5. Schematic figures of  $\Sigma$  and  $\Gamma$ . They are close at 0 but do not intersect.



and divide  $\Gamma_{\text{local}}$  and  $\Sigma_{\text{local}}$  into upper and lower parts, respectively, as

$$(233) \quad \begin{aligned} \Gamma_{\text{local}}^{\text{upper}} &= \Gamma_{\text{local}} \cap \mathbb{C}_+, & \Gamma_{\text{local}}^{\text{lower}} &= \Gamma_{\text{local}} \cap \mathbb{C}_-, \\ \Sigma_{\text{local}}^{\text{upper}} &= \Sigma_{\text{local}} \cap \mathbb{C}_+, & \Sigma_{\text{local}}^{\text{lower}} &= \Sigma_{\text{local}} \cap \mathbb{C}_-. \end{aligned}$$

By (167), (168), (178), (179) and (180), we have that for  $z \in \Gamma_{\text{local}}$  and  $w \in \Sigma_{\text{local}}$ ,

$$(234) \quad F(z, w) = \begin{cases} (1 + \mathcal{O}(n^{-3/4}))(1 + \mathcal{O}(|z|) + \mathcal{O}(|w|)), & \text{if } z \in \Gamma_{\text{local}}^{\text{upper}} \text{ and } w \in \Sigma_{\text{local}}^{\text{upper}}, \\ (-e^{2\tau\pi i} + \mathcal{O}(n^{-3/4}))(1 + \mathcal{O}(|z|) + \mathcal{O}(|w|)), & \text{if } z \in \Gamma_{\text{local}}^{\text{lower}} \text{ and } w \in \Sigma_{\text{local}}^{\text{lower}}, \\ ((-1)^n + \mathcal{O}(n^{-3/4}))(1 + \mathcal{O}(|z|) + \mathcal{O}(|w|)), & \text{if } z \in \Gamma_{\text{local}}^{\text{upper}} \text{ and } w \in \Sigma_{\text{local}}^{\text{lower}}, \\ ((-1)^n e^{4\tau\pi i} + \mathcal{O}(n^{-3/4}))(1 + \mathcal{O}(|z|) + \mathcal{O}(|w|)), & \text{if } z \in \Gamma_{\text{local}}^{\text{lower}} \text{ and } w \in \Sigma_{\text{local}}^{\text{upper}}. \end{cases}$$

Note that for  $z$  in the upper half-plane around 0, 0 is a triple zero of  $I'(z)$  by Lemma A.2, and the Taylor expansions of  $I(z)$  and  $\tilde{I}(z)$  are

$$(235) \quad I(z) = \tilde{I}(z) = I(0) + \frac{1}{24}\tilde{g}^{(4)}(0)z^4 + \mathcal{O}(z^5),$$

where  $\tilde{g}(z)$  is defined as the analytic continuation of  $g(z)$  across  $(-\alpha, \alpha)$  as defined in (227); see Lemma A.2. By (154),

$$(236) \quad \begin{aligned} \tilde{g}^{(4)}(0) &= \frac{1}{\alpha^3}((1 + k^2)\mathbf{E} - (1 - k^2)\mathbf{K}) \\ &= \frac{k^2}{\alpha^3} \int_0^1 \frac{(1 - s^2) + (1 - k^2s^2)}{\sqrt{(1 - s^2)(1 - k^2s^2)}} ds > 0. \end{aligned}$$

For  $z$  in the lower half-plane around 0, the Taylor expansion of  $I(z)$  and  $\tilde{I}(z)$  are, by (228) and (227),

$$(237) \quad \begin{aligned} \tilde{I}(z) &= I(0) - \pi i + \frac{1}{24}\tilde{g}^{(4)}(0)z^4 + \mathcal{O}(z^5), \\ I(z) &= I(0) - \pi i + 2\pi i z + \frac{1}{24}\tilde{g}^{(4)}(0)z^4 + \mathcal{O}(z^5). \end{aligned}$$

We make the change of variables

$$(238) \quad z = (\frac{1}{6}g^{(4)}(0))^{-1/4}n^{-1/4}u, \quad w = (\frac{1}{6}g^{(4)}(0))^{-1/4}n^{-1/4}v,$$

and let

$$(239) \quad d = (\frac{1}{6}g^{(4)}(0))^{1/4}.$$

Then by the Taylor expansions (235) and (237), for  $z \in \Gamma_{\text{local}}^{\text{upper}}$  and  $w \in \Sigma_{\text{local}}^{\text{upper}}$ ,

$$(240) \quad \begin{aligned} \tilde{I}(w) &= I(0) + \frac{1}{4n}v^4 + \mathcal{O}\left(\frac{v^5}{n^{5/4}}\right), \\ I(z) &= I(0) + \frac{1}{4n}u^4 + \mathcal{O}\left(\frac{u^5}{n^{5/4}}\right), \end{aligned}$$

and for  $z \in \Gamma_{\text{local}}^{\text{lower}}$  and  $w \in \Sigma_{\text{local}}^{\text{lower}}$ , noting that  $\text{Im } z = -d^{-1}n^{-1/4}$  for  $z \in \Gamma_{\text{local}}^{\text{lower}}$ ,

$$(241) \quad \begin{aligned} \tilde{I}(w) &= I(0) + \frac{1}{4n}v^4 + \mathcal{O}\left(\frac{v^5}{n^{5/4}}\right) - \pi i, \\ I(z) &= I(0) + \frac{2\pi}{dn^{1/4}} + \frac{1}{4n}u^4 + \mathcal{O}\left(\frac{u^5}{n^{5/4}}\right) + (2\pi \text{Re } z - \pi)i. \end{aligned}$$

By the asymptotics (240), (241) and (234), together with (27), we have that

$$(242) \quad \begin{aligned} &\oint_{\Gamma_{\text{local}}^{\text{upper}}} dz \oint_{\Sigma_{\text{local}}} dw e^{-nI(z)+n\tilde{I}(w)} \frac{e^{n^{1/2}(d^2/2)(\tau_i z^2 - \tau_j w^2) - in^{1/4}d(\xi z - \eta w)}}{z - w} F(z, w) \\ &= dn^{1/4} \oint_{\Gamma_{\text{local}}^{\text{upper}}} dz \oint_{\Sigma_{\text{local}}} dw \frac{e^{v^4/4 - (\tau_j/2)u^2 + i\eta v}}{e^{u^4/4 - (\tau_i/2)u^2 + i\xi u}} \\ &\quad \times \frac{1 + \mathcal{O}(u/n^{1/4}) + \mathcal{O}(v/n^{1/4})}{u - v} \\ &= \frac{1}{dn^{1/4}} \left( \oint_{\Gamma_P} du \oint_{\Sigma_P} dv \frac{e^{v^4/4 - (\tau_j/2)u^2 + i\eta v}}{e^{u^4/4 - (\tau_i/2)u^2 + i\xi u}} \frac{1}{u - v} + \mathcal{O}(n^{-1/4}) \right) \\ &= \frac{4\pi^2 i}{dn^{1/4}} (\tilde{K}_{-\tau_j, -\tau_i}^{\text{Pearcey}}(\eta, \xi) + \mathcal{O}(n^{-1/4})). \end{aligned}$$

On the other hand, from the comparison of formulas (240) and (241), the formula of  $\text{Re } I(z)$  on  $\Gamma_{\text{local}}^{\text{lower}}$  has a term  $2\pi/(dn^{1/4})$  that does not appear in the formula of  $\text{Re } I(z)$  on  $\Gamma_{\text{local}}^{\text{upper}}$ , we have

$$(243) \quad \begin{aligned} &\oint_{\Gamma_{\text{local}}^{\text{lower}}} dz \oint_{\Sigma_{\text{local}}} dw e^{-nI(z)+n\tilde{I}(w)} \frac{e^{n^{1/2}(d^2/2)(\tau_i z^2 - \tau_j w^2) - in^{1/4}d(\xi z - \eta w)}}{z - w} F(z, w) \\ &= \frac{4\pi^2 i}{dn^{1/4}} \mathcal{O}(e^{-2\pi n^{4/3}/d}). \end{aligned}$$

For  $z \in \Gamma_{\text{local}}^{\text{upper}} \setminus \Gamma_{\text{local}}^{\text{upper}}$  and  $w \in \Sigma_{\text{local}}^{\text{upper}} \setminus \Sigma_{\text{local}}^{\text{upper}}$ , by the property that  $\text{Re } I(z)$  attains its global minimum on  $\tilde{\Gamma}$  at 0 and  $\text{Re } \tilde{I}(z) = \text{Re } I(z)$  attains its global maximum on  $\tilde{\Sigma}$  at 0, and the local behavior of  $I(z) = \tilde{I}(z)$  at 0 in the upper half-plane,

we have that

$$(244) \quad \operatorname{Re} I(z) > \operatorname{Re} I(z_0) + \varepsilon n^{-8/9} \quad \text{for } z \in \Gamma^{\text{upper}} \setminus \Gamma_{\text{local}}^{\text{upper}},$$

$$(245) \quad \operatorname{Re} \tilde{I}(w) < \operatorname{Re} I(z_0) - \varepsilon n^{-8/9} \quad \text{for } w \in \Sigma^{\text{upper}} \setminus \Sigma_{\text{local}}^{\text{upper}}.$$

For  $z \in \Gamma^{\text{lower}}$  and  $w \in \Sigma^{\text{lower}}$ , on the other hand, by the formula (225) of  $I(z)$  and  $\tilde{I}(w)$  and the property that  $\operatorname{Re} g(z) = \operatorname{Re} g(\bar{z})$  that follows from the definition (139) of  $g(z)$ , we obtain that

$$(246) \quad \operatorname{Re} I(z) > \operatorname{Re} I(\bar{z}), \quad \operatorname{Re} \tilde{I}(w) = \operatorname{Re} \tilde{I}(\bar{w}) \quad \text{for } z, w \in \mathbb{C}_-,$$

and it applies for all  $w \in \Sigma^{\text{lower}}$  and  $z \in \Gamma^{\text{upper}}$  except for  $z = \pm M$ . Also we have the estimate for  $F(z, w)$  that for all  $z \in D(\delta, \varepsilon, n) \cup E(\varepsilon; n, \tau)$  and  $w \in D(\delta, \varepsilon, n) \cup E(\varepsilon)$ , where  $\delta \in [0, 1)$ ,  $\varepsilon > 0$  is a small positive number, and  $D(\delta, \varepsilon, n), E(\varepsilon), E(\varepsilon; n, \tau)$  are defined in (166) and (177), by Propositions 3.6 and 3.9,

$$(247) \quad F(z, w) = \mathcal{O}(1) \quad \text{if } z \in D(\delta, \varepsilon, n) \cup E(\varepsilon; n, \tau) \text{ and } w \in D(\delta, \varepsilon, n) \cup E(\varepsilon).$$

Then using (244), (245), (246) and (247), we have that for some  $\varepsilon > 0$

$$(248) \quad \oint_{\Gamma} dz \oint_{\Sigma} dwe^{-nI(z)+n\tilde{I}(w)} \frac{e^{n^{1/2}(d^2/2)(\tau_i z^2 - \tau_j w^2) - in^{1/4} d(\xi z - \eta w)}}{z - w} F(z, w) \\ = \oint_{\Gamma_{\text{local}}} dz \oint_{\Sigma_{\text{local}}} dwe^{-nI(z)+n\tilde{I}(w)} \\ \times \frac{e^{n^{1/2}(d^2/2)(\tau_i z^2 - \tau_j w^2) - in^{1/4} d(\xi z - \eta w)}}{z - w} F(z, w) + \frac{1}{dn^{1/4}} o(e^{-\varepsilon n^{1/9}}).$$

Next, we estimate  $K_{t_i, t_j}^{\text{minor}}(x, y)$ . Using the fact that  $\operatorname{Re} I(z)$  attains its global minimum on  $\tilde{\Gamma}$  at 0 and  $\pm M$  are the ends of  $\tilde{\Gamma}$ , there is a  $c_1 > 0$  such that

$$(249) \quad \operatorname{Re} \tilde{I}(0) = \operatorname{Re} I(0) = \operatorname{Re} I(M) - c_1 = \operatorname{Re} I(-M) - c_1.$$

