Asymptotics of free fermions in a quadratic well at finite temperature and the Moshe–Neuberger–Shapiro random matrix model

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Abstract

We derive the local statistics of the canonical ensemble of free fermions in a quadratic potential well at finite temperature, as the particle number approaches infinity. This free fermion model is equivalent to a random matrix model proposed by Moshe, Neuberger and Shapiro. Limiting behaviors obtained before for the grand canonical ensemble are observed in the canonical ensemble: We have at the edge the phase transition from the Tracy–Widom distribution to the Gumbel distribution via the Kardar–Parisi–Zhang (KPZ) crossover distribution, and in the bulk the phase transition from the sine point process to the Poisson point process. A similarity between this model and a class of models in the KPZ universality class is explained. We also derive the multi-time correlation functions and the multi-time gap probability formulas for the free fermions along the imaginary time.

1 Introduction

In this paper we consider the spinless free fermions on $\mathbb{R}^1$ in quadratic potential well (aka harmonic oscillators) at finite temperature. This model was defined by Moshe, Neuberger and Shapiro [26] in the 1990's, further studied by Johansson [19] in the 2000's, and very recently considered in the physics literature by Dean, Le Doussal, Majumdar, Schehr et al [11], [12], [22]. See also [23] for a dynamical version of the model, and [10] for a generalization to other symmetry types.

The most interesting question on this model (later called the MNS model) is the limiting behavior of the fermions at the edge or in the bulk as the number of particles $n \to \infty$. From the physical point of view, the existing result is already rather complete. When the temperature is low enough, the limiting distribution of the rightmost particle is given by the celebrated Tracy–Widom distribution, and when the temperature is high enough, the limiting distribution is given by the Gumbel distribution. At the critical temperature, the limiting distribution is found to be the crossover distribution in the 1-dimensional Kardar–Parisi–Zhang (KPZ) universality class. For particles in the bulk, analogous results are obtained which interpolate between the sine point process and the Poisson point process.

The original version proposed by Moshe, Neuberger and Shapiro is the *canonical ensemble* of the model, but all the asymptotic results available currently in the mathematical literature

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are for the *grand canonical ensemble* of the model (although several recent works in the physical literature [11], [12], [22] consider the canonical ensemble). It is a universally accepted wisdom in statistical physics that the physical properties of the grand canonical ensemble are the same as those of the canonical ensemble, as the particle number approaches infinity. In the case of the MNS model, the grand canonical ensemble has a special mathematical feature that it is a *determinantal point process*, which makes it easier to analyze mathematically than the canonical ensemble. The goal of this paper is to analyze the canonical ensemble of the MNS model, and rigorously prove that the limiting results obtained for the grand canonical ensemble hold for the canonical ensemble as well.

Our purpose is not rigor for rigor's sake. As suggested by the title, the canonical ensemble of the MNS model is associated to a random matrix model (later referred as the MNS random matrix model) whose dimension is equal to the number of particles in the MNS model. Such a relation is not preserved when we move to the grand canonical ensemble. Also in the course of our derivation, we find that the algebraic as well as the analytic properties of the canonical ensemble of the MNS model are analogous to those of the q-Whittaker processes, which are a subclass of the extensively studied Macdonald processes [6]. The q-Whittaker processes contain many interacting particle models in the KPZ universality class as specializations. Although the q-Whittaker processes are in some sense integrable, they are considerably more difficult than determinantal processes. The similarity between probability models in KPZ universality class and free fermions at positive temperature has been noticed in [17], but the relation is via determinantal process. We hope that our analysis of the canonical ensemble of the MNS model sheds light on the study of the q-Whittaker processes and more generally the KPZ universality class.

1.1 q-analog Notation

In this paper, we need the following q-analog notation, which converge to their common counterparts as $q \to 1_-$.

The q-Pochhammer symbol is

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 0,1,2,\dots,\infty. \tag{1}$$

For $x \in \mathbb{C}$ with $|x| < 1$, the basic hypergeometric function ${}_2\phi_1\left[\begin{smallmatrix}a,b\\c\end{smallmatrix};q,x\right]$ is defined by the series [16, Formula (1.2.14)]

$${}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q,x\right] = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}x^n. \tag{2}$$

The basic hypergeometric function can be analytically continued [16, Section 4.3], [4, Section 10.9], [28, Sections 17.4 and 17.6], to a meromorphic function on $\mathbb{C}$ with simple poles $1, q^{-1}, q^{-2}, \dots$, by the alternative expression given in [4, Formula (10.10.12)] or equivalently [16, Formula (1.5.4)]. We note that the basic hypergeometric function has the Watson's contour integral representation [16, Section 4.1]. Here we remark that formally the Watson's contour integral [16, Formula (4.2.2)] for ${}_2\phi_1\left[\begin{smallmatrix}a,b\\c\end{smallmatrix};q,x\right]$ has a branch cut along the real axis. But since we know that the function ${}_2\phi_1\left[\begin{smallmatrix}a,b\\c\end{smallmatrix};q,x\right]$ is meromorphic in x, the integral is a well-defined on $\mathbb{C} \setminus \{1, q^{-1}, q^{-2}, \dots\}$.

In this paper, we only use a special case of the basic hypergeometric function, such that $b = q$ in (2). Then ${}_2\phi_1\left[\begin{smallmatrix}a,q\\c\end{smallmatrix};q,x\right] = F(aq^{-1}, cq^{-1}; x)$ that is defined in [15, Formula (1.1')], and the analytic continuation is by [15, Section 1.3]. In this special case, Watson's contour integral representation becomes

$$ {}_2\phi_1 \left[\begin{matrix} a, q \\ c \end{matrix} ; q, z \right] = \frac{(a;q)_\infty}{(c;q)_\infty} \frac{-1}{2\pi i} \int_{-i\cdot\infty}^{i\cdot\infty} \frac{(cq^s;q)_\infty}{(aq^s;q)_\infty} \frac{\pi(-z)^s}{\sin(\pi s)} ds, \tag{3} $$

where the contour satisfies that the poles $0, 1, 2, \dots$ are on its right, and the poles $-1, -2, \dots$ and $-n - (\log a + 2\pi i k)/\log q$ with $k \in \mathbb{Z}$ and $n = 0, 1, 2, \dots$ are on its left.

1.2 Definition of the MNS model

First recall the one-dimension harmonic oscillator in quantum mechanics. The time-independent Hamiltonian of the free particle in a quadratic potential well is, on the position space,

$$ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2} x^2. \tag{4} $$

In this paper, we assume $\hbar = 1$, $m = 1/2$, and $\omega = 1$, and then

$$ H = -\frac{\partial^2}{\partial x^2} + \frac{x^2}{4}. \tag{5} $$

The eigenfunctions of the Hamiltonian H defined in (5) are

$$ \varphi_k(x) = \left(\frac{1}{\sqrt{2\pi} k!} \right)^{1/2} H_k(x) e^{-x^2/4}, \quad k = 0, 1, 2, \dots, \tag{6} $$

where $H_k(x)$ is the Hermite polynomial, such that it is a monic polynomial of degree k and

$$ \int_{-\infty}^{\infty} \varphi_k(x) \varphi_j(x) dx = \int_{-\infty}^{\infty} H_k(x) H_j(x) e^{-x^2/2} dx = \sqrt{2\pi} k! \delta_{kj}. \tag{7} $$

See [1, Chapter 22] for basic properties of Hermite polynomials. Note that in [1], polynomial $H_n(x)$ is denoted as $He_n(x)$, while the notation $H_n(x)$ is reserved for a slightly different polynomial, see [1, 22.5.18]. The eigenvalue/energy level for eigenstate $\varphi_k(x)$ is $k + 1/2$, $(k = 0, 1, 2, \dots,)$ since

$$ H\varphi_k(x) = \left(-\frac{d^2}{dx^2} + \frac{x^2}{4} \right) \varphi_k(x) = \left(k + \frac{1}{2} \right) \varphi_k(x). \tag{8} $$

Suppose n identical fermions are independent harmonic oscillators, or in other words they are free fermions in the quadratic potential well. The fermion system has eigenstates indexed by $(k_1, k_2, \dots, k_n)$ where $0 \leq k_1 < k_2 < \dots < k_n$ are integers, such that the energy level of the eigenstate is $k_1 + k_2 + \dots + k_n + n/2$, and the eigenfunction is given by the Slater determinant

$$ \Phi_{k_1,\dots,k_n}(x_1, \dots, x_n) = \frac{1}{\sqrt{n!}} \begin{vmatrix} \varphi_{k_1}(x_1) & \dots & \varphi_{k_1}(x_n) \\ \vdots & & \vdots \\ \varphi_{k_n}(x_1) & \dots & \varphi_{k_n}(x_n) \end{vmatrix}. \tag{9} $$

In this eigenstate, the density function for the n particles is $|\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)|^2$. (This density function is normalized, see (55).)

For a quantum system at temperature T, all eigenstates occur at a certain chance according to the *Boltzmann distribution*, such that the probability for an eigenstate with energy level E to occur is $Z^{-1}e^{-E/(\kappa T)}$ where Z is the normalization constant and κ is the Boltzmann constant that we assume to be 1 later [29, Section 6.2]. Hence for the n-particle canonical ensemble of the MNS model, that is, n free fermions in the quadratic potential well, if the temperature is $T > 0$, and if we denote

$$q = e^{-1/(\kappa T)} = e^{-1/T}, \tag{10}$$

the probability for eigenstate $(k_1, k_2, \dots, k_n)$ to occur is $Z_n(q)^{-1} q^{k_1+\cdots+k_n+n/2}$, where

$$Z_n(q) = \sum_{0 \le k_1 < k_1 < \cdots < k_n} q^{k_1+\cdots+k_n+n/2} = \frac{q^{n^2/2}}{(q;q)_n}, \tag{11}$$

and we have that the density function for the n particles is

$$\begin{aligned} P_n(x_1, \dots, x_n) &= \frac{1}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} |\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)|^2 q^{k_1+\cdots+k_n+n/2} \\ &= \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} |\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)|^2 q^{k_1+\cdots+k_n}. \end{aligned} \tag{12}$$

The equivalence of the two expressions in (11) may not be obvious, so we present a short proof at the beginning of Section 2.

The n-particle canonical ensemble of the MNS model at temperature $T = -(\log q)^{-1} > 0$, which is called simply the MNS model if there is no possibility of confusion, is the main topic of this paper. Although it is defined in the language of quantum mechanics, all our analysis is based on the density function (12), so it is harmless to understand the MNS model as a particle model with density (12). We note that in the limit $T \to 0$, the density function $P_n(x_1, \dots, x_n)$ degenerates into $|\Phi_{0,1,\dots,n-1}(x_1, \dots, x_n)|^2$, the density function for the ground state of the quantum system. One readily recognizes that this $T \to 0$ limiting density is the density of eigenvalues of a random matrix in the Gaussian Unitary Ensemble (GUE) [3, Section 2.5], that is, the random Hermitian matrix model defined below in (15). It is then not a surprise that for general $T > 0$, then density (12) is also the eigenvalue density function of a random matrix ensemble.

1.3 MNS random matrix model

The random matrix model defined by Moshe, Neuberger and Shapiro [26] is an unitarily invariant generalization of the GUE with a continuous parameter, such that as the parameter varies, the limiting local statistics of the MNS random matrix model interpolate between the sine point process that is the hallmark of random Hermitian matrices including the GUE, and the Poisson point process.

The space of n-dimensional Hermitian matrices has a natural measure

$$dX = \prod_{i=1}^{n} dx_{ii} \prod_{1 \le j < k \le n} d\Re x_{jk} d\Im x_{jk}, \tag{13}$$

where $X = (x_{jk})_{j,k=n}^{n}$. Let U be a random unitary matrix in $\mathrm{U}(n)$ with respect to the Haar measure. We say that a random Hermitian matrix H is an MNS random matrix if [26, Formulas (1) and (2)]

$$\begin{aligned} P(H)dH &= \frac{1}{C(n,b)} e^{\operatorname{Tr} H^2} e^{-b \operatorname{Tr}([U,H][H,H]^\dagger)} dH \\ &= \frac{1}{C(n,b)} e^{-(2b+1)\operatorname{Tr} H^2} \left[\int_{\mathrm{U}(n)} dU e^{2b \operatorname{Tr}(UHH^\dagger H)} \right] dH. \end{aligned} \tag{14}$$

By comparing the eigenvalue distribution of H and the known density function of free fermions in a quadratic potential well at finite temperature, Moshe, Neuberger and Shapiro observe the following relation.

Proposition 1. *[26, Formula (4)] Suppose the n-dimensional Hermitian random matrix is defined by* (14)*, and suppose the parameter $b = q/(1-q)^2$ with $q \in (0,1)$. Then the joint probability density function of the eigenvalues of $\sqrt{\frac{1}{2}(1-q)/(1+q)}H$ is the same as the density function $P_n(x_1, \dots, x_n)$ defined in* (12)*.*

We note that if we denote the $q \to 0$ limit of $2^{-1/2}H$ by X, then X has the density function

$$P(X)dX = \frac{1}{2^{n/2}\pi^{n^2/2}} \exp\left(-\frac{1}{2}\operatorname{Tr}(X^2)\right) dX, \tag{15}$$

or equivalently, $X_{ii} = N(0, 1/2)$, $\Re X_{jk} = N(0,1)$, $\Im X_{jk} = N(0,1)$ for $1 \le i \le n$ and $1 \le j < k \le n$, and they are independent. This is the celebrated GUE ensemble in dimension n [3, Section 2.5].

The authors of [26] give half of the proof to Proposition 1, and point out that the other half is available in physics literature, see [8]. For the sake of our readers, we provide a brief proof of the part omitted in [26] in Appendix A.

1.4 Statement of results

As the particle number $n \to \infty$, we are interested in the limiting distribution of the rightmost particle in the MNS model. The probability that the position of the rightmost particle is

$$\mathbb{P}_n(\max(x_1, \dots, x_n) \le s) = \mathbb{P}_n(x_1, \dots, x_n \in (-\infty, s]), \tag{16}$$

a special case of the *gap probability*, which is the probability $\mathbb{P}_n(x_1, \dots, x_n \in A)$ for a measurable set $A \subseteq \mathbb{R}$.

We are also interested in the limiting local statistics of particles in the bulk. The gap probability is not an efficient way to describe the local statistics in the bulk, and we compute the limiting *m-correlation functions*, which are defined as

$$R_n^{(m)}(x_1, \dots, x_m) = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^m} \mathbb{P}_n(\text{there is at least one particle in each } [x_i, x_i + \Delta x),\ i = 1, 2, \dots, m). \tag{17}$$

Since the eigenvalue distribution of the MNS random matrix model is also given in (12), the gap probability (16) and the m-correlation functions (17) are the same for the eigenvalues of MNS random matrix model.

For the MNS (random matrix) model, the gap probability and m-correlation functions can be explicitly computed by a contour integral.

Theorem 1. *Given the joint distribution $P_n(x_1, \dots, x_n)$ in* (12) *for n-particles, we have the follows:*

(a) The gap probability is

$$\mathbb{P}_n(x_1, \dots, x_n \in A) = \frac{1}{2\pi i} \oint_0 F(z) \det(I - \mathbf{K}(z;q)\chi_{A^c}) \frac{dz}{z}, \tag{18}$$

where

$$F(z) = q^{-n(n-1)/2}(q;q)_n \frac{(-z;q)_\infty}{z^n}, \tag{19}$$

and $\mathbf{K}(z;q)$ *is the integral operator on* $L^2(\mathbb{R})$*, defined by*

$$\mathbf{K}(z;q)(f)(x) = \int_{\mathbb{R}} K(x,y;z;q)f(y)dy, \quad K(x,y;z;q) = \sum_{k=0}^{\infty} \frac{q^k z}{1+q^k z}\varphi_k(x)\varphi_k(y). \tag{20}$$

(b) *The* m*-correlation function is*

$$R_n^{(m)}(x_1,\dots,x_m) = \frac{1}{2\pi i}\oint_0 F(z)\det(K(x_i,x_j;z;q))_{i,j=1}^m \frac{dz}{z}, \tag{21}$$

and $K(x_i,x_j;z;q)$ *is defined in* (20).

Remark 1. The kernel function $K(x,y;z;q)$ has the double contour integral representation

$$K(x,y;z;q) = \frac{z}{1+z}\frac{e^{\frac{y^2-x^2}{4}}}{(2\pi i)^2}\int_{-i\infty}^{i\infty} ds \oint_{\Gamma_s} dt \frac{e^{(s-x)^2/2}}{e^{(t-y)^2/2}} \,{}_2\phi_1\left[\begin{matrix}-z,q\\-zq\end{matrix};q,\frac{qs}{t}\right], \tag{22}$$

where Γ_s is a positive oriented contour around 0 such that 0 and all the poles $q^k s$ $(k=1,2,\dots)$ are enclosed in Γ_s. The equivalence of (20) and (22) is proved in the end of Section 2.1. This kernel formula is comparable to the kernel formulas (234) occurring in Section 5 for q-Whittaker processes and related particle models.

Remark 2. A formula similar to (21) has appeared recently in the physical literature [12, equation (86)].

We note that in a determinantal process, the m-correlation function is determined by a correlation kernel, and the gap probability is a Fredholm determinant involving the correlation kernel. Moreover, the gap probability has a series expansion by the integral of m-correlation functions, see [34]. Although the MNS model is not a determinantal process, the gap probability and m-correlation functions are also related. See Section 2.3 for detail.

For the rightmost particle in the MNS model, or equivalently, the largest eigenvalue in the MNS random matrix model, we state the limiting distribution in two regimes. If the parameter q is in a compact subset of $(0,1)$, the limiting distribution is the celebrated Tracy–Widom distribution, with the probability distribution function defined by the Fredholm determinant of $\mathbf{K}_{\mathrm{Airy}}$, an operator on $L^2(\mathbb{R})$ with kernel $K_{\mathrm{Airy}}(x,y)$. ($\mathbf{P}_t$ is the projection operator such that $\mathbf{P}_t f(x) = f(x)\chi_{(t,\infty)}(x)$.)

$$F_{\mathrm{GUE}}(t) = \det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{Airy}}\mathbf{P}_t), \quad \text{and} \quad K_{\mathrm{Airy}}(x,y) = \int_0^\infty \mathrm{Ai}(x+r)\,\mathrm{Ai}(y+r)dr. \tag{23}$$

If the parameter q is scaled to be close to 1, such that $1-q = \mathcal{O}(n^{-1/3})$ as $n\to\infty$, the limiting distribution is the so-called crossover distribution that occurs in the weak coupling regime of the Kardar–Parisi–Zhang (KPZ) equation [2], [9], [30], and interpolates the Tracy–Widom distribution and the Gumbel distribution [19]. Its probability distribution function is defined by the Fredholm determinant of $\mathbf{K}_{\mathrm{cross}}(c)$, an operator on $L^2(\mathbb{R})$ depending on a continuous parameter $c\in\mathbb{R}$, such that its kernel is $K_{\mathrm{cross}}(x,y;c)$ given below:

$$F_{\mathrm{cross}}(t;c) = \det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{cross}}(c)\mathbf{P}_t), \quad \text{and} \quad K_{\mathrm{cross}}(x,y;c) = \int_{-\infty}^\infty \frac{e^{-cr}}{1+e^{-cr}}\mathrm{Ai}(x-r)\,\mathrm{Ai}(y-r)dr. \tag{24}$$

It is clear that as the parameter $c\to-\infty$, $F_{\mathrm{cross}}(t;c)\to F_{\mathrm{GUE}}(t)$. Our $K_{\mathrm{cross}}(x,y;c)$ is the correlation kernel of the "interpolating process" in [19].

Theorem 2. *Suppose as $n \to \infty$, s depends on n as*

$$s \equiv s_n = 2\sqrt{n} + tn^{-1/6}. \tag{25}$$

Then we have the follows.

(a) Suppose $q \in (0,1)$ is independent of n,

$$\lim_{n\to\infty} \mathbb{P}(\max(x_1, \dots, x_n) \le s_n) = F_{\mathrm{GUE}}(t). \tag{26}$$

(b) Suppose $q = \exp(-cn^{-1/3})$ depending on n, where $c > 0$ is a constant,

$$\lim_{n\to\infty} \mathbb{P}(\max(x_1, \dots, x_n) \le s_n) = F_{\mathrm{cross}}(t; c). \tag{27}$$

For the particles/eigenvalues in the bulk, we also consider their limiting behavior in two regimes. If the parameter q is in a compact subset of $(0,1)$, the positions of particles in an $\mathcal{O}(n^{-1/2})$ window converge to the sine point process [3, Sections 3.5 and 4.2], with the m-correlation functions defined by the correlation kernel

$$R^{(m)}_{\sin}(x_1, \dots, x_m) = \det(K_{\sin}(x_i, x_j))^m_{i,j=1}, \quad \text{where} \quad K_{\sin}(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}. \tag{28}$$

If the parameter is scaled to be close to 1, such that $1 - q = \mathcal{O}(n^{-1})$, the positions of particles in an $\mathcal{O}(n^{-1})$ window converge to a determinantal point process that interpolates the sine process and the Poisson process, with the m-correlation function defined by the correlation kernel depending on a positive continuous parameter

$$R^{(m)}_{\mathrm{inter}}(x_1, \dots, x_m; a) = \det(K_{\mathrm{inter}}(x_i, x_j; a))^m_{i,j=1}, \quad \text{where} \quad K_{\mathrm{inter}}(x, y; a) = \int_0^\infty \frac{\cos(\pi(x-y)t)}{ae^{t^2}+1} dt. \tag{29}$$

We note that as $a \to 0_+$, if $x = \xi/\sqrt{-\log a}$ and $y = \eta/\sqrt{-\log a}$, then

$$\lim_{a\to 0_+} K_{\mathrm{inter}}\left(\frac{\xi}{\sqrt{-\log a}}, \frac{\eta}{\sqrt{-\log a}}; a\right) dy = K_{\sin}(\xi, \eta) d\eta, \quad \text{for } \xi, \eta \text{ in a compact subset of } \mathbb{R}. \tag{30}$$

Our correlation kernel K_{inter} is the same as the kernel L_c in [19, Theorem 1.9] up to a change of scaling.