By the approximation (237) for  $w \in \Sigma_{\text{local}}$  of  $\tilde{I}(w)$ , the estimate (245) and (246) for  $w \in \Sigma \setminus \Sigma_{\text{local}}$ , and (249), using the asymptotic formula (167) of  $p_n(s)$ , we have that for all  $w \in \Sigma$  and  $t_i, t_j, x, y$  expressed by (31),

$$(250) \quad p_n(w) e^{-t_j n w^2 / 2} \frac{-e^{-i y n w}}{1 - e^{2\pi i n w - 2\tau \pi i}} \\ = e^{(T - 2t_c) n M^2 / 4} \mathcal{O}(e^{n(g(M) - T M^2 / 4 - c_1 + \varepsilon)}),$$

where  $\varepsilon$  is an arbitrarily small positive number. Similarly, for  $s \in \mathbb{R} \setminus [-M, M]$ , using the asymptotics formula (167) of  $p_{n-1}(s)$  and Proposition 3.9, we have

$$(251) \quad \frac{1}{h_{n,n-1}^{(T;\tau)}} p_{n-1}(s) e^{-(T-t_i)ns^2/2} e^{ixns} = e^{-(T-2t_c)ns^2/4} \mathcal{O}(e^{n(\operatorname{Re} g_+(s) - Ts^2/4 - l + \varepsilon')}),$$

where  $\varepsilon'$  is an arbitrarily small positive number. By the inequalities (140),

$$(252) \quad \operatorname{Re} g_+(s) - \frac{Ts^2}{4} \leq \frac{l}{2} \quad \text{for } s \in \mathbb{R} \setminus [-M, M].$$

Hence, for all  $w \in \Sigma$  and  $s \in \mathbb{R} \setminus [-M, M]$ ,

$$(253) \quad \frac{1}{h_{n,n-1}^{(T;\tau)}} p_{n-1}(s) p_n(w) e^{-(T-t_i)ns^2/2 - t_jnw^2/2} \frac{-e^{ixns - iynw}}{1 - e^{2\pi inw - 2\tau\pi i}} = \mathcal{O}(e^{n(-c_1 + \varepsilon + \varepsilon')}).$$

If the factor  $p_{n-1}(s)p_n(w)$  in (253) is changed into  $p_n(s)p_{n-1}(w)$ , the estimate (253) still holds. So for all  $w \in \Sigma$  and  $s \in \mathbb{R} \setminus [-M, M]$ ,

$$(254) \quad \frac{1}{h_{n,n-1}^{(T;\tau)}} \frac{p_n(s)p_{n-1}(w) - p_{n-1}(s)p_n(w)}{s - w} e^{-(T-t_i)ns^2/2 - t_jnw^2/2} \times \frac{-e^{ixns - iynw}}{1 - e^{2\pi inw - 2\tau\pi i}} = \mathcal{O}(e^{n(-c_1 + \varepsilon + \varepsilon')}).$$

Note that the integrand in (219) vanishes rapidly as  $w \in \infty$  along  $\Sigma$ . Thus, we have

$$(255) \quad K_{t_i, t_j}^{\text{minor}}(x, y) = \mathcal{O}(e^{n(-c_1 + \varepsilon + \varepsilon')}).$$

The asymptotics (242), (243), (248) and (255), together with (230) and (217), yield

$$(256) \quad \tilde{K}_{t_i, t_j}(x, y) = \frac{n^{3/4}}{d} (\tilde{K}_{-\tau_j, -\tau_i}^{\text{Pearcey}}(\eta, \xi) + \mathcal{O}(n^{-1/4})).$$

At last,  $\mathring{W}_{[i,j]}(x, y)$  is defined in (112) with explicit formula given in (133). It is 0 when  $t_j \leq t_i$  and when  $t_j > t_i$ , a standard approximation technique gives that

$$(257) \quad \mathring{W}_{[i,j]}(x, y) = \frac{n^{3/4}}{d} \frac{1}{\sqrt{2\pi(\tau_j - \tau_i)}} e^{-(\eta - \xi)^2 / (2(\tau_j - \tau_i))} (1 + \mathcal{O}(n^{-1/4})).$$

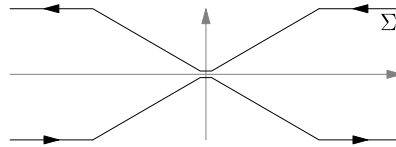


FIG. 6. Schematic figure of  $\Sigma$ .

Comparing (256) and (257) with (25) and (26), we obtain (211).

5.3. *Limiting tacnode process.* With notation defined in (34), we write (224) as

$$(258) \quad \tilde{K}_{t_i, t_j}(x, y) = \frac{n}{2\pi} \oint_{\Sigma} dz \oint_{\Sigma} dw J(z, w),$$

where

$$(259) \quad \begin{aligned} J(z, w) &= \left( e^{-(nT/4)(z^2+w^2)} \frac{p_n(z)p_{n-1}(w) - p_{n-1}(z)p_n(w)}{h_{n, n-1}^{(T; \tau)}(z-w)} \right) \\ &\times e^{n^{2/3}(d^2/2)(\tau_i z^2 - \tau_j w^2)} e^{-in^{1/3}d(\xi z - \eta w)} \\ &\times \frac{e^{\pi i(nz - \tau)} e^{\pi i(nw - \tau)}}{(e^{2\pi i(nz - \tau)} - 1)(e^{2\pi i(nw - \tau)} - 1)}. \end{aligned}$$

In this section, we define the shape of  $\Sigma$  as follows. First, the part of  $\Sigma$  in the first quadrant consists of a horizontal ray from  $\infty \cdot e^0$  to  $1 + i$ , a line segment from  $\sqrt{3} + i$  to  $(\sqrt{3} + i)d^{-1}n^{-1/3}$ , and a line segment from  $(\sqrt{3} + i)d^{-1}n^{-1/3}$  to  $id^{-1}n^{-1/3}$ . The part of  $\Sigma$  in the second quadrant is a reflection of that in the first quadrant about the imaginary axis, and the part of  $\Sigma$  in the lower half-plane is a reflection of that in the upper half-plane about the real axis.  $\Sigma \cap \mathbb{C}_+$  is oriented from right to left, and  $\Sigma \cap \mathbb{C}_-$  is from left to right. See Figure 6. We denote  $\Sigma_{\text{local}}, \Sigma_{\text{local}}^{\text{upper}}, \Sigma_{\text{local}}^{\text{lower}}$  as

$$(260) \quad \begin{aligned} \Sigma_{\text{local}} &= \Sigma \cap \{z \in \mathbb{C} \mid |z| < n^{-1/4}\}, & \Sigma_{\text{local}}^{\text{upper}} &= \Sigma_{\text{local}} \cap \mathbb{C}_+, \\ \Sigma_{\text{local}}^{\text{lower}} &= \Sigma_{\text{local}} \cap \mathbb{C}_-. \end{aligned}$$

To make the discussion about the apparent singularity  $(z - w)^{-1}$  easier, we integrate  $z$  on  $\Sigma$  and  $w$  on  $\Sigma + \frac{i}{2}d^{-1}n^{-2/3}$  that is obtained by shifting  $\Sigma$  above by  $\frac{i}{2}d^{-1}n^{-1/3}$ ; see Figure 7.

Applying the asymptotic formula (175) to the integrand of (258) and taking the change of variables

$$(261) \quad z = \frac{u}{dn^{1/3}}, \quad w = \frac{v}{dn^{1/3}}$$

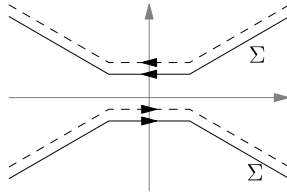


FIG. 7. Locally around 0 with scaling  $n^{-1/3}$ . The solid curve is  $\Sigma$  and the dashed curve is  $\Sigma + \frac{i}{2}d^{-1}n^{-2/3}$ .

we have

$$\begin{aligned}
 & \frac{n}{2\pi} \oint_{\Sigma_{\text{local}}} dz \oint_{\Sigma_{\text{local}} + (i/2)d^{-1}n^{-2/3}} dw J(z, w) \\
 (262) \quad &= \frac{n^{2/3}}{4\pi^2 id} \oint_{\Sigma_T^*} du \oint_{\Sigma_T^* + i/2} dv \frac{e^{1/2(\tau_i u^2 - \tau_j v^2) - i(\xi u - \eta v)}}{u - v} \\
 & \times \left( \frac{1}{1 - e^{2\pi i(nz - \tau)}} \right)^T \Psi(u; \sigma)^{-1} \Psi(v; \sigma) \\
 & \times \left( \frac{1}{1 - e^{-2\pi i(nw - \tau)}} \right) (1 + \mathcal{O}(n^{-1/4})),
 \end{aligned}$$

where  $\Sigma_T^*$  is the large but finite contour

$$(263) \quad \Sigma_T^* = \Sigma_T \cap N_{dn^{1/12}}(0) \quad \text{where } N_{dn^{1/12}}(0) = \{z \mid |z| < dn^{1/12}\},$$

and  $\Sigma_T$  is shown in Figure 3. Note that for  $z \in \Sigma_{\text{local}}^{\text{upper}}$ , or equivalently,  $u \in \Sigma_T^* \cap \mathbb{C}_+$ ,

$$\begin{aligned}
 (264) \quad & \frac{1}{1 - e^{2\pi i(nz - \tau)}} = 1 + \mathcal{O}(e^{-2n^{2/3}/d}), \\
 & \frac{1}{1 - e^{-2\pi i(nz - \tau)}} = \mathcal{O}(e^{-2n^{2/3}/d}),
 \end{aligned}$$

and for  $z \in \Sigma_{\text{local}}^{\text{lower}}$ , or equivalently,  $u \in \Sigma_T^* \cap \mathbb{C}_-$ ,

$$\begin{aligned}
 (265) \quad & \frac{1}{1 - e^{2\pi i(nz - \tau)}} = \mathcal{O}(e^{-2n^{2/3}/d}), \\
 & \frac{1}{1 - e^{-2\pi i(nz - \tau)}} = 1 + \mathcal{O}(e^{-2n^{2/3}/d}).
 \end{aligned}$$

For  $w \in \Sigma_{\text{local}}^{\text{upper}} + \frac{i}{2}d^{-1}n^{-2/3}$  or  $\Sigma_{\text{local}}^{\text{upper}} + \frac{i}{2}d^{-1}n^{-2/3}$ , we have analogous result for  $(1 - e^{\pm 2\pi i(nw - \tau)})^{-1}$ , and omit the explicit formulas. Substituting (264) and (265)

and their counterparts for  $w$  into (262) and using the fact that  $\det \Psi(\zeta; \sigma) \equiv 1$ , we find that

$$\begin{aligned}
 & \frac{n}{2\pi} \oint_{\Sigma_{\text{local}}} dz \oint_{\Sigma_{\text{local}}+(i/2)d^{-1}n^{-2/3}} dw J(z, w) \\
 &= \frac{n^{2/3}}{4\pi^2 i d} \oint_{\Sigma_T^*} du \oint_{\Sigma_T^*+i/2} dv e^{((\tau_i/2)u^2 - (\tau_j/2)v^2) - i(\xi u - \eta v)} \\
 (266) \quad & \times \frac{f(u; \sigma)g(v; \sigma) - g(u; \sigma)f(v; \sigma)}{(u - v)} (1 + \mathcal{O}(n^{-1/4})) \\
 &= \frac{n^{2/3}}{d} K_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta) (1 + \mathcal{O}(n^{-1/4})),
 \end{aligned}$$

where  $f(u; \sigma)$  and  $g(u; \sigma)$  are defined in (28).