Theorem 3. *(a) Suppose $n \to \infty$, $q \in (0,1)$ is independent of n, and $x_1, \dots, x_m$ depend on n as*

$$x_i = 2x\sqrt{n} + \frac{\pi \xi_i}{(1-x^2)^{1/2}\sqrt{n}}, \quad i = 1, \dots, m, \tag{31}$$

where ξ_i are constants and $x \in (-1,1)$. Then

$$\lim_{n\to\infty} \left(\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}\right)^m R^{(m)}_n(x_1, \dots, x_m) = R^{(m)}_{\sin}(\xi_1, \dots, \xi_m). \tag{32}$$

(b) Suppose $n \to \infty$, $q = e^{-c/n}$, and $x_1, \dots, x_m$ depend on n as

$$x_i = 2x\sqrt{n} + \frac{\pi \xi_i}{\sqrt{n/c}}, \quad i = 1, \dots, m, \tag{33}$$

where ξ_i are constants and $x \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \left(\frac{\pi}{\sqrt{n/c}}\right)^m R^{(m)}_n(x_1, \dots, x_m) = R^{(m)}_{\mathrm{inter}}\left(\xi_1, \dots, \xi_m; \frac{e^{cx^2}}{e^c - 1}\right). \tag{34}$$

Remark 3. (i) As $q \to 0$, the MNS random matrix model (14) converges to the GUE (15). The Tracy–Widom limit at the edge and the sine limit in the bulk for GUE is a well known result in random matrix theory [3, Chapter 3].

(ii) Our limiting results for the canonical ensemble of the MNS model agree with those obtained in recent physical works [11], [12], as well as results for the grand canonical ensemble [19]. Although the canonical ensemble is not a determinantal point process, as $n \to \infty$ its scaling limits at the edge and in the bulk are both determinantal point processes.

(iii) Since the MNS model can be interpreted as a random matrix model, we would like to expect some universality result in the local statistics. However, in the regime $1 - q = \mathcal{O}(n^{-1})$, Theorem 3(b) shows that the limiting local correlation functions depend on x, the limiting position of the particles. This is different from most other random matrix models, and is a feature which was not observed in earlier studies of the grand canonical ensemble [19], although the kernel $K_{\text{inter}}(x_i, x_j; \frac{e^{cx^2}}{e^c - 1})$ is a specialization of the one obtained recently in [12, equation (274)] for free fermions in d dimensions with general potentials.

We note that the 1-correlation function yields the empirical probability density function $\rho_n(x)$, since

$$\rho_n(x) = \frac{1}{n} R_n^{(1)}(x). \tag{35}$$

From (32) we obtain that if q is fixed, then the limiting empirical probability density function is

$$\lim_{n\to\infty} 2\sqrt{n}\rho_n(2\sqrt{n}x) = \frac{2}{\pi}\sqrt{1 - x^2}, \quad x \in (-1, 1). \tag{36}$$

Here we use the simple property that $K_{\sin}(x, x) = 1$. This shows that the limiting empirical probability density of the eigenvalues is the semicircle law, the same as that of the GUE random matrix. On the other hand,

$$K_{\text{inter}}(x, x; a) = \frac{-\sqrt{\pi}}{2} \operatorname{Li}_{1/2}(-a^{-1}), \tag{37}$$

where $\operatorname{Li}_{1/2}$ is the polylogarithm [28, 25.12.11]. Hence if $q = e^{-c/n}$,

$$\lim_{n\to\infty} 2\sqrt{n}\rho_n(2\sqrt{n}x) = \frac{-1}{\sqrt{\pi c}} \operatorname{Li}_{1/2}(e^{-cx^2} - e^{c(1-x^2)}). \tag{38}$$

This limiting distribution on the right-hand of (38) is supported on $\mathbb{R}$, but as $c \to +\infty$, it converges to the semicircle law on the right-hand side of (36) which is supported on $[-1, 1]$. The limiting empirical probability density function (38) agrees with [11, Formula (8)]. The asymptotics of $\operatorname{Li}_{1/2}$ can be found in [36].

1.5 Generalizations and related models

The most interesting feature of the MNS model is that its rightmost particle has a similar distribution to the edge particle of several interacting particle models related to Kardar–Parisi–Zhang (KPZ) universality class. In fact, the similarity is not only at the level of limiting distribution, but also at the level of algebraic structure for the finite systems. However, this similarity will be clear only after some technical results are established, so we refer the reader to Section 5 for detail. It is also worth noticing that the very recent preprint [10] suggests random matrix generalizations to the MNS random matrix model. Below we discuss the dynamical generalization of the MNS model and compare it with the nonintersecting Brownian motions on a circle.

1.5.1 Relation to time-periodic Ornstein–Uhlenbeck (OU) processes and the multi-time correlations

The presentation of the relation to OU process and the multi-time correlations is based on the recent preprint [23], and the physical concepts are explained therein.

It is not a surprise that the quantum harmonic oscillator is related to the Ornstein–Uhlenbeck (OU) process. The imaginary time propagator of the harmonic oscillator, or more precisely, the particle with Hamiltonian (5), is, by letting with $\hbar = 1$, $m = 1/2$ and $\omega = 1$ in [23, Formula (17)],

$$G(y, \tau \mid x, 0) = \sum_{k=0}^{\infty} \varphi_k(x)\varphi_k(y)e^{-k\tau}, \tag{39}$$

where τ is the imaginary time. Consider the OU process defined by the stochastic differential equation

$$dx(t) = -x(t)dt + \sqrt{2}dW(t) \tag{40}$$

where $W(t)$ is the Wiener process. (Note that our OU process differs from that defined by [23, Formula (1)] by the choice of constants in the stochastic differential equation.) The imaginary time propagator in (39) is equal to the OU propagator up to a conjugation, see [23, Formula (19)]. Hence free fermions in a quadratic potential well are related to the nonintersecting OU processes, due to the analogy between the Slater determinant for the former and the Karlin–McGregor formula for the latter. In particular, the ground state of free fermions in a quadratic potential has the same probability density function as that of the one-time distribution of the stationary nonintersecting OU processes, that is, the limit of nonintersecting OU processes starting from time $-M$ and ending at time M as $M \to \infty$ and both the starting and ending positions are close to the origin, since their probability density functions are both time invariant and identical to that of the eigenvalues of a GUE random matrix, see [23, Formulas (7) and (62)].

A key observation in [23] is that the probability density function of the MNS model, or more precisely, the density $P_n(x_1, \dots, x_n)$ defined by (5)–(12), is the same as the stationary distribution of the n particles in nonintersecting OU processes defined in (40) with time-periodic boundary condition and the period $\beta = 1/(\kappa T) = 1/T$, see Figure 1. To explain the stationary distribution, we consider the OU processes $x_1(t), \dots, x_n(t)$, such that they are conditioned not to intersect during time $[0, \beta]$, and they satisfy $x_k(0) = x_k(\beta) = y_k$. Suppose $y_1, \dots, y_n$ has a joint probability distribution F, then for any $\tau \in (0, \beta)$, $x_1(\tau), \dots, x_n(\tau)$ has a joint probability distribution F_τ that depends on τ and F. By explicit computation we verify that $F_\tau = F$ for all $\tau \in (0, \beta)$ if and only if F has the probability density function $P_n(y_1, \dots, y_n)$ given in (12). Hence we claim that the distribution of the free fermions at temperature T given by (12) is the stationary distribution of nonintersecting OU processes with time period $1/T$. Also we call the nonintersecting OU processes with time period $1/T$ stationary if its marginal distribution at time 0 is given by P_n in (12).

As a quantum mechanical ensemble, we can consider the dynamics of the MNS model. As often happens, the dynamics of the MNS model along imaginary time is mathematically easier. In [23], the multi-time joint probability density function of the MNS model along imaginary time is derived, and also the multi-time correlation functions along imaginary time. To be precise, suppose that $\tau_1 < \tau_2 < \cdots < \tau_m$ are in the interval $[0, \beta)$ and they denote the imaginary times, the multi-time joint probability density function is obtained in [23, Formula (79)] and the multi-time correlation functions are obtained in [23, Formula (83)]. Moreover, the multi-time distribution of the stationary nonintersecting OU processes with time period $1/T$ is the same as that of the MNS model along imaginary time, see [23, Section VII.A].

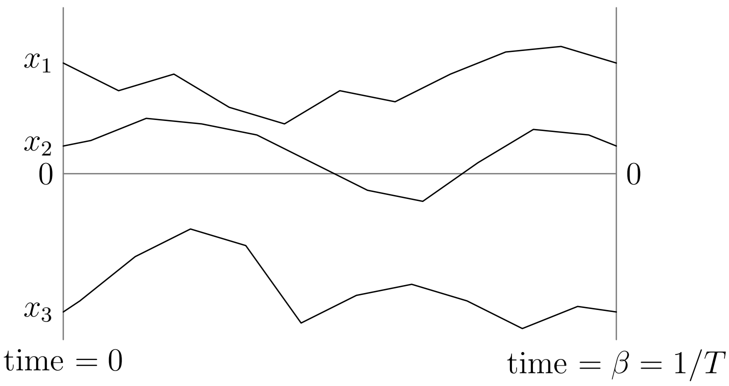

Figure 1: Schematic paths for three particles in nonintersecting OU processes with time period $\beta = 1/T$.

Proposition 2. *[23, Formulas (79) and (83), and Section VII.A]*

- *Let n free fermions be in the quadratic potential well, defined by the Hamiltonian* (5)*, at temperature $T > 0$. Suppose $0 \le \tau_1, \tau_2, \dots, \tau_m < \beta = 1/T$. Then the joint probability density of the fermions at imaginary times $\tau_1, \dots, \tau_m$ is, if $\tau_1 < \tau_2 < \cdots < \tau_m$,*

$$P(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}) = \frac{q^{n/2}}{Z_n(q)} \left[\prod_{l=1}^{m-1} \det \left(G(x_j^{(l+1)}, \tau_{l+1} - \tau_l \mid x_k^{(l)}, 0) \right)_{j,k=1}^{n} \right] \times \det \left(G(x_j^{(1)}, \beta - (\tau_m - \tau_1) \mid x_k^{(m)}, 0) \right)_{j,k=1}^{n}, \quad (41)$$

  where $\mathbf{x}^{(l)} = (x_1^{(l)}, \dots, x_n^{(l)})$ are the positions of the fermions at time τ_l; the multi-time correlation function of the fermions at imaginary times $\tau_1, \dots, \tau_m$ is

$$R^{(m)}(x_1, \dots, x_m; \tau_1, \dots, \tau_m) = \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \det \left(K_{k_1,\dots,k_n}(x_i, x_j; \tau_i, \tau_j) \right)_{i,j=1}^{m}, \quad (42)$$

  where the kernel function $K_{k_1,\dots,k_n}(x_i, x_j; \tau_i, \tau_j)$ will be defined in (256) *in Section 6.*

- *Let $x_1(t), \dots, x_n(t)$ be n independent OU processes defined in* (40)*. Condition them to be nonintersecting over time $[0, \beta = 1/T]$, and $x_k(0) = x_k(\beta) = y_k$, with the positions $y_1, \dots, y_n$ be random variables with joint probability density function $P_n(y_1, \dots, y_n)$ defined in* (12)*. Then the joint probability density function of the particles at times $\tau_1, \tau_2, \dots, \tau_m \in [0, \beta)$ is given by* (41) *if $\tau_1 < \tau_2 < \cdots < \tau_m$, and the multi-time correlation function is given by* (42)*.*

Here we remark that since the finite-temperature Green's function for a quantum system at temperature $T > 0$ is (anti)periodic in imaginary time with period $\beta = 1/(\kappa T)$ [14, Chapter 7], it suffices to consider multi-time joint probability density function and correlation functions at imaginary times in $[0, \beta)$.

We can simplify the multi-time correlation function (42) in to a form analogous to (21), and derive a formula for the multi-time gap probability that is analogous to (18). Before giving our results, we introduce some notations. Define

$$E(x,y;\tau,\sigma)=\begin{cases}0 & \text{if } \tau\geq\sigma,\\ \displaystyle\sum_{k=0}^{\infty}\varphi_k(x)\varphi_k(y)e^{k(\tau-\sigma)}=\frac{1}{\sqrt{2\pi(1-e^{2(\tau-\sigma)})}} & \\ \times\exp\left(\dfrac{-(1+e^{2(\tau-\sigma)})(x^2+y^2)+4e^{\tau-\sigma}xy}{4(1-e^{2(\tau-\sigma)}}\right) & \text{if } \tau<\sigma.\end{cases} \tag{43}$$

Note that for $\tau<\sigma$, $E(x,y;\tau,\sigma)=G(y,\tau-\sigma\mid x,0)$, the imaginary time propagator defined in (39). Then define

$$K_n(x,y;\tau,\sigma;z;q)=\sum_{k=0}^{\infty}\frac{q^kz}{1+q^kz}\left[\varphi_k(x)\varphi_k(y)e^{k(\tau-\sigma)}-\frac{1}{n}E(x,y;\tau,\sigma)\right], \tag{44}$$

of which the function $K(x,y;z;q)$ in (20) is the $\tau=\sigma$ specialization. Furthermore, we define the integral operator $\mathbf{K}(\tau_1,\dots,\tau_m;z;q)$ on $L^2(\mathbb{R}\times\{1,\dots,m\})$ whose kernel is represented by an $m\times m$ matrix $(K(x_i,x_j;\tau_i,\tau_j;z;q))_{i,j=1}^m$, where $K(x,y;\tau,\sigma;z;q)$ is defined in (44). To be concrete, for a function f on $\mathbb{R}\times\{1,\dots,m\}$, we denote it by $(f(x;1),\dots,f(x;m))$ where $f(x;k)$ is a function on $\mathbb{R}\times\{k\}$, and have

$$(\mathbf{K}(\tau_1,\dots,\tau_m;z;q)f)(x;k)=\sum_{j=1}^{m}\int_{\mathbb{R}}K(x,y;\tau_k,\tau_j;z;q)f(y;j)dy. \tag{45}$$

At last if $A_1,\dots,A_m\subseteq\mathbb{R}$ are measurable sets, we denote $\chi_{A_1,\dots,A_m}$, the projection operator on $L^2(\mathbb{R}\times\{1,2,\dots,m\})$, such that

$$(\chi_{A_1,\dots,A_m}f)(x;k)=\begin{cases}f(x;k) & \text{if } x\in A_k \text{ for } k=1,\dots,m,\\ 0 & \text{otherwise.}\end{cases} \tag{46}$$

Our result is as follows:

Theorem 4. *Consider either the n free fermions at temperature $1/T$ or the n-particle time-periodic nonintersecting OU processes with period $1/T$ that is defined in Proposition 2. Let $\tau_1,\dots,\tau_m\in[0,1/T)$ be either the imaginary times for free fermions or the times for OU processes.*

(a) The multi-time m-correlation function (42) *at $\tau_1,\dots,\tau_m\in(0,1/T)$, as stated in Proposition 2, can be written as*

$$R^{(m)}(x_1,\dots,x_m;\tau_1,\dots,\tau_m)=\frac{1}{2\pi i}\oint_0 F(z)\det\left(K_n(x_i,x_j;\tau_i,\tau_j;z;q)\right)_{i,j=1}^m dz, \tag{47}$$

where $F(z)$ is defined in (19) *and $K_n(x_i,x_j;\tau_i,\tau_j;z;q)$ is defined in* (44).

(b) Suppose $\tau_1,\dots,\tau_m$ are distinct. Let $A_1,\dots,A_m\subseteq\mathbb{R}$ be measurable sets. The gap probability that at imaginary time τ_k, all fermions are in A_k, or that at time τ_k, all particles in OU processes are in A_k, is given by

$$\mathbb{P}(A_1,\dots,A_m;\tau_1,\dots,\tau_m)=\frac{1}{2\pi i}\oint_0 F(z)\det(I-\mathbf{K}(\tau_1,\dots,\tau_m;z;q)\chi_{A_1^c,\dots,A_m^c}), \tag{48}$$

where the operators $\mathbf{K}(\tau_1,\dots,\tau_m;z;q)$ and $\chi_{A_1^c,\dots,A_m^c}$ are defined in (45) *and* (46).

We note that if $\tau_1=\cdots=\tau_m$, Theorem 4(a) degenerates to Theorem 1(b). Hence we can compute the limits of the multi-time correlation functions and gap probability, which will be done in a subsequent publication.

1.5.2 Nonintersecting Brownian motions on a circle

The feature of the m-correlation function formula (21) for the MNS model is that it is equal to a contour integral whose integrand is a $m \times m$ determinant depending on a formal correlation kernel. This feature occurs in another model, nonintersecting Brownian motions on a circle with a fixed winding number, studied by the authors in [25]. To see the analogy, we recall that the counterpart m-correlation function in [25] is $(R^{(n)}_{0\to T})_\omega(a^{(1)}_1,\dots,a^{(1)}_{k_1};\dots;a^{(m)}_1,\dots,a^{(m)}_{k_m};t_1,\dots,t_m)$ defined in [25, Formulas (36) and (136)], where ω is the winding number. (The correlation function $(R^{(n)}_{0\to T})_\omega$ is a multi-time correlation function, more analogous to $R^{(m)}(x_1,\dots,x_m;\tau_1,\dots,\tau_m)$ defined in (42) and re-expressed in (47), to which $R^{(m)}_n(x_1,\dots,x_m)$ defined in (17) is a special case.) By [25, Formula (135)], we have that[1]

$$(R^{(n)}_{0\to T})_\omega(a^{(1)}_1,\dots,a^{(1)}_{k_1};\dots;a^{(m)}_1,\dots,a^{(m)}_{k_m};t_1,\dots,t_m) = \frac{(-1)^{n+1}}{2\pi i}\oint_0 R^{(n)}_{0\to T}\left(a^{(1)}_1,\dots,a^{(1)}_{k_1};\dots;a^{(m)}_1,\dots,a^{(m)}_{k_m};t_1,\dots,t_m;\frac{\log z}{2\pi i}\right)\frac{dz}{z^{\omega+1}}. \quad (49)$$

Then by [25, Formula (116)]

$$R^{(n)}_{0\to T}(a^{(1)}_1,\dots,a^{(1)}_{k_1};\dots;a^{(m)}_1,\dots,a^{(m)}_{k_m};t_1,\dots,t_m;\tau) = \det\left(K_{t_i,t_j}(a^{(i)}_{l_i},a^{(j)}_{l'_j})\right)_{i,j=1,\dots,m,l_i=1,\dots,k_i,l'_j=1,\dots,k_j}, \quad (50)$$

where $K_{t_i,t_j}(a^{(i)}_{l_i},a^{(j)}_{l'_j})$ are given in [25, Formula (117)] and depend on τ, see [25, Remark 2.2]. We note that the right-hand side of (49) is a holomorphic function since $K_{t_i,t_j}(a^{(i)}_{l_i},a^{(j)}_{l'_j})$ in (50) is analytic in $e^{2\pi i\tau}$.

The formal similarity of correlation functions between the MNS model and the nonintersecting Brownian motions on a circle is intuitively explained by their periodicities. The MNS model is related to the nonintersecting OU processes with time periodicity, see Section 1.5.1, so it is comparable to the nonintersecting Brownian motions with space periodicity.

Another similarity of the two models is as follows. The grand canonical ensemble of the MNS model, which is the superposition of (the canonical ensemble of) the MNS model according to the Boltzmann distribution, is a determinantal process [11], and its counterpart, the nonintersecting Brownian motions on a circle regardless the winding number, are also a determinantal process [25, Section 2.3].

Outline

In Section 2 we prove Theorem 1. First we prove Theorem 1(a) in Section 2.1, and then prove Theorem 1(b) in Section 2.2. In Section 2.3 we give an alternative proof of Theorem 1(a) based on Theorem 1(b). Although the proof in Section 2.3 is shorter, since that in Section 2.1 has analogy in the study of particle models related to KPZ universality class, see Section 5, we also present it. In Sections 3 and 4 we prove Theorems 2 and 3 respectively. In Section 5 we discuss some particle models related to KPZ universality class. In Section 6 we prove Theorem 4 for the dynamic generalization of the MNS model. In Appendix A we present a proof of Proposition 1.

Acknowledgment

We thank Grégory Schehr for helpful discussion on the dynamics of the MNS model.

[1] In [25, Formula (135)], the symbol o in the third line should be ω.