By the estimates of  $p_n(z)$  and  $p_{n-1}(z)/h_{n,n-1}^{(T; \tau)}$  in Proposition 3.7 and (176) in Proposition 3.8, we have that for all  $z \in \Sigma$  and  $w \in \Sigma + \frac{i}{2}d^{-1}n^{-2/3}$

$$(267) \quad \frac{p_n(z)p_{n-1}(w) - p_{n-1}(z)p_n(w)}{h_{n,n-1}^{(T; \tau)}(z - w)} = e^{ng(z)+ng(w)-nl} \mathcal{O}(n^{2/3}),$$

where  $g(z)$  is defined in (157) and the  $n^{2/3}$  factor comes from  $(z - w)^{-1}$ . Hence, for all  $z \in \Sigma$  and  $w \in \Sigma + \frac{i}{2}d^{-1}n^{-2/3}$ ,

$$\begin{aligned}
 (268) \quad |J(z, w)| &= e^{n(\tilde{g}(z) - (T/4)z^2 + \pi iz) + n(\tilde{g}(w) - (T/4)w^2 + \pi iw)} e^{n^{2/3}d^2(\tau_i z^2 - \tau_j w^2)} \\
 &\times e^{-in^{1/3}d(\xi z - \eta w)} \mathcal{O}(n^{2/3}),
 \end{aligned}$$

where  $\tilde{g}$  is defined by  $g$  in (227). By direct calculation, we have that for  $z \in \Sigma \setminus \Sigma_{\text{local}}$ ,  $\text{Re}(\tilde{g}(z) - T^2z^2/4 + \pi iz)$  decreases as  $z$  moves away from 0. Hence, by standard argument of steepest-descent method and the result of (266), we have that

$$\begin{aligned}
 (269) \quad \tilde{K}_{t_i, t_j}(x, y) &= \frac{n}{2\pi} \oint_{\Sigma} dz \oint_{\Sigma+(i/2)d^{-1}n^{-2/3}} dw J(z, w) \\
 &= \frac{n}{2\pi} \oint_{\Sigma_{\text{local}}} dz \oint_{\Sigma_{\text{local}}+(i/2)d^{-1}n^{-2/3}} dw J(z, w) + \mathcal{O}(e^{-cn^{1/4}}) \\
 &= \frac{n^{2/3}}{d} \tilde{K}_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) (1 + \mathcal{O}(n^{-1/4})),
 \end{aligned}$$

where  $c$  is a positive constant.

At last,  $\mathring{W}_{[i, j]}(x, y)$  is defined in (112) with explicit formula given in (133). It is 0 when  $t_j \leq t_i$  and when  $t_j > t_i$ , a standard approximation technique gives that

$$(270) \quad \mathring{W}_{[i, j]}(x, y) = \frac{n^{2/3}}{d} \frac{1}{\sqrt{2\pi(\tau_j - \tau_i)}} e^{-(\eta - \xi)^2 / (2(\tau_j - \tau_i))} (1 + \mathcal{O}(n^{-1/3})).$$

Comparing (269) and (270) with (29) and (26), we prove (35).

5.4. *Proof of Theorem 1.4.* For notational simplicity, we only consider the limiting 2-correlation functions, such that  $t_1, t_2 \in (0, T)$  are two times and  $x, y$  are two locations on  $\mathbb{T}$ . We assume that  $t_1, t_2, x, y$  are expressed by (31) with  $i = 1, j = 2$ , and then

$$(271) \quad \begin{aligned} & (R_{0 \rightarrow T}^{(n)})_\omega(x; y; t_1, t_2) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{(\Delta x)^2} \mathbb{P} \left( \begin{array}{l} \text{there is a particle in } [x, x + \Delta x] \text{ at time } t_1, \\ \text{there is a particle in } [y, y + \Delta x] \text{ at time } t_2, \\ \text{and the total winding number is } \omega \end{array} \right). \end{aligned}$$

From (135), we have

$$(272) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \sum_{\omega \in \mathbb{Z}} (R_{0 \rightarrow T}^{(n)})_\omega(x; y; t_1, t_2) e^{2\pi\omega(\tau + \epsilon(n))i} \left( \frac{d}{n^{3/4}} \right)^2 \\ &= \lim_{n \rightarrow \infty} \frac{R_n(T; \tau)}{R_n(T; \epsilon(n))} R_{0 \rightarrow T}^{(n)}(x; y; t_1, t_2; \tau) \left( \frac{d}{n^{3/4}} \right)^2, \end{aligned}$$

where  $R_{0 \rightarrow T}^{(n)}(x; y; t_1, t_2; \tau)$  is a special case of the  $\tau$ -deformed joint correlation function defined in (115).

By the determinantal formula (116) and the asymptotic result (211), we have for all  $\tau \in [0, 1]$ ,

$$(273) \quad \lim_{n \rightarrow \infty} R_{0 \rightarrow T}^{(n)}(x; y; t_1, t_2; \tau) = \begin{vmatrix} K_{-\tau_1, -\tau_1}^{\text{Pearcey}}(\xi, \xi) & K_{-\tau_2, -\tau_1}^{\text{Pearcey}}(\eta, \xi) \\ K_{-\tau_1, -\tau_2}^{\text{Pearcey}}(\xi, \eta) & K_{-\tau_2, -\tau_2}^{\text{Pearcey}}(\eta, \eta) \end{vmatrix},$$

and on the other hand by (184) and (24), we have

$$(274) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \frac{R_n(T; \tau)}{R_n(T; \epsilon(n))} \\ &= \lim_{n \rightarrow \infty} \sum_{\omega \in \mathbb{Z}} \mathbb{P}(\text{Total winding number equals } \omega) e^{2\pi\omega(\tau + \epsilon(n))i} \\ &= \sum_{\omega \in \mathbb{Z}} q^{\omega^2} \sqrt{\frac{\pi}{2\tilde{\mathbf{K}}}} e^{2\pi\omega(\tau + \epsilon(n))i}. \end{aligned}$$

Hence, a comparison of Fourier coefficients on both sides of (272) shows that

$$(275) \quad \begin{aligned} & \lim_{n \rightarrow \infty} (R_{0 \rightarrow T}^{(n)})_\omega(x; y; t_1, t_2) \left( \frac{d}{n^{3/4}} \right)^2 \\ &= q^{\omega^2} \sqrt{\frac{\pi}{2\tilde{\mathbf{K}}}} \begin{vmatrix} K_{-\tau_1, -\tau_1}^{\text{Pearcey}}(\xi, \xi) & K_{-\tau_2, -\tau_1}^{\text{Pearcey}}(\eta, \xi) \\ K_{-\tau_1, -\tau_2}^{\text{Pearcey}}(\xi, \eta) & K_{-\tau_2, -\tau_2}^{\text{Pearcey}}(\eta, \eta) \end{vmatrix}, \end{aligned}$$

which is the desired result. Thus, we prove Theorem 1.4 in the  $n = 2$  case.



**6. Interpolation problem and Riemann–Hilbert problem associated to discrete Gaussian orthogonal polynomials.**

6.1. *Equilibrium measure and the g-function.* In this subsection, we prove the results presented in Section 3.1 for the supercritical case  $T > T_c$ . The existence and uniqueness of the equilibrium measure associated to the potential  $Tx^2/2$  that satisfies the minimization problem (137) and (138) is proved in Kuijlaars (2000), along with several analytic properties. Thus, if we find a probability measure  $\nu_T$  with continuous density function  $\rho_T(x)$  such that the associated  $g$ -function satisfies the variational condition (140), then it is the unique equilibrium measure. For  $T \leq T_c = \pi^2$ , it is straightforward to verify that the well-known semicircle law (142) and the  $g$ -function (157) satisfy the variational condition (140), so the equilibrium measure is given by (142). Thus, this subsection is dedicated to the construction of the equilibrium measure and the derivative of the  $g$ -function for  $T > T_c = \pi^2$ . The  $g$ -function is then determined by its derivative up to the Lagrange multiplier  $l$ . Our strategy is to construct a probability measure  $\nu_T$  with continuous density  $d\nu_T(x) = \rho_T(x) dx$  together with the derivative of the associated  $g$ -function, such that  $\nu_T$  is supported on an interval  $[-\beta, \beta]$ , and has a saturated region  $[-\alpha, \alpha]$ , that is,  $\rho_T(x) = 0$  for  $x \in \mathbb{R} \setminus (-\beta, \beta)$ ,  $\rho_T(x) = 1$  for  $x \in [-\alpha, \alpha]$  and  $0 < \rho_T(x) < 1$  for  $x \in (-\beta, \alpha) \cup (\alpha, \beta)$ , and then verify that the probability measure satisfies the variational condition (140). Therefore, we conclude that the construction of the equilibrium measure is valid.

The derivative of the  $g$ -function is expressed as

$$(276) \quad g'(z) = \int_{-\beta}^{\beta} \frac{1}{z-x} \rho_T(x) dx, \quad z \in \mathbb{C} \setminus [-\beta, \beta],$$

and so the equilibrium measure  $\nu_T = \rho_T(x) \chi_{[-\beta, \beta]}(x) dx$  is given as

$$(277) \quad \rho_T(x) = \frac{-1}{\pi} \operatorname{Im} g'_+(x) = \frac{1}{\pi} \operatorname{Im} g'_-(x) \quad \text{for } x \in [-\beta, \beta],$$

where  $g'_+(x)$  and  $g'_-(x)$  are the limiting values from the upper and lower half-planes, respectively. That the measure  $\nu_T$  has total measure 1 is equivalent to

$$(278) \quad g'(z) = \frac{1}{z} + \mathcal{O}(z^{-2}) \quad \text{as } z \rightarrow \infty.$$

The variational problem (140) implies

$$(279) \quad g'_+(x) + g'_-(x) = Tx \quad \text{for } x \in (-\beta, -\alpha) \cup (\alpha, \beta),$$

$$(280) \quad g'_+(x) - g'_-(x) = -2\pi i \quad \text{for } x \in (-\alpha, \alpha).$$

To construct  $g'(z)$ , we use the incomplete elliptic integrals  $F(z; k)$  and  $E(z; k)$  and the complete elliptic integrals  $\mathbf{K} = \mathbf{K}(k)$  and  $\mathbf{E} = \mathbf{E}(k)$  introduced in (144)

and (16). They have the properties that

$$(281) \quad F_+(x; k) + F_-(x; k) = 2\mathbf{K}, \quad E_+(x; k) + E_-(x; k) = 2\mathbf{E} \quad \text{for } x \in (1, k^{-1}),$$

$$(282) \quad F_+(x; k) + F_-(x; k) = -2\mathbf{K}, \quad E_+(x; k) + E_-(x; k) = -2\mathbf{E} \quad \text{for } x \in (-k^{-1}, -1),$$

$$(283) \quad F_+(x; k) - F_-(x; k) = 2i\mathbf{K}', \quad E_+(x; k) - E_-(x; k) = 2i(\mathbf{K}' - \mathbf{E}') \quad \text{for } x \in \mathbb{R} \setminus (-k^{-1}, k^{-1}).$$

Here, we use the notation

$$(284) \quad \mathbf{K}' = \mathbf{K}(k'), \quad \mathbf{E}' = \mathbf{E}(k') \quad \text{where } k' = \sqrt{1 - k^2}.$$

Identities (281) and (282) can be checked by straightforward computation, and (283) can be checked with the help of Gradshteyn and Ryzhik (2007), 3.152-9, page 280 and 3.169-17, page 209. The identity (283) also comes from the imaginary periods of  $F(z; k)$  and  $E(z; k)$ ; see Erdélyi et al. (1981), Section 13.7, page 314. For fixed  $\alpha$  and  $\beta$ , let

$$(285) \quad k = \frac{\alpha}{\beta}.$$

With the help of Legendre's relation [Erdélyi et al. (1981), Section 13.8, page 320, Formula (15)],

$$(286) \quad \mathbf{K}\mathbf{E}' + \mathbf{K}'\mathbf{E} - \mathbf{K}\mathbf{K}' = \frac{\pi}{2},$$

we find that when  $g'(z)$  is given by

$$(287) \quad g'(z) = \begin{cases} \frac{Tz}{2} + 2\mathbf{E}F\left(\frac{z}{\alpha}; k\right) - 2\mathbf{K}E\left(\frac{z}{\alpha}; k\right) - \pi i, & \text{for } z \in \mathbb{C}_+, \\ \frac{Tz}{2} + 2\mathbf{E}F\left(\frac{z}{\alpha}; k\right) - 2\mathbf{K}E\left(\frac{z}{\alpha}; k\right) + \pi i, & \text{for } z \in \mathbb{C}_-, \end{cases}$$

it satisfies (279) and (280), and it is also well defined on  $(-\infty, -\beta) \cup (\beta, \infty)$  by analytic continuation. To make (278) hold, we need to choose the correct values for  $\alpha$  and  $\beta$ . As  $z \rightarrow \infty$ , the asymptotic behaviors of  $F(z; k)$  and  $E(z; k)$  are

$$(288) \quad F(z; k) = i\mathbf{K}' + \frac{1}{kz} + \mathcal{O}(z^{-2}),$$

$$(289) \quad E(z; k) = kz + i(\mathbf{K}' - \mathbf{E}') + \frac{k^{-1} - k}{2z} + \mathcal{O}(z^{-2}).$$

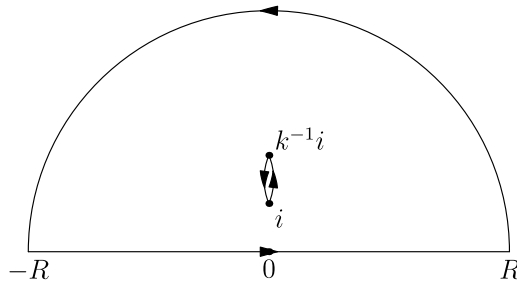


FIG. 8. The contours  $C_R$  and  $C_{1,k^{-1}}$ .