2 Gap probability and correlation functions

In our derivation below, we need two integral representations of Hermite polynomials:

$$H_k(x) = \frac{1}{\sqrt{2\pi}i}\int_{-i\infty}^{i\infty} e^{\frac{1}{2}(s-x)^2} s^k ds, \tag{51}$$

$$\frac{1}{k!}H_k(x)e^{-x^2/2} = \frac{1}{2\pi i}\oint_\Gamma e^{-\frac{1}{2}(t-x)^2} t^{-k}\frac{dt}{t}, \tag{52}$$

where Γ is a contour around 0 with positive orientation. The integral formula (52) can be found in [1, 22.10.9], and (51) is equivalent to [1, 22.10.15]. Thus we have

$$\begin{aligned}\varphi_k(x) &= \left(\frac{1}{\sqrt{2\pi}k!}\right)^{1/2} e^{-x^2/4}\frac{1}{\sqrt{2\pi}i}\int_{-i\infty}^{i\infty} e^{\frac{1}{2}(s-x)^2} s^k ds \\ &= \left(\frac{1}{\sqrt{2\pi}k!}\right)^{-1/2} e^{x^2/4}\frac{1}{\sqrt{2\pi}2\pi i}\oint_\Gamma e^{-\frac{1}{2}(t-x)^2} t^{-k-1}dt.\end{aligned} \tag{53}$$

From [1, 22.14.17], we have

$$|\varphi_n(x)| \leq \frac{k}{2^{1/4}\pi^{1/4}}, \quad k \approx 1.086435. \tag{54}$$

Before proving Theorem 1, as a warm-up we show that (12) is in fact a probability density and check the formula (11) for $Z_n(q)$. For distinct $k_1, \dots, k_n$, by the Andréif formula and the orthonormality of φ_k,

$$\begin{aligned}&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\Phi_{k_1,\dots,k_n}(x_1,\dots,x_n)|^2 dx_1\cdots dx_n \\ &= \frac{1}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\begin{vmatrix}\varphi_{k_1}(x_1) & \dots & \varphi_{k_1}(x_n) \\ \vdots & & \vdots \\ \varphi_{k_n}(x_1) & \dots & \varphi_{k_n}(x_n)\end{vmatrix}^2 dx_1\cdots dx_n \\ &= 1,\end{aligned} \tag{55}$$

which shows that $Z_n(q)$ is the correct normalization constant in (12). By the inductive formula

$$\begin{aligned}\frac{Z_n(q)}{q^{n/2}} &= \sum_{0\leq k_1<k_2<\cdots<k_n} q^{k_1+\cdots+k_n} \\ &= \sum_{k_1=0}^{\infty} q^{k_1}q^{(k_1+1)(n-1)} \sum_{0\leq l_1<l_2<\cdots<l_{n-1}} q^{l_1+\cdots+l_{n-1}} \\ &= \sum_{k_1=0}^{\infty} q^{(k_1+1)n-1}\frac{Z_{n-1}(q)}{q^{(n-1)/2}},\end{aligned} \tag{56}$$

where $l_j = k_{j+1} - k_1 - 1$ for $j = 1, 2, \dots, n-1$, and the result

$$Z_1(q) = q^{1/2}\sum_{k_1=0}^{\infty} q^{k_1} = \frac{q^{1/2}}{1-q}, \tag{57}$$

we derive the second identity in (11).

2.1 Gap probability

Let $A \subseteq \mathbb{R}$ be a measurable set. We consider the probability that all the n particles are in A, which we denote by $\mathbb{P}_n(x_1, \dots, x_n \in A)$. We have

$$\begin{aligned}\mathbb{P}_n(x_1, \dots, x_n \in A) &= \int_A \cdots \int_A P_n(x_1, \dots, x_n) dx_1 \cdots dx_n \\ &= \frac{q^{n/2}}{Z_n(q)} \int_A \cdots \int_A \sum_{0 \le k_1 < k_2 < \cdots < k_n} |\Phi_{k_1, \dots, k_n}(x_1, \dots, x_n)|^2 q^{k_1 + \cdots + k_n} dx_1 \cdots dx_n \\ &= \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \int_A \cdots \int_A |\Phi_{k_1, \dots, k_n}(x_1, \dots, x_n)|^2 q^{k_1 + \cdots + k_n} dx_1 \cdots dx_n.\end{aligned} \tag{58}$$

Note that by the Andréif formula,

$$\begin{aligned}&\int_A \cdots \int_A \begin{vmatrix} \varphi_{k_1}(x_1) & \dots & \varphi_{k_1}(x_n) \\ \vdots & & \vdots \\ \varphi_{k_1}(x_1) & \dots & \varphi_{k_1}(x_n) \end{vmatrix}^2 q^{k_1 + \cdots + k_n} dx_1 \cdots dx_n \\ &= n! q^{k_1 + \cdots + k_n} \det \left(\langle \varphi_{k_i}(x), \varphi_{k_j}(x) \rangle_A \right)_{i,j=1}^n \\ &= n! \det \left(\langle q^{k_i} \varphi_{k_i}(x), \varphi_{k_j}(x) \rangle_A \right)_{i,j=1}^n ,\end{aligned} \tag{59}$$

where

$$\langle f(x), g(x) \rangle_A = \int_A f(x) g(x) dx. \tag{60}$$

Hence

$$\mathbb{P}_n(x_1, \dots, x_n \in A) = \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} \det \left(\langle q^{k_i} \varphi_{k_i}(x), \varphi_{k_j}(x) \rangle_A \right)_{i,j=1}^n . \tag{61}$$

Recall the integral operator $\mathbf{K}(z;q)$ defined in (21). We now introduce another integral operator $\mathbf{M}(q)$ acting on $L^2(\mathbb{R})$, depending on the parameter $q \in (0,1)$. It is defined by

$$\mathbf{M}(q)(f)(x) = \int_{\mathbb{R}} M(x, y; q) f(y) dy, \quad M(x, y; q) = \sum_{k=0}^{\infty} q^k \varphi_k(x) \varphi_k(y). \tag{62}$$

Let $A \subseteq \mathbb{R}$ be a measurable set, and let χ_A be the projection onto $L^2(A)$. It is straightforward to check by definition that $\mathbf{M}(q)$ and $\mathbf{K}(z;q)$ are trace class operators for $0 < q < 1$, and then $\mathbf{M}(q)\chi_A$ and $\mathbf{K}(z;q)\chi_{A^c}$ are also trace class operators [32]. Hence the Fredholm determinants $\det(I + z\mathbf{M}(q)\chi_A)$ and $\det(I - \mathbf{K}(z;q)\chi_{A_c})$ are well defined. We have the following relation between $\mathbf{M}(q)$ and $\mathbf{K}(z;q)$.

Lemma 1. *Let $q \in (0,1)$. For any $z \in \mathbb{C}$, and for any measurable $A \subseteq \mathbb{R}$, the following identity holds:*

$$(I + z\mathbf{M}(q)\chi_A) = (I + z\mathbf{M}(q))(I - \mathbf{K}(z;q)\chi_{A^c}). \tag{63}$$

Hence

$$\det(I + z\mathbf{M}(q)\chi_A) = \det(I + z\mathbf{M}(q)) \det(I - \mathbf{K}(z;q)\chi_{A^c}). \tag{64}$$

Proof. Since the Hermite functions $\{\varphi_k(x)\}_{k=0}^{\infty}$ form an orthonormal basis for $L^2(\mathbb{R})$, it is easy to see that

$$\mathbf{M}(q^k) = \mathbf{M}(q)^k. \tag{65}$$

We define the resolvent operator $\mathbf{R}(z;q)$ by

$$I - \mathbf{R}(z;q) = (I + z\mathbf{M}(q))^{-1}. \tag{66}$$

If $|z| < 1$, we have that $\mathbf{R}(z;q)$ is a well-defined integral operator and

$$\mathbf{R}(z;q) = -\sum_{l=1}^{\infty}(-z\mathbf{M}(q))^l. \tag{67}$$

Assuming for now that $|z|< 1$, by the estimate (54), we have that uniformly for all $x, y \in \mathbb{R}$

$$\begin{aligned} K(x,y;z;q) &= \sum_{k=0}^{\infty} q^k z\varphi_k(x)\varphi_k(y) \sum_{l=0}^{\infty}(-1)^l z^l q^{lk} \\ &= \sum_{l=1}^{\infty}(-1)^{l+1} z^l \sum_{k=0}^{\infty} q^{kl}\varphi_k(x)\varphi_k(y) \\ &= \sum_{l=1}^{\infty}(-1)^{l+1} z^l M(x,y;q^l). \end{aligned} \tag{68}$$

This implies the identity that

$$\mathbf{K}(z;q) = \mathbf{R}(z;q), \tag{69}$$

for all $|z| < 1$. Using the identity $\mathbf{K}(z;q)\chi_{A^c} = \mathbf{R}(z;q)\chi_{A^c}$ we find

$$\begin{aligned} (I + z\mathbf{M}(q))(I - \mathbf{K}(z;q)\chi_{A^c}) &= I + z\mathbf{M}(q) - \mathbf{R}(z;q)\chi_{A^c} - \mathbf{M}(q)\mathbf{R}(z;q)\chi_{A^c} \\ &= I + z\mathbf{M}(q)\chi_A + (z\mathbf{M}(q) - \mathbf{R}(z;q) - \mathbf{M}(q)\mathbf{R}(z;q)\chi_{A^c} \\ &= I + z\mathbf{M}(q)\chi_A, \end{aligned} \tag{70}$$

where in the last step we use $z\mathbf{M}(q) - \mathbf{R}(z;q) - \mathbf{M}(q)\mathbf{R}(z;q) = 0$, which is a consequence of (66). Hence we prove (63) in the case $|z| < 1$. Since the integral operator $\mathbf{K}(z;q)$ is well defined for all $z \in \mathbb{C}$, by analytic continuation (63) holds for all $z \in \mathbb{C}$. $\square$

We expand the Fredholm determinant $\det(I + z\mathbf{M}(q)\chi_A)$ into a series of multiple integrals by [32, Theorem 3.10], and then simplify it by the Cauchy–Binet identity as follows.

$$\begin{aligned} \det(I + z\mathbf{M}(q)\chi_A) &= 1 + \frac{z}{1!}\int_A M(x,x;q)dx + \frac{z^2}{2!}\int_A dx_1 \int_A dx_2 \det(M(x_i,x_j;q))_{i,j=1}^2 + \cdots \\ &= 1 + z\left(\sum_{0\le k_1} \langle q^{k_1}\varphi_{k_1}, \varphi_{k_1}\rangle_A\right) \\ &\quad + z^2\left(\sum_{0\le k_1<k_2} \begin{vmatrix} \langle q^{k_1}\varphi_{k_1}, \varphi_{k_1}\rangle_A & \langle q^{k_1}\varphi_{k_1}, \varphi_{k_2}\rangle_A \\ \langle q^{k_2}\varphi_{k_2}, \varphi_{k_1}\rangle_A & \langle q^{k_2}\varphi_{k_2}, \varphi_{k_2}\rangle_A \end{vmatrix}\right) \\ &\quad + z^3\left(\sum_{0\le k_1<k_2<k_3} \begin{vmatrix} \langle q^{k_1}\varphi_{k_1}, \varphi_{k_1}\rangle_A & \langle q^{k_1}\varphi_{k_1}, \varphi_{k_2}\rangle_A & \langle q^{k_1}\varphi_{k_1}, \varphi_{k_3}\rangle_A \\ \langle q^{k_2}\varphi_{k_2}, \varphi_{k_1}\rangle_A & \langle q^{k_2}\varphi_{k_2}, \varphi_{k_2}\rangle_A & \langle q^{k_2}\varphi_{k_2}, \varphi_{k_3}\rangle_A \\ \langle q^{k_3}\varphi_{k_3}, \varphi_{k_1}\rangle_A & \langle q^{k_3}\varphi_{k_3}, \varphi_{k_2}\rangle_A & \langle q^{k_3}\varphi_{k_3}, \varphi_{k_3}\rangle_A \end{vmatrix}\right) \\ &\quad + \cdots. \end{aligned} \tag{71}$$