The constant term in (288) is obtained by

$$\begin{aligned}
 \lim_{\substack{z \rightarrow \infty \\ z \in \mathbb{C}_+}} F(z; k) &= \int_0^{i \cdot \infty} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \\
 (290) \qquad \qquad \qquad &= i \int_0^\infty \frac{dt}{\sqrt{(1+t^2)(1+k^2t^2)}} \\
 &= i F(1; \sqrt{1-k^2}) = i \mathbf{K}',
 \end{aligned}$$

where evaluation of the elliptic integral is done by Gradshteyn and Ryzhik (2007), 3.152-2, page 279. The  $z^{-1}$  term in (288) follows the asymptotics of the integrand in the defining formula (144) of  $F(z; k)$ . The  $z$  term in (289) is obvious, and the constant term is given by

$$\begin{aligned}
 \lim_{\substack{z \rightarrow \infty \\ z \in \mathbb{C}_+}} E(z; k) - kz &= \int_0^{i \cdot \infty} \left( \sqrt{\frac{1-k^2s^2}{1-s^2}} - k \right) ds \\
 (291) \qquad \qquad \qquad &= \frac{i}{2} \int_{-\infty}^\infty \left( \sqrt{\frac{1+k^2t^2}{1+t^2}} - k \right) dt.
 \end{aligned}$$

To evaluate the integral on the right-hand side of (291), we define the pair of contours (see Figure 8),

$$\begin{aligned}
 C_R &= [-R, R] \cup \{Re^{i\theta} \mid \theta \in [0, \pi]\} \quad \text{counterclockwise,} \\
 (292) \quad C_{1,k^{-1}} &= \text{contour starting from } i, \text{ along the right-hand side of the} \\
 &\quad \text{imaginary axis, to } k^{-1}i, \text{ and then along the left-hand side} \\
 &\quad \text{of the imaginary axis, back to } i.
 \end{aligned}$$

Then by the contour integral technique and Gradshteyn and Ryzhik (2007), 3.169-17, page 309,

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \sqrt{\frac{1+k^2t^2}{1+t^2}} - k \, dt \\
 &= \lim_{R \rightarrow \infty} \oint_{C_R} \sqrt{\frac{1+k^2t^2}{1+t^2}} \, dt = \oint_{C_{1,k^{-1}}} \sqrt{\frac{1+k^2t^2}{1+t^2}} \, dt \\
 (293) \quad &= 2 \int_1^{k^{-1}} \sqrt{\frac{1-k^2t^2}{1-t^2}} \, dt \\
 &= 2(F(1; \sqrt{1-k^2}) - E(1; \sqrt{1-k^2})) = 2(\mathbf{K}' - \mathbf{E}'),
 \end{aligned}$$

and we get the result. The  $z^{-1}$  term of (289) is obtained analogously to the  $z^{-1}$  term of (288).

Then as  $z \rightarrow \infty$  in  $\mathbb{C}_+$ ,

$$\begin{aligned}
 (294) \quad g'(z) &= \left(\frac{T}{2} - \frac{2k\mathbf{K}}{\alpha}\right)z + 2i\left(\mathbf{K}'\mathbf{E} + \mathbf{K}\mathbf{E}' - \mathbf{K}\mathbf{K}' - \frac{\pi}{2}\right) \\
 &+ 2\alpha\left(\frac{\mathbf{E}}{k} - \frac{(1-k^2)\mathbf{K}}{2k}\right)\frac{1}{z} + \mathcal{O}(z^{-2}).
 \end{aligned}$$

Note that the constant term of  $g'(z)$  vanishes automatically by Legendre’s relation (286). For  $k = \alpha/\beta$ , the identity (278) is satisfied when  $\alpha$  and  $\beta$  are given by (145) and (148).

By Lemma 3.2, the relation (148) is a 1–1 correspondence between  $T > T_c = \pi^2$  and  $k \in (0, 1)$ . Thus, for each  $T = T(k) > T_c$ , there are well-defined  $\alpha, \beta$  and  $\rho_T$  given by (145) and (278). By the construction of  $\rho_T$ , especially (279), we have that the measure  $d\nu_T$  with density  $\rho_T(x)\chi_{[-\beta,\beta]}(x)$  has total measure 1, and satisfies the variational condition on  $[\alpha, \beta]$  given that the Lagrange multiplier  $l$  is properly chosen and  $0 < \rho_T(x) < 1$  for all  $x \in (\alpha, \beta)$ . By the symmetry of  $\nu_T$  about the origin, we finish the verification that  $\nu_T$  is the equilibrium measure. Additionally, we have the following lemma, which states that the equilibrium measure is *regular* in the sense of Bleher and Liechty (2011).

LEMMA 6.1. (a)  $0 < \rho_T(x) < 1$  for all  $x \in (\alpha, \beta)$ .

(b)  $g_+(x) + g_-(x) - \frac{Tx^2}{2} - l > 0$  for  $x \in [0, \alpha)$ .

(c)  $2g(x) - \frac{Tx^2}{2} - l < 0$  for  $x \in (\beta, \infty)$ .

(d) There exist constants  $c_1$  and  $c_2$  such that

$$(295) \quad \rho_T(x) = c_1\sqrt{\beta-x}(1 + \mathcal{O}((\beta-x))) \quad \text{as } x \rightarrow \beta \text{ from the left}$$

and

$$(296) \quad 1 - \rho_T(x) = c_2 \sqrt{x - \alpha} (1 + \mathcal{O}((x - \alpha)))$$

*as  $x \rightarrow \alpha$  from the right.*

We finish this subsection by the proof of Lemmas 3.2 and 6.1.

PROOF OF LEMMA 3.2. The two limits in (149) are straightforward to check from the integral formulas (16) of  $\mathbf{K}$  and  $\mathbf{E}$ . To see the monotonicity, we use Byrd and Friedman (1971), 710.00 and 710.02, page 282, and have

$$(297) \quad \begin{aligned} \frac{d}{dk}(\mathbf{K}(k)\mathbf{E}(k)) &= \frac{\mathbf{E}^2 - (1 - k^2)\mathbf{K}^2}{k(1 - k^2)} \\ &= \frac{1}{k(1 - k^2)} \int_0^1 \frac{\sqrt{1 - k^2s^2} - \sqrt{1 - k^2}}{\sqrt{(1 - s^2)(1 - k^2s^2)}} ds > 0, \end{aligned}$$

which proves the monotonicity.  $\square$

PROOF OF LEMMA 6.1. For part (a), since  $\rho_T(x)$  on  $(0, \alpha)$  is expressed by  $\Lambda_0(x; k)$  for  $x \in (0, 1)$  in (151), we only need to show that  $\Lambda_0(x; k) \in (0, 1)$  for  $x \in (0, 1)$ . By Byrd and Friedman (1971), 151.01, page 36, we have that  $\Lambda_0(0; k) = 0$  and  $\Lambda_0(1; k) = 1$ . We need only to show that  $\Lambda_0(x; k)$  is strictly increasing on  $(0, 1)$ . By Byrd and Friedman (1971), 730.04, page 284, this is implied by the inequality  $\mathbf{E} - (1 - k^2)x^2\mathbf{K} > 0$  for all  $x \in (0, 1)$ . The inequality is proved as

$$(298) \quad \mathbf{E} - (1 - k^2)x^2\mathbf{K} = \int_0^1 \frac{\sqrt{1 - k^2s^2} - x^2\sqrt{1 - k^2}}{\sqrt{(1 - s^2)(1 - k^2s^2)}} > 0.$$

For parts (b) and (c), we note that since  $g_+(x) + g_-(x) - Tx^2/2 - l$ , which becomes  $2g(x) - Tx^2/2 - l$  for  $x > \beta$ , is a continuous function and  $l$  is chosen so that it vanishes for  $x \in [\alpha, \beta]$ , it suffices to show the inequalities

$$(299) \quad \frac{1}{4}(g'_+(x) + g'_-(x) - Tx) = \mathbf{E}F\left(\frac{x}{\alpha}; k\right) - \mathbf{K}E\left(\frac{x}{\alpha}; k\right) < 0,$$

$x \in (0, \alpha),$

$$(300) \quad \begin{aligned} \frac{1}{4}(2g'(x) - Tx) &= -\mathbf{E} \int_{k^{-1}}^{x/\alpha} \frac{ds}{\sqrt{(s^2 - 1)(k^2s^2 - 1)}} \\ &\quad - \mathbf{K} \int_{k^{-1}}^{x/\alpha} \frac{\sqrt{k^2s^2 - 1}}{\sqrt{s^2 - 1}} ds < 0, \quad x \in (\beta, \infty). \end{aligned}$$

The inequality (300) obviously holds. To prove (299), we use Byrd and Friedman (1971), 414.01, page 229,

$$\begin{aligned}
 & \mathbf{E}F\left(\frac{x}{\alpha}; k\right) - \mathbf{K}E\left(\frac{x}{\alpha}; k\right) \\
 (301) \quad &= \frac{1}{\beta x} \sqrt{(\beta - x^2)(\alpha^2 - x^2)} \left( \mathbf{K} - \Pi_1\left(-\frac{x^2}{\beta^2}; k\right) \right) \\
 &= \frac{-1}{\beta x} \sqrt{(\beta - x^2)(\alpha^2 - x^2)} \int_0^1 \frac{(x^2/\beta^2)s^2 ds}{(1 - (x^2/\beta^2)s^2)\sqrt{(1 - s^2)(1 - k^2s^2)}},
 \end{aligned}$$

which is clearly negative for  $x \in (0, \alpha)$ .

Part (d) is easy to check using formula (150).  $\square$

6.2. *Interpolation problem and outline of the steepest descent analysis.* The orthogonal polynomials (90) are encoded in the following interpolation problem (IP). For a given  $n = 0, 1, \dots$ , find a  $2 \times 2$  matrix-valued function  $\mathbf{P}_n(z) = (\mathbf{P}_n(z)_{ij})_{1 \leq i, j \leq 2}$  with the following properties:

1. *Analyticity:*  $\mathbf{P}_n(z)$  is an analytic function of  $z$  for  $z \in \mathbb{C} \setminus L_{n,\tau}$ .

2. *Residues at poles:* At each node  $x \in L_{n,\tau}$ , the elements  $\mathbf{P}_n(z)_{11}$  and  $\mathbf{P}_n(z)_{21}$  of the matrix  $\mathbf{P}_n(z)$  are analytic functions of  $z$ , and the elements  $\mathbf{P}_n(z)_{12}$  and  $\mathbf{P}_n(z)_{22}$  have a simple pole with the residues,

$$(302) \quad \text{Res}_{z=x} \mathbf{P}_n(z)_{j2} = \frac{1}{n} e^{-nTx^2/2} \mathbf{P}_n(x)_{j1}, \quad j = 1, 2.$$

3. *Asymptotics at infinity:* There exists a function  $r(x) > 0$  on  $L_{n,\tau}$  such that

$$(303) \quad \lim_{x \rightarrow \infty} r(x) = 0,$$

and such that as  $z \rightarrow \infty$ ,  $\mathbf{P}_n(z)$  admits the asymptotic expansion,

$$(304) \quad \mathbf{P}_n(z) \sim \left( I + \frac{\mathbf{P}_1}{z} + \frac{\mathbf{P}_2}{z^2} + \dots \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix},$$

$$z \in \mathbb{C} \setminus \left[ \bigcup_{x \in L_{n,\tau}} D(x, r(x)) \right],$$

where  $D(x, r(x))$  denotes a disk of radius  $r(x) > 0$  centered at  $x$  and  $I$  is the identity matrix.

The unique solution to the IP is

$$(305) \quad \mathbf{P}_n(z) = \begin{pmatrix} p_{n,n}^{(T;\tau)}(z) & (Cp_{n,n}^{(T;\tau)})(z) \\ (h_{n,n-1}^{(T;\tau)})^{-1} p_{n,n-1}^{(T;\tau)}(z) & (h_{n,n-1}^{(T;\tau)})^{-1} (Cp_{n,n-1}^{(T;\tau)})(z) \end{pmatrix},$$

where the weighted discrete Cauchy transform  $C$  is defined in (162). The normalizing constants in (92) and the recurrence coefficients (93) are encoded in the matrices  $\mathbf{P}_1$  and  $\mathbf{P}_2$  in the expansion (304). Namely, we have

$$(306) \quad h_{n,n}^{(T;\tau)} = [\mathbf{P}_1]_{12}, \quad (h_{n,n-1}^{(T;\tau)})^{-1} = [\mathbf{P}_1]_{21}$$

and

$$(307) \quad \beta_{n,n-1}^{(T;\tau)} = \frac{[\mathbf{P}_2]_{21}}{[\mathbf{P}_1]_{21}} - [\mathbf{P}_1]_{11}.$$

The steepest descent analysis of the IP for a general class of orthogonal polynomials is described in Bleher and Liechty (2011) in the case  $\tau = 0$  [see also Baik et al. (2007) for polynomials orthogonal on a finite lattice]. For the discrete Gaussian orthogonal polynomials the analysis for a general  $\tau$  was given in Liechty (2012) for the case  $T = T_c + o(1)$  as  $n \rightarrow \infty$ . The analysis consists of a sequence of transformations

$$(308) \quad \mathbf{P}_n \rightarrow \mathbf{R}_n \rightarrow \mathbf{T}_n \rightarrow \mathbf{S}_n \rightarrow \mathbf{X}_n.$$

The first transformation  $\mathbf{P}_n \rightarrow \mathbf{R}_n$  reduces the IP to a Riemann–Hilbert problem (RHP). The second transformation  $\mathbf{R}_n \rightarrow \mathbf{T}_n$  uses the  $g$ -function to give a RHP which approaches the identity matrix as  $z \rightarrow \infty$ . The third transformation  $\mathbf{T}_n \rightarrow \mathbf{S}_n$  is local and involves transformations only close to the support of the equilibrium measure. The RHP for  $\mathbf{S}_n$  can be approximated by RHPs for which we can write explicit solutions in different regions of the complex plane, and  $\mathbf{X}_n$  is uniformly close to the identity matrix.

In the supercritical case  $T > T_c$ , one can make the reduction to a RHP in the following way. Fix some  $\varepsilon > 0$  and some  $0 < \delta < 1$ . Let  $\Gamma_+$  (resp.,  $\Gamma_-$ ) be a contour from  $e^{i0} \cdot \infty$  to  $e^{i\pi} \cdot \infty$  (resp.,  $e^{-i\pi} \cdot \infty$  to  $e^{i0} \cdot \infty$ ) which lies in the upper (resp., lower) half-plane and sits at a distance  $\varepsilon n^{-\delta}$  from the real line except close to the turning points  $\pm\alpha$  and  $\pm\beta$ , where it maintains a fixed distance  $\varepsilon$  from these points; see Figure 9. We let  $\Omega_{\pm}^{\Delta}$  be the region bounded by the real line and  $\Gamma_{\pm}$  with  $|\operatorname{Re} z| < \alpha$ , and  $\Omega_{\pm}^{\nabla}$  the region bounded by the real line and  $\Gamma_{\pm}$  with  $|\operatorname{Re} z| > \alpha$ . We make the reduction of the IP to a RHP and the transformations to the RHP as in Liechty (2012); see Liechty (2012), Figure 2 and equations (4.27), (4.28), (4.32).

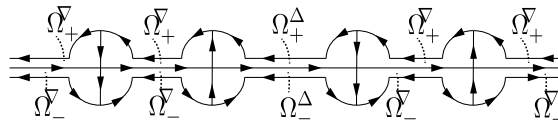


FIG. 9. The contour  $\Sigma_S$ . The horizontal line is  $\mathbb{R}$  and the vertical segments pass through  $\pm\alpha$  and  $\pm\beta$ . The remaining pieces of the contour are  $\Gamma_{\pm}$  which pass close to  $\mathbb{R}$  at a distance of  $\varepsilon n^{-\delta}$  except close to the turning points  $\pm\alpha$  and  $\pm\beta$ . The regions  $\Omega_{\pm}^{\Delta}$  are bounded by the real line and  $\Gamma_{\pm}$  with  $|\operatorname{Re} z| < \alpha$ , and the regions  $\Omega_{\pm}^{\nabla}$  are bounded by the real line and  $\Gamma_{\pm}$  with  $|\operatorname{Re} z| > \alpha$ .

Note that the lattice shift parameter which we call  $\tau$  is called  $(-\alpha)$  in [Liechty \(2012\)](#).

Let us briefly describe the explicit transformations involved in the steepest descent analysis. Introduce the functions

$$(309) \quad \Pi(z) := \frac{\sin(n\pi z - \tau\pi)}{n\pi}, \quad G(z) := g_+(z) - g_-(z),$$

where  $g_{\pm}(z)$  are defined first on  $\mathbb{R}$  as the limiting values of the  $g$ -function from  $\mathbb{C}_{\pm}$ , and then extended to a small neighborhood of  $\mathbb{R}$  by analytic continuation. Notice that the function  $G(z)$  is also given by the integral formula (141). The transformations described above involve the matrices

$$(310) \quad \begin{aligned} \mathbf{D}_{\pm}^u(z) &= \begin{pmatrix} 1 & -\frac{e^{-(nT/2)z^2}}{n\Pi(z)}e^{\pm i\pi(nz-\tau)} \\ 0 & 1 \end{pmatrix}, \\ \mathbf{D}_{\pm}^l(z) &= \begin{pmatrix} \Pi(z)^{-1} & 0 \\ -ne^{(nT/2)z^2}e^{\pm i\pi(nz-\tau)} & \Pi(z) \end{pmatrix}, \\ j_{\pm}(z) &= \begin{pmatrix} 1 & 0 \\ e^{\mp nG(z)} & 1 \end{pmatrix}, \\ \mathbf{A}_{\pm}(z) &= \begin{pmatrix} \mp \frac{1}{2n\pi i}e^{\mp i\pi(nz-\tau)} & 0 \\ 0 & \mp 2n\pi i e^{\pm i\pi(nz-\tau)} \end{pmatrix}. \end{aligned}$$

After the first two transformations of the IP, the matrix  $\mathbf{S}_n(z)$  is defined as

$$(311) \quad \mathbf{S}_n(z) = \begin{cases} e^{-(nl/2)\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix} \mathbf{P}_n(z) \mathbf{D}_{\pm}^l(z) \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix}^{-1} \\ \quad \times e^{-n(g(z)-l/2)\sigma_3} \mathbf{A}_{\pm}(z), \\ \quad \text{for } z \in \Omega_{\pm}^{\Delta}, \\ e^{-(nl/2)\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix} \mathbf{P}_n(z) \mathbf{D}_{\pm}^u(z) \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix}^{-1} \\ \quad \times e^{-n(g(z)-l/2)\sigma_3} j_{\pm}(z)^{\mp 1}, \\ \quad \text{for } z \in \Omega_{\pm}^{\nabla} \text{ and } \alpha \leq |\operatorname{Re} z| \leq \beta, \\ e^{-(nl/2)\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix} \mathbf{P}_n(z) \mathbf{D}_{\pm}^u(z) \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix}^{-1} \\ \quad \times e^{-n(g(z)-l/2)\sigma_3}, \\ \quad \text{for } z \in \Omega_{\pm}^{\nabla} \text{ and } |\operatorname{Re} z| \geq \beta, \\ e^{-(nl/2)\sigma_3} \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix} \mathbf{P}_n(z) \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix}^{-1} e^{-n(g(z)-l/2)\sigma_3}, \\ \quad \text{otherwise,} \end{cases}$$

where  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is the third Pauli matrix. This matrix function satisfies the following RHP.



- $\mathbf{S}_n(z)$  is an analytic function of  $z$  for  $z \in \mathbb{C} \setminus \Sigma_S$ , where  $\Sigma_S$  consists  $\mathbb{R}$ ,  $\Gamma_+$ , and  $\Gamma_-$ , along with the four vertical segments  $[\pm\beta - i\varepsilon, \pm\beta + i\varepsilon]$  and  $[\pm\alpha - i\varepsilon, \pm\alpha + i\varepsilon]$ , oriented as shown in Figure 9.
- For  $z \in \Sigma_S$ , the function  $\mathbf{S}_n(z)$  satisfies the jump conditions

$$(312) \quad \mathbf{S}_{n+}(z) = \mathbf{S}_{n-}(z)j_S(z),$$

where

$$(313) \quad j_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } z \in (-\beta, -\alpha) \cup (\alpha, \beta), \\ \begin{pmatrix} e^{-i\Omega_n} & 0 \\ \mathcal{O}(e^{-n^{1-\delta}c(z)}) & e^{i\Omega_n} \end{pmatrix}, & \text{for } z \in (-\alpha, \alpha), \\ \begin{pmatrix} 1 & \mathcal{O}(e^{-n^{1-\delta}c(z)}) \\ \mathcal{O}(e^{-n^{1-\delta}c(z)}) & 1 \end{pmatrix}, & \text{for } z \text{ on the rest of } \Sigma_S, \end{cases}$$

where

$$(314) \quad \Omega_n := \pi(n + 1 - 2\tau),$$

and  $c(z)$  is a nonnegative continuous function on  $\Sigma_S$  which may vanish only at the turning points  $\pm\alpha$  and  $\pm\beta$ .

- As  $z \rightarrow \infty$ ,

$$(315) \quad \mathbf{S}_n(z) = I + \frac{\mathbf{S}_1}{z} + \frac{\mathbf{S}_2}{z^2} + \dots.$$

Notice that the errors in the off diagonal terms in (313) are subexponential, but still smaller than any power of  $n$ . In the usual method of steepest descent [Bleher and Liechty (2011)], these terms are exponentially small, but our analysis is slightly different in that we have taken the contours  $\Gamma_{\pm}$  to be very close to  $\mathbb{R}$ . The reason is that in Proposition 3.6 the asymptotic formulas are given for  $z \in D(\delta, \varepsilon, n)$ , which is the region above  $\Gamma_+$  and below  $\Gamma_-$ .