With the help of (11), (12) and (61) thus find

$$\det(I + z\mathbf{M}(q)\chi_A) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{q^{n(n-1)/2}(q;q)_n} \mathbb{P}_n(x_1, \dots, x_n \in A), \tag{72}$$

and arrive at the formula for any dimension n,

$$\mathbb{P}_n(x_1, \dots, x_n \in A) = q^{n(n-1)/2}(q;q)_n \frac{1}{2\pi i} \oint_0 \det(I + z\mathbf{M}(q)\chi_A) \frac{dz}{z^{n+1}}. \tag{73}$$

In order to do asymptotic analysis it is convenient to work with the operator $\mathbf{K}(z;q)$ rather than $\mathbf{M}(q)$. Since the kernel of $\mathbf{M}(q)$ is diagonalized by φ_n, the determinant is simple to compute:

$$\det(I + z\mathbf{M}(q)) = \prod_{k=0}^{\infty}(1 + q^k z) = (-z;q)_\infty. \tag{74}$$

Thus substituting (64) and (74) into (73), we obtain the formula (18) and prove Theorem 1(a). In particular, when $A = (-\infty, s]$, (16) implies

$$\mathbb{P}(\max(x_1, \dots, x_n) \le s) = q^{-n(n-1)/2}(q;q)_n \frac{1}{2\pi i} \oint_0 \frac{(-z;q)_\infty}{z^n} \det(I - \mathbf{K}(z;q)\chi_{(s,\infty)}) \frac{dz}{z}. \tag{75}$$

To finish this subsection, we prove the double contour integral formula (22) for $K(x,y;z;q)$.

Proof of (22). By (20) and the integral expressions of φ_k in (51) and (52), we can prove without much difficulty the convergence

$$K(x,y;z;q) = \lim_{M\to+\infty} K^{(M)}(x,y;z;q), \tag{76}$$

where (with Γ being a fixed contour enclosing 0)

$$K^{(M)}(x,y;z;q) = \sum_{k=0}^{\infty} \frac{e^{(y^2-x^2)/4}}{(2\pi i)^2} \int_{-iM}^{iM} ds \oint_\Gamma \frac{dt}{t} \frac{e^{(s-x)^2/2}}{e^{(t-y)^2/2}} \frac{z}{1+q^k z} \left(\frac{qs}{t}\right)^k. \tag{77}$$

By deforming Γ to a circle Γ_s depending on s, such that its center is 0 and its radius is $\sqrt{q}|s|+1$, we have by (2)

$$\begin{aligned} K^{(M)}(x,y;z;q) &= \frac{e^{(y^2-x^2)/4}}{(2\pi i)^2} \int_{-iM}^{iM} ds \oint_{\Gamma_s} \frac{dt}{t} \frac{e^{(s-x)^2/2}}{e^{(t-y)^2/2}} \sum_{k=0}^{\infty} \frac{z}{1+q^k z} \left(\frac{qs}{t}\right)^k \\ &= \frac{z}{1+z} \frac{e^{(y^2-x^2)/4}}{(2\pi i)^2} \int_{-iM}^{iM} ds \oint_{\Gamma_s} \frac{dt}{t} \frac{e^{(s-x)^2/2}}{e^{(t-y)^2/2}} {}_2\phi_1 \left[\begin{matrix} -z, q \\ -zq \end{matrix}; q, \frac{qs}{t} \right]. \end{aligned} \tag{78}$$

Taking the limit $M \to +\infty$ on the right-hand side of (78), we obtain the right-hand side of (22), if Γ_s is the circular contour defined above. Hence we prove (22) and verify Remark 1. □

2.2 Correlation functions

By the joint probability density function (12), we find that the m-correlation function of the model can be expressed as

$$R^{(m)}(x_1, \dots, x_m) = \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \dots < k_n} q^{k_1 + \dots + k_n} R^{(m)}_{k_1,\dots,k_n}(x_1, \dots, x_m), \tag{79}$$

where $R^{(m)}_{k_1,\dots,k_n}(x_1,\dots,x_m)$ is the m-correlation function for the n-particle model with density function $|\Phi_{k_1,\dots,k_n}(x_1,\dots,x_n)|^2$. Since $\Phi_{k_1,\dots,k_n}(x_1,\dots,x_n)$ is defined in the determinantal form (9), by a general result for determinantal process [20, Proposition 2.11], we have that

$$R^{(m)}_{k_1,\dots,k_n}(x_1,\dots,x_m) = \det(K_{k_1,\dots,k_n}(x_i,x_j))^m_{i,j=1}, \quad \text{where} \quad K_{k_1,\dots,k_n}(x,y) = \sum_{i=1}^n \varphi_{k_i}(x)\varphi_{k_i}(y). \tag{80}$$

It is straightforward to write

$$\begin{aligned} R^{(m)}_{k_1,\dots,k_n}(x_1,\dots,x_m) = {}& \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \det \begin{vmatrix} \varphi_{k_{i_1}}(x_1)\varphi_{k_{i_1}}(x_1) & \dots & \varphi_{k_{i_m}}(x_1)\varphi_{k_{i_m}}(x_m) \\ \varphi_{k_{i_1}}(x_2)\varphi_{k_{i_1}}(x_1) & \dots & \varphi_{k_{i_m}}(x_2)\varphi_{k_{i_m}}(x_m) \\ \vdots & & \vdots \\ \varphi_{k_{i_1}}(x_m)\varphi_{k_{i_1}}(x_1) & \dots & \varphi_{k_{i_m}}(x_2)\varphi_{k_{i_m}}(x_m) \end{vmatrix} \\ = {}& \sum_{\substack{j_1<j_2<\cdots<j_m \\ \{j_1,\dots,j_m\}\subseteq\{k_1,\dots,k_n\}}} \hat{R}^{(m)}_{j_1,\dots,j_m}(x_1,\dots,x_m), \end{aligned} \tag{81}$$

where

$$\begin{aligned} \hat{R}^{(m)}_{j_1,\dots,j_m}(x_1,\dots,x_m) = {}& \det\left(\sum_{i=1}^m \varphi_{j_i}(x_k)\varphi_{j_i}(x_l)\right)^m_{k,l=1} \\ = {}& \sum_{\kappa,\lambda\in S_n} \operatorname{sgn}(\lambda) \prod_{i=1}^m \varphi_{j_{\kappa(i)}}(x_i)\varphi_{j_{\kappa(i)}}(x_{\lambda(i)}). \end{aligned} \tag{82}$$

Hence we can write write

$$R^{(m)}(x_1,\dots,x_m) = \frac{q^{n/2}}{Z_n(q)} \sum_{0\le j_1<\cdots<j_m} C_{j_1,\dots,j_m} \hat{R}^{(m)}_{j_1,\dots,j_m}(x_1,\dots,x_m), \tag{83}$$

where

$$C_{j_1,\dots,j_m} = \sum_{\substack{0\le k_1<\cdots<k_n \\ \{k_1,\dots,k_n\}\supseteq\{j_1,\dots,j_m\}}} q^{k_1+\cdots+k_n}. \tag{84}$$

Now we state the explicit formula of $C_{j_1,\dots,j_m}$ and postpone its proof to the end of this section:

$$C_{j_1,\dots,j_m} = q^{j_1+\cdots+j_m} \frac{1}{2\pi i} \oint_0 \frac{dz}{z^{n-m+1}} \left[\prod_{k=0}^\infty (1+q^k z)\right] \left[\prod_{i=1}^m (1+q^{j_i}z)^{-1}\right]. \tag{85}$$

We note that the contour in (85) can be any one that encloses 0 in positive orientation, for the integrand has only one pole at 0.

On the other hand, we consider the kernel $K(x,y;z;q)$ defined in (20). It is straightforward to get by the Cauchy–Binet identity that for all $z \neq -q^{-k}$, $k = 0,1,2,\dots$,

$$\det(K(x_i,x_j;z;q))^m_{i,j=1} = \sum_{0\le j_1<\cdots<j_m} q^{j_1+\cdots+j_m} z^m \left[\prod_{i=1}^m (1+q^{j_i}z)^{-1}\right] \hat{R}^{(m)}_{j_1,\dots,j_m}(x_1,\dots,x_m). \tag{86}$$

Hence by comparing (83), (85) and (86), we prove the desired identity (21), where the contour can be any one that encloses 0 in positive orientation. By (86) we have that the points $z = -q^{-k}$ are first order poles of $\det(K(x_i,x_j;z;q))^m_{i,j=1}$. Hence they are not the poles of the integrand.

Remark 4. If $m > n$ in (21), we can see by (86) that $R^{(m)}(x_1, \dots, x_m) = 0$ for all $x_1, \dots, x_m$. This confirms that there are no more than n particles in the model.

Proof of equation (85). In this proof, we use the notational convention that $j_0 = -1$ and $j_{m+1} = \infty$.