6.2.1. *Model RHP.* The model RHP appears when we drop in the jump matrix  $j_S(z)$  the terms that vanish as  $n \rightarrow \infty$ :

- $\mathbf{M}(z)$  is analytic in  $\mathbb{C} \setminus [-\beta, \beta]$ .
- $\mathbf{M}_+(z) = \mathbf{M}_-(z)j_M(z)$  for  $z \in [-\beta, \beta]$ , where

$$(316) \quad j_M(z) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in (-\beta, -\alpha) \cup (\alpha, \beta), \\ \begin{pmatrix} e^{-i\Omega_n} & 0 \\ 0 & e^{i\Omega_n} \end{pmatrix}, & z \in (-\alpha, \alpha). \end{cases}$$

- As  $z \rightarrow \infty$ ,

$$(317) \quad \mathbf{M}(z) \sim I + \frac{\mathbf{M}_1}{z} + \frac{\mathbf{M}_2}{z^2} + \dots.$$

The solution to this RHP is described in terms of Jacobi theta functions, and is presented in [Bleher and Liechty \(2010\)](#), Section 8.

Consider the function  $u(z)$  defined in (165). This function is analytic for  $z \in \mathbb{C} \setminus [-\beta, \beta]$ . On that interval it satisfies certain jump conditions [see [Bleher and Liechty \(2010\)](#), Section 8]. We will use the Jacobi theta functions  $\vartheta_j(z)$ , ( $j = 3, 4$ ) with elliptic nome  $q$  given by (21). The solution is slightly different for  $n$  odd and  $n$  even. Using the notation  $\epsilon(n)$  introduced in (4), we can write the solution in the following uniform way:

$$(318) \quad \mathbf{M}(z) = \mathbf{F}(\infty)^{-1} \begin{pmatrix} \frac{\gamma(z) + \gamma^{-1}(z)}{2} \frac{\vartheta_3(u(z) - \pi/4 - \pi(\tau - \epsilon(n)))}{\vartheta_3(u(z) - \pi/4)} & \\ \frac{\gamma(z) - \gamma^{-1}(z)}{2i} \frac{\vartheta_3(u(z) + \pi/4 - \pi(\tau - \epsilon(n)))}{\vartheta_3(u(z) + \pi/4)} & \\ \frac{\gamma(z) - \gamma^{-1}(z)}{-2i} \frac{\vartheta_3(u(z) + \pi/4 + \pi(\tau - \epsilon(n)))}{\vartheta_3(u(z) + \pi/4)} & \\ \frac{\gamma(z) + \gamma^{-1}(z)}{2} \frac{\vartheta_3(u(z) - \pi/4 + \pi(\tau - \epsilon(n)))}{\vartheta_3(u(z) - \pi/4)} & \end{pmatrix},$$

where

$$(319) \quad \mathbf{F}(\infty) = \begin{pmatrix} \frac{\vartheta_3(\pi(\tau - \epsilon(n)))}{\vartheta_3(0)} & 0 \\ 0 & \frac{\vartheta_3(\pi(\tau - \epsilon(n)))}{\vartheta_3(0)} \end{pmatrix}.$$

The entries of the matrix

$$(320) \quad \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix}^{-1} \mathbf{M}(z) \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \end{pmatrix},$$

are listed in (169)–(172). Notice that the ratios of theta functions in (318) and (319) become trivial when  $\tau = \epsilon(n)$ . The coefficient  $\mathbf{M}_1$  in the expansion of  $\mathbf{M}(z)$  at  $z = \infty$  is

$$(321) \quad \mathbf{M}_1 = \begin{pmatrix} \frac{\pi\beta\vartheta_3'(\pi(\tau - \epsilon(n)))}{4\tilde{\mathbf{K}}\vartheta_3(\pi(\tau - \epsilon(n)))} & -\frac{\beta - \alpha}{2i} \frac{\vartheta_3(0)\vartheta_4(\pi(\tau - \epsilon(n)))}{\vartheta_4(0)\vartheta_3(\pi(\tau - \epsilon(n)))} \\ \frac{\beta - \alpha}{2i} \frac{\vartheta_3(0)\vartheta_4(\pi(\tau - \epsilon(n)))}{\vartheta_4(0)\vartheta_3(\pi(\tau - \epsilon(n)))} & \frac{\pi\beta\vartheta_3'(\pi(\tau - \epsilon(n)))}{4\tilde{\mathbf{K}}\vartheta_3(\pi(\tau - \epsilon(n)))} \end{pmatrix},$$

and the (21)-entry of the coefficient  $\mathbf{M}_2$  is

$$(322) \quad [\mathbf{M}_2]_{21} = \frac{\pi\beta(\beta - \alpha)\vartheta_3(0)\vartheta_4'(\pi(\tau - \epsilon(n)))}{8i\vartheta_3(\pi(\tau - \epsilon(n)))\vartheta_4(0)\tilde{\mathbf{K}}}.$$

Notice that according the RHP for  $\mathbf{M}(z)$ ,  $\det \mathbf{M}(z)$  is entire. Since  $\det \mathbf{M}(\infty) = 1$ , it follows from Liouville’s theorem that  $\det \mathbf{M}(z) \equiv 1$ .

6.2.2. *The local solution at  $\pm\alpha$  and  $\pm\beta$ .* Consider small disks  $D(\pm\alpha, \varepsilon)$  and  $D(\pm\beta, \varepsilon)$  around  $\pm\alpha$  and  $\pm\beta$  with radius  $\varepsilon$ . We seek a local parametrix  $\mathbf{U}(z)$  in these disks satisfying:

- $\mathbf{U}(z)$  is analytic in  $\{D(\pm\alpha, \varepsilon) \cup D(\pm\beta, \varepsilon)\} \setminus \Sigma_S$ .
- For  $z \in \{D(\pm\alpha, \varepsilon) \cup D(\pm\beta, \varepsilon)\} \cap \Sigma_S$ ,  $\mathbf{U}(z)$  satisfies the jump condition  $\mathbf{U}_+(z) = \mathbf{U}_-(z)j_S(z)$ .
- On the boundary of the disks,  $\mathbf{U}(z)$  satisfies

$$(323) \quad \mathbf{U}(z) = \mathbf{M}(z)(I + \mathcal{O}(n^{-1})), \quad z \in \partial D(\pm\alpha, \varepsilon) \cup \partial D(\pm\beta, \varepsilon).$$

The solution is given explicitly in terms of Airy functions, and we do not describe it here.

6.2.3. *The final transformation of the RHP.* We now consider the contour  $\Sigma_X$ , which consists of the circles  $\partial D(\pm\beta, \varepsilon)$  and  $\partial D(\pm\alpha, \varepsilon)$ , all oriented counterclockwise, together with the parts of  $\Sigma_S \setminus \{[-\beta, \alpha] \cup [\alpha, \beta]\}$  which lie outside of the disks  $D(\pm\beta, \varepsilon)$ ,  $D(\pm\alpha, \varepsilon)$ . Let

$$(324) \quad \begin{aligned} \mathbf{X}_n(z) &= \begin{cases} \mathbf{S}_n(z)\mathbf{M}(z)^{-1}, & \text{for } z \text{ outside the disks } D(\pm\beta, \varepsilon), D(\pm\alpha, \varepsilon), \\ \mathbf{S}_n(z)\mathbf{U}(z)^{-1}, & \text{for } z \text{ inside the disks } D(\pm\beta, \varepsilon), D(\pm\alpha, \varepsilon). \end{cases} \end{aligned}$$

Then  $\mathbf{X}_n(z)$  satisfies a RHP with jumps on the contour  $\Sigma_X$  which are uniformly close to the identity matrix, and  $\mathbf{X}_n(\infty) = I$ . The solution to this RHP is given explicitly in terms of a Neumann series. Due to the fact that the contours  $\Gamma_{\pm}$  and the real line are very close to one another (at a distance of the order  $n^{-\delta}$ ), we find that  $\mathbf{X}_n(z)$  satisfies

$$(325) \quad \mathbf{X}_n(z) \sim I + \mathcal{O}\left(\frac{1}{n^{1-\delta}(|z| + 1)}\right) \quad \text{as } n \rightarrow \infty,$$

uniformly for  $z \in \mathbb{C} \setminus \Sigma_X$ , which is a weaker error than the  $\mathcal{O}(n^{-1})$  error in [Bleher and Liechty \(2011\)](#).

6.3. *Proofs of Propositions 3.6, 3.9 and 3.10.* We can invert the explicit transformations of the IP in different regions of the complex plane using (324) and (311). The asymptotic formula (325) then gives asymptotic formulas for  $\mathbf{P}_n(z)$ . Considering  $z$  in the region  $D(\delta, \varepsilon, n)$  proves Proposition 3.6. Considering  $z \in E(\varepsilon)$ , and taking  $\delta = 0$  proves Proposition 3.9. For Proposition 3.10, we can invert the explicit transformations with  $\delta = 0$ , and Proposition 3.10 then follows from (306), (307), and the expansions of  $\mathbf{M}(z)$  at  $z = \infty$  given in (321) and (322).

APPENDIX A: CONSTRUCTION OF STEEPEST-DESCENT CONTOURS  $\tilde{\Gamma}$  AND  $\tilde{\Sigma}$

In this appendix, we show that the first and second derivatives of  $I(z)$ , defined in (225), vanish at  $z = 0$ , and construct two contours  $\tilde{\Gamma}$  and  $\tilde{\Sigma}$  lying in the region  $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} \mid \text{Im } z \geq 0\}$  and passing through 0, such that  $\tilde{\Sigma}$  is from  $e^0 \cdot \infty$  to  $e^{\pi i} \cdot \infty$  and  $\tilde{\Gamma}$  is from  $M$  to  $-M$  where  $M > \beta$ . We require that  $\text{Re } I(z)$  attains its unique global maximum on  $\tilde{\Sigma}$  at 0, and attains its unique global minimum on  $\tilde{\Gamma}$  at 0. Since  $\text{Re } I(z)$  is symmetric about the imaginary axis, we only need to construct  $\tilde{\Gamma} \cap D$  and  $\tilde{\Sigma} \cap D$  where

$$(326) \quad D = \{z \in \mathbb{C} \mid \text{Re } z \geq 0 \text{ and } \text{Im } z \geq 0\}$$

and construct the other parts of them by reflection.

To simplify the notation, we take a change of variable

$$(327) \quad u = \frac{z}{\alpha}.$$

Then we have that, by (156), (225) and (287) [Byrd and Friedman (1971), 140.01, page 33]

$$(328) \quad I'(z) = -2\mathbf{K} \left( Z(u) - \left( 1 - \frac{\mathbf{E}}{\mathbf{K}} \right) u \right)$$

where  $Z(u) = Z(u; k) = E(u; k) - \frac{\mathbf{E}}{\mathbf{K}} F(u; k)$ .

REMARK A.1. Here, the arguments of  $Z(u; k)$ , the Jacobi Zeta function, are different from those in Byrd and Friedman (1971) such that our  $u$  is equal to  $\sin \beta$  for the  $\beta$  in  $Z(\beta, k)$  in Byrd and Friedman (1971), 140.02, 03. The Jacobi Zeta function also appears in (199), where the arguments have same meaning as those in Byrd and Friedman (1971), 140.01, but the parameter is  $\tilde{k}$  instead of  $k$ .

Below we collect some results about  $Z(u)$ .

LEMMA A.1. (a)  $Z(u)$  is analytic in  $D$ ,

$$(329) \quad Z'(0) = 1 - \frac{\mathbf{E}}{\mathbf{K}}, Z''(0) = 0 \quad \text{and} \quad Z(u) = ku - \frac{\pi i}{2\mathbf{K}} + \mathcal{O}(u^{-1})$$

as  $u \rightarrow \infty$  in  $D$ .

(b) For  $x \in [0, 1]$ ,  $Z(x)$  is a real-valued function such that

$$(330) \quad Z(0) = Z(1) = 0 \quad \text{and} \quad Z''(x) < 0 \quad \text{for all } u \in (0, 1).$$

(c) For  $x \in [1, k^{-1}]$ ,  $Z(x)$  is a pure imaginary-valued function such that

$$(331) \quad \operatorname{Im} Z(1) = 0, \quad \operatorname{Im} Z\left(\frac{1}{k}\right) = -\frac{\pi}{2\mathbf{K}} \quad \text{and} \quad \frac{d}{dx} \operatorname{Im} Z(x) < 0$$

*for all*  $x \in \left(1, \frac{1}{k}\right)$ .

(d) For  $x \in [k^{-1}, \infty)$ ,  $Z(x) + \pi i / (2\mathbf{K})$  is a real-valued function such that

$$(332) \quad Z\left(\frac{1}{k}\right) + \frac{\pi i}{2\mathbf{K}} = 0 \quad \text{and} \quad \frac{d}{dx} \left( Z(x) + \frac{\pi i}{2\mathbf{K}} \right) > 0,$$

$$\frac{d^2}{dx^2} \left( Z(x) + \frac{\pi i}{2\mathbf{K}} \right) < 0 \quad \text{for all } x \in \left(\frac{1}{k}, \infty\right).$$

(e) For  $y \in [0, \infty)$ ,  $Z(iy)$  is a pure imaginary-valued function such that

$$(333) \quad Z(0) = 0 \quad \text{and} \quad \frac{d}{dy} \operatorname{Im} Z(iy) > 0, \quad \frac{d^2}{dy^2} \operatorname{Im} Z(iy) > 0$$

*for all*  $y \in (0, \infty)$ .

PROOF. The linear term in the asymptotics in part (a) of Lemma A.1 is a direct consequence of the explicit formula of  $Z(u)$  in  $D$ ,

$$(334) \quad Z(u) = \int_0^u \frac{(1 - \mathbf{E}/\mathbf{K}) - k^2 s^2}{\sqrt{(1 - s^2)(1 - k^2 s^2)}} ds,$$

which is given by (328) and (144). In the integrand of (334) the sign of the square root is chosen as  $\sqrt{(1 - s^2)(1 - k^2 s^2)} \sim 1$  as  $s$  approaches 0 from the region  $D$ . To compute the constant term, it suffices to compute the asymptotics of  $Z(iy) - iky = E(iy; k) - (\mathbf{E}/\mathbf{K})F(iy; k) - iky$  as  $y \rightarrow +\infty$ . By Gradshteyn and Ryzhik (2007), 3.152-1, page 279,  $\lim_{y \rightarrow \infty} F(iy; k) = i\mathbf{K}'$ , and by the computation in equations (291) and (293),  $\lim_{y \rightarrow \infty} E(iy; k) - iky = i(\mathbf{K}' - \mathbf{E}')$ . Then an application of Legendre’s relation (286) yields the result.

From the formula (334), it is clear that:  $Z(0) = 0$ ;  $Z(x)$  is real valued for  $x \in [0, 1]$ ;  $\operatorname{Re} Z(x)$  is constant for  $x \in [1, k^{-1}]$ ;  $\operatorname{Im} Z(x)$  is constant for  $x \in [k^{-1}, \infty)$ ; and  $Z(iy)$  is pure imaginary for  $y \in [0, \infty)$ . It is also straightforward to see that

$$(335) \quad Z(1) = E(1; k) - \frac{\mathbf{E}}{\mathbf{K}}F(1; k) = \mathbf{E} - \frac{\mathbf{E}}{\mathbf{K}}\mathbf{K} = 0,$$

and with the help of Byrd and Friedman (1971), 111.09, page 11, and the Legendre’s relation (286),

$$(336) \quad \begin{aligned} Z(k^{-1}) &= E(k^{-1}; k) - \frac{\mathbf{E}}{\mathbf{K}}F(k^{-1}; k) = \mathbf{E} + i(\mathbf{K}' - \mathbf{E}') - \frac{\mathbf{E}}{\mathbf{K}}(\mathbf{K} + i\mathbf{K}') \\ &= \frac{i}{\mathbf{K}}(\mathbf{K}\mathbf{K}' - \mathbf{K}\mathbf{E}' - \mathbf{E}\mathbf{K}') = -\frac{\pi i}{2\mathbf{K}}. \end{aligned}$$

Thus, all the identities in (330), (331), (332), (333) are all proved.

To consider the values of  $Z'(0)$  and  $Z''(0)$ , and the inequalities of  $Z'(u)$  in (331), (332), (333), we write from (334)

$$(337) \quad Z'(u) = \frac{(1 - \mathbf{E}/\mathbf{K}) - k^2u^2}{\sqrt{(1 - u^2)(1 - k^2u^2)}} ds.$$

Note that for  $u \geq 1$ ,

$$(338) \quad \left(1 - \frac{\mathbf{E}}{\mathbf{K}}\right) - k^2u^2 = \frac{1}{\mathbf{K}} \int_0^1 \frac{k^2(s^2 - u^2)}{\sqrt{(1 - s^2)(1 - k^2s^2)}} ds < 0,$$

we obtain the inequality parts of (331), (332), (333) and the evaluation of  $Z'(0)$  and  $Z''(0)$  in (329).

To consider the inequalities of  $Z''(u)$  in (330), (332) and (333), we can write  $Z'(u)$  as

$$(339) \quad Z'(u) = \frac{k}{\mathbf{K}} \sqrt{\frac{1 - u^2}{k^{-2} - u^2}} \int_0^1 \frac{(s^2 - u^2)/(1 - u^2)}{\sqrt{(1 - s^2)(1 - k^2s^2)}} ds,$$

for  $u \in (0, 1)$  and  $u \in (k^{-1}, \infty)$ , where in either case the square root is taken positive value. We observe that  $Z'(u)$  is a decreasing function on  $(0, 1)$  since  $(1 - u^2)/(k^{-2} - u^2)$  and  $(s^2 - u^2)/(1 - u^2)$  are both increasing, while  $Z'(u)$  is also a decreasing function on  $(k^{-1}, \infty)$  by exactly the same reason. Similarly, writing

$$(340) \quad \frac{d}{dy} \operatorname{Im} Z(iy) = \frac{k}{\mathbf{K}} \sqrt{1 + y^2} k^{-2} + y^2 \int_0^1 \frac{(s^2 + y^2)/(1 + y^2)}{\sqrt{(1 + s^2)(1 + k^2s^2)}} ds,$$

we observe that  $\frac{d}{dy} \operatorname{Im} Z(iy)$  is increasing for all  $y \in (0, \infty)$ . This proves the inequality of  $Z''(u)$  in (330), (332) and (333).  $\square$

LEMMA A.2. *The function  $I'(z)$  has only one zero  $z = 0$  in the region  $D$  that is a third-order zero, and  $I^{(4)}(0) > 0$ .*

PROOF. From (329), it is clear that  $u = 0$  is a zero of  $Z(u) - (1 - \mathbf{E}/\mathcal{F})u$  with order at least 3, and then by (328) the same holds for  $I'(z)$ . On the other hand,  $I^{(4)}(z) = \tilde{g}^{(4)}(z)$ , and the explicit computation (236) of  $\tilde{g}^{(4)}(0)$  shows that  $I^{(4)}(0) > 0$ . Below we show that the function  $Z(u) - (1 - \mathbf{E}/\mathcal{F})u$  has only one zero  $u = 0$  in  $D$ , and complete the proof.

We note that by the results in Lemma A.1,  $Z(u) - (1 - \mathbf{E}/\mathcal{F})u$  has no zero in either  $\{z = x|x > 0\}$  or  $\{z = iy|y > 0\}$ , and it does not vanish as  $u \rightarrow \infty$ . So to prove that  $Z(u) - (1 - \mathbf{E}/\mathcal{F})u$  has no other zero in  $D$ , we define a region (see Figure 10)

$$(341) \quad D_R(1) = \{u \in D ||u| \leq R\} \setminus \{u \in \mathbb{C} | \operatorname{Re} u < 1 \text{ and } \operatorname{Im} u < R^{-1}\},$$

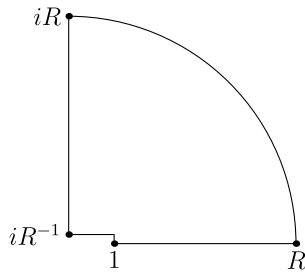


FIG. 10. The region  $D_R(1)$ .

where  $R$  is a positive number, and need only to show that for however large  $R$ ,  $Z(u) - (1 - \mathbf{E}/\mathbf{K})u$  has no zero in the interior of  $D_R(1)$ .

By the results in Lemma A.1, we have that if  $R$  is large enough, then  $Z$  is a homeomorphic mapping on  $\partial D_R(1)$ . Then by a basis result for univalent functions,  $Z$  maps the interior of  $D_R(1)$  into the region enclosed by  $Z(\partial D_R(1))$  that does not contain 0, see Figure 11. Then by a continuity argument, if  $Z(u) - (1 - \mathbf{E}/\mathbf{K})u$  has a zero in the interior of  $D_R(1)$ , there must be a  $t \in (0, 1 - \mathbf{E}/\mathbf{K})$  such that  $Z(u) - tu$  has a zero on  $\partial D_R(1)$ , but by the results in Lemma A.1, for all such  $t$ ,  $Z(u) - tu$  does not vanish on  $\partial D_R(1)$  given that  $R$  is large enough. Thus, we show that  $Z(u) - (1 - \mathbf{E}/\mathbf{K})u$  has no zero other than 0 in  $D$  by contradiction.  $\square$

Now we construct  $\tilde{\Gamma}$ . By (329) and (330), we know that  $\text{Re}(Z(u) - (1 - \mathbf{E}/\mathbf{K})u)$  is decreasing on  $[0, 1]$ . By Lemma A.1(c) and that  $(1 - \mathbf{E}/\mathbf{K}) > 0$ , we also have that  $\text{Re}(Z(u) - (1 - \mathbf{E}/\mathbf{K})u)$  is decreasing on  $[1, k^{-1}]$ . By the relation (328),  $\text{Re } I(z)$  is decreasing on the interval  $[0, \beta]$ . Thus, it suffices to define  $\tilde{\Gamma} \cap D$  as the

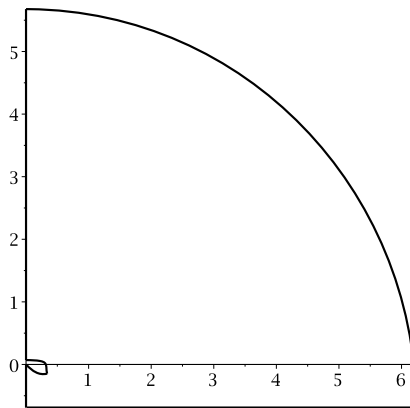


FIG. 11. The shape of  $Z(\partial D_R(1))$  with  $k = 0.9$  and  $R = 7$ .

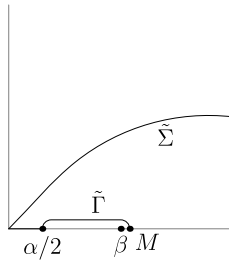


FIG. 12. Schematic graphs of  $\tilde{\Sigma}$  and  $\tilde{\Gamma}$  in  $D$ .

interval  $[0, \beta]$  if we allow  $M = \beta$ . Practically, for the convenience of the asymptotic analysis in Section 5.2, we let  $M$  be slightly bigger than  $\beta$  and define  $\tilde{\Gamma} \cap D$  by a deformation of the interval  $[0, M]$  such that  $[0, \alpha/2]$  is part of  $\tilde{\Gamma} \cap D$  and  $(\alpha/2, M)$  is lifted above slightly; see Figure 12.

In the construction of  $\tilde{\Sigma} \cap D$  and  $\tilde{\Gamma} \cap D$ , we use techniques in planar dynamical systems. Regarding  $\text{Re } I(z)$  as a function defined on the Cartesian plane whose coordinates are  $\text{Re } z$  and  $\text{Im } z$ , we define the gradient field

$$(342) \quad \nabla \text{Re } I(z) = \left( \frac{\partial}{\partial x} \text{Re } I(z), \frac{\partial}{\partial y} \text{Re } I(z) \right)$$

where  $x = \text{Re } z, y = \text{Im } z$ .