By definition,

$$C_{j_1,\dots,j_m} = q^{j_1+\cdots+j_m} \sum_{\substack{0\le l_0 \le j_1,\ 0\le l_1\le j_2-j_1-1,\dots,0\le l_{m-1}\le j_m-j_{m-1}-1,\\ 0\le l_m = n-m-(l_0+l_1+\cdots+l_{m-1})}} \prod_{i=0}^{m} g_i(l_i), \tag{87}$$

where for $k = 0, 1, \dots, m$

$$g_k(l) = \sum_{j_k < i_1 < \cdots < i_l < j_{k+1}} q^{i_1+\cdots+i_l}. \tag{88}$$

For $k = 0, 1, \dots, m$, letting

$$G_k(z) = \sum_{l=0}^{j_{k+1}-j_k-1} g_k(l) z^l, \tag{89}$$

we find

$$C_{j_1,\dots,j_m} = q^{j_1+\cdots+j_m} \frac{1}{2\pi i} \oint_0 \frac{dz}{z^{n-m+1}} \prod_{k=0}^{m} G_k(z). \tag{90}$$

Now we compute $G_k(z)$. In the same way as the inductive computation of $Z_n(q)$ in (56)–(57), we have

$$g_m(l) = \frac{q^{\binom{l}{2}}}{(q;q)_l} q^{l(j_m+1)}. \tag{91}$$

and similarly we can obtain that for $k = 0, 1, \dots, m-1$

$$g_k(l) = \begin{bmatrix} j_{k+1} - j_k - 1 \\ l \end{bmatrix}_q q^{\binom{l}{2}} q^{l(j_k+1)}, \tag{92}$$

with the understanding that $j_0 = -1$. Hence by [4, Corollary 10.2.2(b)] for the $k = m$ case, and [4, Corollary 10.2.2(c)] for the $k = 0, 1, \dots, m-1$ cases, we have

$$G_k(z) = (-q^{j_k+1}z; q)_{j_{k+1}-j_k-1} = \prod_{l=j_k+1}^{j_{k+1}-1} (1 + q^l z). \tag{93}$$

Hence

$$\prod_{k=0}^{m} G_k(z) = \left[\prod_{l=0}^{\infty} (1 + q^l z) \right] \left[\prod_{i=1}^{m} (1 + q^{j_i} z)^{-1} \right], \tag{94}$$

and we prove (85) by plugging (94) into (90). □

2.3 Alternative derivation of gap probability by correlation functions

For $0 < k_1 < k_2 < \cdots < k_n$, we have by [20, Proposition 2.11]

$$\begin{aligned} &\int_A \cdots \int_A |\Phi_{k_1,\dots,k_n}(x_1, \dots, x_n)|^2 dx_1 \cdots dx_n \\ &= 1 + \sum_{m=1}^{n} \frac{(-1)^m}{m!} \int_{A^c} \cdots \int_{A^c} \det \left(K_{k_1,\dots,k_n}(x_i, x_j) \right)_{i,j=1}^m dx_1 \cdots dx_m \\ &= 1 + \sum_{m=1}^{n} \frac{(-1)^m}{m!} \int_{A^c} \cdots \int_{A^c} R^{(m)}_{k_1,\dots,k_n}(x_1, \dots, x_m) dx_1 \cdots dx_m, \end{aligned} \tag{95}$$

where $K_{k_1,\ldots,k_n}(x,y)$ is defined in (80) and $R^{(m)}_{k_1,\ldots,k_n}(x_1,\ldots,x_m)$ is defined in (79).

Plugging (95) into (58), and using (79), we have

$$\mathbb{P}_n(x_1,\ldots,x_n\in A)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\int_{A^c}\cdots\int_{A^c}R^{(m)}(x_1,\ldots,x_m)dx_1\cdots dx_m, \tag{96}$$

where $R^{(m)}(x_1,\ldots,x_m)$ is defined in (17). Note that here we let m run over all positive integers, since $R^{(m)}(x_1,\ldots,x_m)=0$ by Remark 4. Then by the integral expression (21) of $R^{(m)}(x_1,\ldots,x_m)$, we have

$$\begin{aligned}\mathbb{P}_n(x_1,\ldots,x_n\in A)&=1+\frac{1}{2\pi i}\oint_0\frac{dz}{z}F(z)\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\int_{A^c}\cdots\int_{A^c}\det\left(K(x_i,x_j;z;q)\right)_{i,j=1}^m\\&=1+\frac{1}{2\pi i}\oint_0\frac{dz}{z}F(z)\left[\det\left(I-\mathbf{K}(z;q)\chi_{A^c}\right)-1\right]\\&=\frac{1}{2\pi i}\oint_0\frac{dz}{z}F(z)\det\left(I-\mathbf{K}(z;q)\chi_{A^c}\right).\end{aligned} \tag{97}$$

Hence (18) is proved.

3 Proof of Theorem 2

Our starting point is formula (75), the special case of (18) with $A=(-\infty,s]$. After the change of variable

$$w=q^nz, \tag{98}$$

formula (75) becomes

$$\mathbb{P}(\max(x_1,\ldots,x_n)\le s)=q^{n(n+1)/2}(q;q)_n\frac{1}{2\pi i}\oint_0(-q^{-n}w;q)_\infty\det(I-\mathbf{P}_s\mathbf{K}(q^{-n}w;q)\mathbf{P}_s)\frac{dw}{w^{n+1}}, \tag{99}$$

where $\mathbf{P}_s$ is the projection onto $L^2(s,\infty)$. It is straightforward to see that

$$(-q^{-n}w;q)_\infty=(-w;q)_\infty w^nq^{-n(n+1)/2}(-q/w;q)_n. \tag{100}$$

Thus we have that the integral in (99) can be written as

$$\frac{1}{2\pi i}\oint_0(q;q)_n(-w;q)_\infty(-q/w;q)_n\det(I-\mathbf{P}_s\mathbf{K}(q^{-n}w;q)\mathbf{P}_s)\frac{dw}{w}. \tag{101}$$

By the triple product identity [4, Theorem 10.4.1]

$$(-w;q)_\infty(-q/w;q)_\infty(q;q)_\infty=\sum_{k=-\infty}^{\infty}q^{\frac{k(k-1)}{2}}w^k, \tag{102}$$

the integral in (99) is written as

$$\frac{(q;q)_n}{(q;q)_\infty}\frac{1}{2\pi i}\oint_0\left(\sum_{k=-\infty}^{\infty}q^{\frac{k(k-1)}{2}}w^k\right)\frac{(-q/w;q)_n}{(-q/w;q)_\infty}\det(I-\mathbf{P}_s\mathbf{K}(q^{-n}w;q)\mathbf{P}_s)\frac{dw}{w}. \tag{103}$$

We take the contour in (103) as $|w|=\sqrt{q}$ and make the change of variable $w=\sqrt{q}e^{i\pi\theta}$. Then (103) becomes

$$\frac{1}{2}\int_{-1}^{1}\left(\sum_{k=-\infty}^{\infty}q^{k^2/2}e^{ik\pi\theta}\right)\det(I-\mathbf{P}_s\mathbf{K}(q^{-n+1/2}e^{i\pi\theta};q)\mathbf{P}_s)F_n(\theta;q)d\theta, \tag{104}$$

where

$$F_n(\theta;q) = \frac{(q;q)_n}{(q;q)_\infty} \frac{(-\sqrt{q}e^{-i\pi\theta};q)_n}{(-\sqrt{q}e^{-i\pi\theta};q)_\infty}. \tag{105}$$

3.1 Preliminary estimates of $\tilde{K}_n(x,y)$

In what follows we will need to compute the limit of the Fredholm determinant in the integrand of (104) as $n \to \infty$ in the scaling limit $s = s_n = 2\sqrt{n} + tn^{-1/6}$ for $t \in \mathbb{R}$. In this scaling

$$\det(I - \mathbf{P}_s \mathbf{K}(q^{-n+1/2}e^{i\pi\theta};q)\mathbf{P}_s) = \det(I - \mathbf{P}_t \tilde{\mathbf{K}} \mathbf{P}_t), \tag{106}$$

where $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}(\theta)$ has the kernel

$$\begin{aligned} \tilde{K}_n(x,y) = \tilde{K}_n(x,y;\theta) &:= n^{-1/6} K_n(2\sqrt{n} + xn^{-1/6}, 2\sqrt{n} + yn^{-1/6}) \\ &= n^{-1/6} \sum_{k=0}^{\infty} c_k \varphi_k(2\sqrt{n} + xn^{-1/6}) \varphi_k(2\sqrt{n} + yn^{-1/6}), \end{aligned} \tag{107}$$

where

$$c_k = c_k(\theta) := \frac{e^{\pi i\theta} q^{k-n+1/2}}{1 + e^{\pi i\theta} q^{k-n+1/2}} = \frac{e^{\pi i\theta/2} \sqrt{q}^{k-n+1/2}}{2\cosh\left(\frac{k-n+1/2}{2}\log q + \frac{i\pi\theta}{2}\right)}, \tag{108}$$

with the dependence on θ suppressed if there is no chance of confusion.

We need to compute the $n \to \infty$ limit of $\tilde{K}_n(x,y)$ for x, y in a compact subset of $\mathbb{R}$, and show that $\tilde{K}_n(x,y)$ vanishes exponentially fast as $\max(x,y) \to +\infty$ and $\min(x,y)$ is bounded below. We will use the following global approximation formula for φ_k, which is from [27, Section 11.4, Exercises 4.2 and 4.3]. For x in a compact subset of $(-1,+\infty)$ and $k = 0, 1, 2, \dots$ uniformly,

$$(k+1/2)^{1/12} \varphi_k(2\sqrt{k+1/2}x) = 2^{1/6} \left(\frac{\zeta(x)}{x^2-1}\right)^{1/4} \left(\mathrm{Ai}\left((2k+1)^{2/3}\zeta(x)\right) + \varepsilon_k(x)\right) \times (1 + \mathcal{O}((k+1/2)^{-1})), \tag{109}$$

such that

(i) the $1 + \mathcal{O}((k+1/2)^{-1})$ factor depends on k only;

(ii) $\zeta(x)$ is a continuous, differentiable and monotonically increasing function on $(-1,+\infty)$. Moreover, it is bounded below as $x \to -1_+$ and has x^2 growth as $x \to +\infty$. The explicit formula of $\zeta(x)$ is given in [27, Section 11.4, Exercise 4.2]. Around 1, it satisfies

$$\zeta(1) = 0 \quad \text{and} \quad \zeta'(1) = 2^{1/3}; \tag{110}$$

(iii) $\varepsilon_n(x)$ is defined in [27, Section 11.4, Exercise 4.2], where it is denoted as $\varepsilon(x)$. From [27, Section 11.2], we have the estimate uniform in k and x,

$$|\varepsilon(x)| = \begin{cases} \mathcal{O}\left((k+1/2)^{-7/6}(-\zeta(x))^{-1/4}\right) & \text{if } x \in (-1,1], \\ \mathcal{O}\left((k+1/2)^{-7/6}\zeta(x)^{-1/4}\exp\left(-\frac{2}{3}(2k+1)\zeta(x)^{3/2}\right)\right) & \text{if } x \in [1,+\infty). \end{cases} \tag{111}$$

To use estimate (109), we also need that by [27, Sections 11.1-2], especially [27, Formulas (2.05), (2.13) and (2.15) in Chapter 11],

$$|\mathrm{Ai}(x)| \le f(x), \quad \text{where} \quad f(x) = \begin{cases} \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-\frac{2}{3}x^{3/2}} & x > 1, \\ 1 & -1 \le x \le 1, \\ \lambda^{1/2}\pi^{-1/2}(-x)^{-1/4} & x < -1, \end{cases} \tag{112}$$

and the constant $\lambda = 1.04\ldots$.