By Lemma A.1(a), (e), we have that for  $y > 0, Z(iy) - (1 - \mathbf{E}/\mathbf{K})iy$  is pure imaginary, and its imaginary part is positive. Then by (328), we conclude that  $\{iy | y > 0\}$  is an upward flow curve of  $\nabla \text{Re } I(z)$ . By Lemma A.1(d) and the relation (328), we have that for all  $x > M > \beta, \text{Im}(Z(x) - (1 - \mathbf{E}/\mathbf{K})x) > 0$  and then the gradient field  $\nabla \text{Re } I(z)$  is transversal to the interval  $[M, \infty)$  and is outward of  $D$ .

Since by Lemma A.2, 0 is a triple zero of  $I'(z)$  and  $I^{(4)}(0) > 0$ , there is a flow curve that ends at 0, with direction  $\pi/4$ , and we denote it as  $\gamma$ . Since the gradient field  $\nabla \text{Re } I(z)$  has no singular point by Lemma A.2, this flow curve is from either the boundary of  $D$  or  $\infty$ . As we showed above, the left edge of  $D$  is a flow curve and at the interval  $[M, \infty)$ , as part of  $\partial D$ , the gradient field is outward, so the  $\gamma$  cannot be from the left edge of  $D$  or  $[M, \infty)$ . If  $\gamma$  is from  $(0, M)$ , then it crosses  $\tilde{\Gamma}$  at a point other than 0, denoted by  $z_0$ . But by the definition of  $\tilde{\Gamma}, \text{Re } I(z_0) > \text{Re } I(0)$ . On the other hand, by the property of the flow curve  $\gamma, \text{Re } I(z_0) < \text{Re } I(0)$ , and we derive a contradiction. Thus,  $\gamma$  cannot be from  $\partial D$ , but is from  $\infty$ . At last by the behavior of  $\nabla \text{Re } I(z)$  given in Lemma A.1(a), we verify that it suffices to let  $\tilde{\Sigma} \cap D = \gamma$ , as shown in Figure 12.



APPENDIX B: PROOF OF PROPOSITION 1.5

Since  $\Psi(\zeta; s)$  satisfies [Hastings and McLeod (1980)]

$$(343) \quad \frac{\partial}{\partial s} \Psi(\zeta; s) = \begin{pmatrix} -i\zeta & q(s) \\ q(s) & i\zeta \end{pmatrix} \Psi(\zeta; s),$$

it is easy to derive the identity that for  $u, v \in \Sigma_T$ ,

$$(344) \quad \begin{aligned} \frac{\partial}{\partial s} \left( \frac{f(u; s)g(v; s) - g(u; s)f(v; s)}{u - v} \right) \\ = -i(f(u; s)g(v; s) + g(u; s)f(v; s)), \end{aligned}$$

where  $f$  and  $g$  are defined by  $\Psi$  by (28). Hence, (30) can be written as

$$(345) \quad \begin{aligned} \tilde{K}_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) &= \frac{1}{4\pi^2} \int_{\Sigma_T} du \int_{\Sigma_T} dv e^{\tau_i u^2/2 - \tau_j v^2/2} e^{-i(u\xi - v\eta)} \\ &\quad \times \int_{\sigma}^{\infty} ds (f(u; s)g(v; s) + g(u; s)f(v; s)) \\ &= \int_{\sigma}^{\infty} ds \left[ \left( \frac{1}{2\pi} \oint_{\Sigma_T} du e^{\tau_i u^2/2 - i\xi u} f(u; s) \right) \right. \\ &\quad \times \left( \frac{1}{2\pi} \oint_{\Sigma_T} dv e^{-\tau_j v^2/2 + i\eta v} g(v; s) \right) \\ &\quad + \left( \frac{1}{2\pi} \oint_{\Sigma_T} du e^{\tau_i u^2/2 - i\xi u} g(u; s) \right) \\ &\quad \left. \times \left( \frac{1}{2\pi} \oint_{\Sigma_T} dv e^{-\tau_j v^2/2 + i\eta v} f(v; s) \right) \right]. \end{aligned}$$

In order to relate formula (345) for the tacnode kernel to the other formula (45) defined by Airy function and related operators, we consider the expressions for the entries of  $\Psi(\zeta; s)$  in terms of Airy functions. Introduce the functions in  $x$  with parameters  $\zeta$  and  $s$ ,

$$(346) \quad \begin{aligned} E_+(x) &= E_+(x; \zeta, s) := e^{i((4/3)\zeta^3 + (s+2x)\zeta)}, \\ E_-(x) &= E_-(x; \zeta, s) := e^{-i((4/3)\zeta^3 + (s+2x)\zeta)} = E_+(x; -\zeta, s). \end{aligned}$$

Then the matrix entries of  $\Psi(\zeta; s)$  are given by the formulas

$$(347) \quad \Psi_{11}(\zeta; s) = \langle E_-, R_s + \delta_0 \rangle_0, \quad \Psi_{21}(\zeta; s) = -\langle E_-, Q_s \rangle_0,$$

$$(348) \quad \Psi_{12}(\zeta; s) = -\langle E_+, Q_s \rangle_0, \quad \Psi_{22}(\zeta; s) = \langle E_+, R_s + \delta_0 \rangle_0,$$

where the inner product  $\langle \cdot, \cdot \rangle_0$ , functions  $R_s, Q_s$ , and the delta function  $\delta_0$  are defined in Section 1.2. The derivation of (347) is essentially given in Baik, Liechty and Schehr (2012). Note that the functions  $\Phi_1(\zeta; s)$  and  $\Phi_2(\zeta; s)$  in Baik, Liechty

and Schehr (2012), Proposition 2.1, are the same as the functions  $\Phi^1(\zeta; s)$  and  $\Phi^2(\zeta; s)$  in Claeyns and Kuijlaars (2006), and the entries  $\Psi_{11}(\zeta; s)$  and  $\Psi_{21}(\zeta; s)$  are the same as the functions  $\Phi_1(\zeta; s)$  and  $\Phi_2(\zeta; s)$  in Claeyns and Kuijlaars (2006). Using the relation given in equation (1.19) of Claeyns and Kuijlaars (2006), equation (347) follows from Proposition 2.1 of Baik, Liechty and Schehr (2012). By the relation (15), (347) implies (348).

Consider now the integrals

$$(349) \quad I_{a,b;s}^\pm(x) := \frac{1}{2\pi} \oint_{\Sigma_T^\pm} e^{a\zeta^2 + ib\zeta} E_\pm(x; \zeta, s) d\zeta,$$

where  $\Sigma_T^+$  (resp.,  $\Sigma_T^-$ ) is the connected piece of  $\Sigma_T$  which lies above (resp., below) the real axis. A simple change of variables gives that

$$(350) \quad \begin{aligned} I_{a,b;s}^+(x) &= \frac{1}{2\pi} \oint_{\Sigma_T^+} e^{i(4/3)\zeta^3 + a\zeta^2 + i(s+2x+b)\zeta} d\zeta \\ &= -2^{-2/3} e^{-a^3/24 - a(s+2x+b)/4} \text{Ai}\left(\frac{s+2x+b}{2^{2/3}} + \frac{a^2}{2^{8/3}}\right), \end{aligned}$$

where we have used the integral representation of the Airy function

$$(351) \quad \text{Ai}(x) = \frac{-1}{2\pi} \oint_{\Sigma_T^+} e^{(i/3)\zeta^3 + ix\zeta} d\zeta.$$

Similarly,

$$(352) \quad I_{a,b;s}^-(x) = 2^{-2/3} e^{-a^3/24 - a(s+2x-b)/4} \text{Ai}\left(\frac{s+2x-b}{2^{2/3}} + \frac{a^2}{2^{8/3}}\right).$$

We can now write the expression (345) in terms of Airy functions and operators only, since the functions  $f$  and  $g$  there are expressed by entries of  $\Psi$ . Notice that in the expressions (347) and (348) for the entries of the matrix  $\Psi$ , the dependence on  $\zeta$  lies solely in the left-hand side of the inner products. Thus, by changing the order of integration, we can write (345) in terms of the integrals  $I_{a,b;s}^\pm(x)$ . Indeed we have

$$(353) \quad \begin{aligned} \tilde{K}_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) &= \int_\sigma^\infty ds [(\langle I_{\tau_i/2, -\xi; s}^-, R_s + \delta_0 \rangle_0 + \langle I_{\tau_i/2, -\xi; s}^+, Q_s \rangle_0) \\ &\quad \times (-\langle I_{-\tau_j/2, \eta; s}^+, R_s + \delta_0 \rangle_0 - \langle I_{-\tau_j/2, \eta; s}^-, Q_s \rangle_0) \\ &\quad + (-\langle I_{\tau_i/2, -\xi; s}^+, R_s + \delta_0 \rangle_0 - \langle I_{\tau_i/2, -\xi; s}^-, Q_s \rangle_0) \\ &\quad \times (\langle I_{-\tau_j/2, \eta; s}^-, R_s + \delta_0 \rangle_0 + \langle I_{-\tau_j/2, \eta; s}^+, Q_s \rangle_0)]. \end{aligned}$$

Notice that in terms of the function  $b_{\tau,z,\sigma}$  defined in (43),

$$(354) \quad \begin{aligned} I_{\tau,z;s}^+(x) &= -2^{-2/3} \pi b_{2^{-4/3}\tau, 2^{-2/3}z, s}(x), \\ I_{\tau,z;s}^-(x) &= 2^{-2/3} \pi b_{2^{-4/3}\tau, -2^{-2/3}z, s}(x). \end{aligned}$$

Hence, formula (353) becomes

$$\begin{aligned}
 & \tilde{K}_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) \\
 &= 2^{-4/3} \int_{\sigma}^{\infty} ds [(\langle b_{2^{-7/3}\tau_i, 2^{-2/3}\xi, s}, R_s + \delta_0 \rangle_0 - \langle b_{2^{-7/3}\tau_i, -2^{-2/3}\xi, s}, Q_s \rangle_0) \\
 &\quad \times (\langle b_{-2^{-7/3}\tau_j, 2^{-2/3}\eta, s}, R_s + \delta_0 \rangle_0 \\
 (355) \quad &\quad - \langle b_{-2^{-7/3}\tau_j, -2^{-2/3}\eta, s}, Q_s \rangle_0) \\
 &\quad + (\langle b_{2^{-7/3}\tau_i, -2^{-2/3}\xi, s}, R_s + \delta_0 \rangle_0 - \langle b_{2^{-7/3}\tau_i, 2^{-2/3}\xi, s}, Q_s \rangle_0) \\
 &\quad \times (\langle b_{-2^{-7/3}\tau_j, -2^{-2/3}\eta, s}, R_s + \delta_0 \rangle_0 \\
 &\quad \quad - \langle b_{-2^{-7/3}\tau_j, 2^{-2/3}\eta, s}, Q_s \rangle_0)],
 \end{aligned}$$

which is, in terms of the function  $\hat{p}_1(z; s, \tau)$  defined in (46),

$$\begin{aligned}
 & \tilde{K}_{\tau_i, \tau_j}^{\text{tac}}(\xi, \eta; \sigma) \\
 &= 2^{-4/3} \int_{\sigma}^{\infty} ds (\hat{p}_1(-2^{-2/3}\xi; s, 2^{-7/3}\tau_i) \hat{p}_1(-2^{-2/3}\eta; s, -2^{-7/3}\tau_j) \\
 (356) \quad &\quad + \hat{p}_1(2^{-2/3}\xi; s, 2^{-7/3}\tau_i) \hat{p}_1(2^{-2/3}\eta; s, -2^{-7/3}\tau_j)) \\
 &= 2^{-2/3} \tilde{\mathcal{L}}_{\text{tac}}(2^{-2/3}\xi, 2^{-2/3}\eta; \sigma, 2^{-7/3}\tau_i, 2^{-7/3}\tau_j),
 \end{aligned}$$

where  $\tilde{\mathcal{L}}_{\text{tac}}$  is defined in (44). It is simple to see that by (26)

$$(357) \quad 2^{-2/3} (\phi_{2 \cdot 2^{-7/3}\tau_i, 2 \cdot 2^{-7/3}\tau_j}(2^{-2/3}\xi, 2^{-2/3}\eta)) = \phi_{\tau_i, \tau_j}(\xi, \eta).$$

Combining (355) and (357) gives (47), and Proposition 1.5 is proved.

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