Below we provide computational results that are used in the proof of both part (a) and part (b) of Theorem 2. Note that we use C to denote a large enough positive constant and γ a small enough positive constant. It is harmless to assume $C = 1000$ and $\gamma = 1/10$.

x, y **in a compact subset.** First we consider the case that $x, y \in [-M/2, M/2]$ where M is a positive constant. Without loss of generality, we assume that $Mn^{1/3}$ is an integer, and then write

$$\tilde{K}_n(x,y) = K_n^{(1,M)}(x,y) + K_n^{(2,M)}(x,y) + K_n^{(3,M)}(x,y), \tag{113}$$

where

$$K_n^{(1,M)}(x,y) = n^{-1/6} \sum_{k=0}^{n-Mn^{1/3}-1} c_k \varphi_k(2\sqrt{n} + xn^{-1/6})\varphi_k(2\sqrt{n} + yn^{-1/6}), \tag{114}$$

$$K_n^{(2,M)}(x,y) = n^{-1/6} \sum_{k=n-Mn^{1/3}}^{n+Mn^{1/3}} c_k \varphi_k(2\sqrt{n} + xn^{-1/6})\varphi_k(2\sqrt{n} + yn^{-1/6}), \tag{115}$$

$$K_n^{(3,M)}(x,y) = n^{-1/6} \sum_{k=n+Mn^{1/3}+1}^{\infty} c_k \varphi_k(2\sqrt{n} + xn^{-1/6})\varphi_k(2\sqrt{n} + yn^{-1/6}). \tag{116}$$

$$\tag{117}$$

The following estimates on the coefficients c_k are uniform in k and θ:

$$c_k = \begin{cases} 1 + \mathcal{O}(q^{Mn^{1/3}}), & k = 0, \ldots, n - Mn^{1/3} - 1, \\ \mathcal{O}(q^{l+Mn^{1/3}}), & k = n + Mn^{1/3} + l, \quad l = 1, 2, \ldots. \end{cases} \tag{118}$$

With the estimates (118) for c_k and (109) for φ_k, it follows that

$$|K_n^{(3,M)}(x,y)| \leq n^{-1/6} C \sum_{l=1}^{\infty} q^{l+Mn^{1/3}} (n + Mn^{1/3} + l + 1/2)^{-1/6} \leq C \frac{n^{-1/3}}{1-q} q^{Mn^{1/3}}, \tag{119}$$

where C is a constant independent of n, x, y, M, θ and q. Similarly,

$$|K_n^{(1,M)}(x,y)| \leq n^{-1/6} C \sum_{k=0}^{n-Mn^{1/3}-1} (k+1/2)^{-1/6} \exp\left(-\frac{2}{3}(2k+1)\zeta\left(\frac{2\sqrt{n} + xn^{-1/3}}{2\sqrt{k+1/2}} \right)^{3/2} \right) \times \exp\left(-\frac{2}{3}(2k+1)\zeta\left(\frac{2\sqrt{n} + yn^{-1/3}}{2\sqrt{k+1/2}} \right)^{3/2} \right), \tag{120}$$

where C is independent of n, x, y, M, θ and q. After some calculation, the sum $K_n^{(1,M)}(x,y)$ is estimated as

$$|K_n^{(1,M)}(x,y)| \leq C \exp(-\gamma M^{3/2}), \tag{121}$$

where C and γ are independent of n, x, y, M, θ and q.

The approximation of $K_n^{(2,M)}(x,y)$ depends on θ and will be given later.

$x \to +\infty$ **and** y **is bounded below.** Let M be the same as above and $N > M$ be a large positive constant, and without loss of generality assume that $Nn^{1/3}$ is an integer. Suppose $x \geq 2N$ and $y \geq -M/2$. We write

$$\tilde{K}_n(x,y) = K_n^{(4,M,N)}(x,y) + K_n^{(2,M)}(x,y) + K_n^{(5,N)}(x,y), \tag{122}$$

where $K_n^{(2,M)}(x,y)$ is defined in (115), and

$$K_n^{(4,M,N)}(x,y) = n^{-1/6} \sum_{\substack{0 \leq k \leq n - Mn^{1/3} - 1 \\ \text{or } n + Mn^{1/3} + 1 \leq k \leq n + Nn^{1/3}}} c_k \varphi_k(2\sqrt{n} + xn^{-1/6}) \varphi_k(2\sqrt{n} + yn^{-1/6}), \tag{123}$$

$$K_n^{(5,N)}(x,y) = n^{-1/6} \sum_{k = n + Nn^{1/3} + 1}^{\infty} c_k \varphi_k(2\sqrt{n} + xn^{-1/6}) \varphi_k(2\sqrt{n} + yn^{-1/6}). \tag{124}$$

Similar to (119), we have the estimate

$$|K_n^{(5,N)}(x,y)| \leq C \frac{n^{-1/3}}{1-q} q^{Nn^{1/3}}, \tag{125}$$

where C is independent of n, x, y, N, θ and q. Similarly, like (119) and (120),

$$\begin{aligned} &|K_n^{(4,M,N)}(x,y)| \\ &\leq n^{-1/6} C \sum_{\substack{0 \leq k \leq n - Mn^{1/3} - 1 \\ \text{or } n + Mn^{1/3} + 1 \leq k \leq n + Nn^{1/3}}} (k + 1/2)^{-1/6} \exp\left(-\frac{2}{3}(2k+1)\zeta\left(\frac{2\sqrt{n} + xn^{-1/3}}{2\sqrt{k + 1/2}} \right)^{3/2} \right) \\ &\leq C \exp(-\gamma N^{3/2}), \end{aligned} \tag{126}$$

where C and γ are independent of n, x, y, M, N, θ and q. Note that in (126) the estimate of $\varphi_k(2\sqrt{n} + xn^{-1/6})$ is the same as in (120), while the estimate of $\varphi_k(2\sqrt{n} + yn^{-1/6})$ is roughly $\mathcal{O}((k + 1/2)^{-1/12})$, as in (119).

Below we prove Theorem 2. We first give full detail for part (b), and then show that a simplified argument works for part (a). The technical core is the estimate of $K_n^{(2,M)}(x,y)$.

3.2 Gap probability for the rightmost particle: $q = e^{-cn^{-1/3}}$

Now consider the scaling $q = e^{-cn^{-1/3}}$ for some $c > 0$. We begin with the following lemma on the asymptotics of the q-Pochhammer symbols appearing in (104).

Lemma 2. *For $q = e^{-cn^{-1/3}}$, we have the estimate uniformly for $\theta \in [-1, 1]$:*

$$\frac{(q;q)_n}{(q;q)_\infty} = 1 + \mathcal{O}(n^{1/3} e^{-cn^{2/3}}), \qquad \frac{(-\sqrt{q}e^{-\pi i \theta}; q)_n}{(-\sqrt{q}e^{-\pi i \theta}; q)_\infty} = 1 + \mathcal{O}(n^{1/3} e^{-cn^{2/3}}), \tag{127}$$

and then uniformly for $\theta \in [-1, 1]$, the function

$$F_n(\theta; q) = 1 + \mathcal{O}(n^{1/3} e^{-cn^{2/3}}). \tag{128}$$

Proof. We only prove the second equation of (127). We have

$$\frac{(\sqrt{q}e^{-\pi i\theta};q)_n}{(\sqrt{q}e^{-\pi i\theta};q)_\infty} = \frac{1}{\prod_{k=0}^{\infty}(1-e^{-\pi i\theta}q^{k+n+1/2})}, \tag{129}$$

thus

$$\begin{aligned}\left|\log\frac{(\sqrt{q}e^{-\pi i\theta};q)_n}{(\sqrt{q}e^{-\pi i\theta};q)_\infty}\right| &\le \sum_{k=0}^{\infty}|\log(1-e^{-\pi i\theta}q^{k+n+1/2})| \\ &< \frac{2q^n}{1-q} = \frac{2e^{-cn^{2/3}}}{1-e^{-cn^{-1/3}}} = \frac{2e^{-cn^{2/3}}}{cn^{-1/3}}(1+\mathcal{O}(n^{-1/3})).\end{aligned} \tag{130}$$

The result is obtained by exponentiating. □

Also note that the Poisson summation formula gives

$$\sum_{k=-\infty}^{\infty} e^{\frac{-cn^{-1/3}k^2}{2}}e^{k\pi i\theta} = n^{1/6}\sqrt{2\pi c^{-1}}\sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}. \tag{131}$$

Applying formulas (106), (127) and (131) to the integral (104), we find that (104) becomes

$$n^{1/6}\int_{-1}^{1}\sqrt{\frac{\pi}{2c}}\left(\sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}\right)\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)F_n(\theta;q)d\theta. \tag{132}$$

Fix a small $\epsilon>0$. We plug the formula (131) into (132), use the estimates in Lemma 2, and split the integral (132) into two parts, I_1 and I_2, where

$$I_1 = n^{1/6}\int_{-1+\epsilon}^{1-\epsilon}\sqrt{\frac{\pi}{2c}}\left(\sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}\right)\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)F_n(\theta;q)d\theta, \tag{133}$$

$$I_2 = n^{1/6}\int_{1-\epsilon}^{1+\epsilon}\sqrt{\frac{\pi}{2c}}\left(\sum_{k=-\infty}^{\infty} e^{-\frac{n^{1/3}\pi^2(2k-\theta)^2}{2c}}\right)\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)F_n(\theta;q)d\theta. \tag{134}$$

In order to evaluate these integrals as $n\to\infty$, we need some estimates on the determinant $\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)$ which are uniform in θ. These are given in the following lemma.

Lemma 3. *(a) For $\theta\in(-1+\epsilon,1-\epsilon)$, the determinant $\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)$ is bounded uniformly in θ as $n\to\infty$. Furthermore it has the limit*

$$\lim_{n\to\infty}\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t) = \det(I-\mathbf{P}_t\mathbf{K}_{\text{cross}}(c;\theta)\mathbf{P}_t), \tag{135}$$

where $\mathbf{K}_{\text{cross}}(c;\theta)$ is the integral operator on $L^2(\mathbb{R})$ with kernel

$$K_{\text{cross}}(x,y;c;\theta) = \int_{-\infty}^{\infty}\frac{e^{i\pi\theta}e^{-cr}}{1+e^{i\pi\theta}e^{-cr}}\operatorname{Ai}(x-r)\operatorname{Ai}(y-r)\,dr. \tag{136}$$

(b) There exist positive constants $\tilde{C}$ such that for all $n\in\mathbb{N}$ and all $\theta\in(1-\epsilon,1+\epsilon)$,

$$|\det(I-\mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)| \le \exp\left(\left(\tilde{C}c^{-1}e^{-ct}\log n\right)^2+\tilde{C}c^{-1}e^{-ct}\log n\right). \tag{137}$$

Given the results of this lemma, it is fairly straightforward to prove Theorem 2(b). Consider I_1 first. Clearly as $n \to \infty$ the dominant term in the infinite sum is $k = 0$, and we have

$$I_1 = n^{1/6}\sqrt{\frac{\pi}{2c}}\int_{-1+\epsilon}^{1-\epsilon} e^{-\frac{n^{1/3}\pi^2\theta^2}{2c}} \det(I - \mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)\left(1 + \mathcal{O}(e^{-\frac{n^{1/3}\pi^2}{2c}})\right) d\theta\,. \tag{138}$$

Since the Fredholm determinant in the integrand has a limit as $n \to \infty$, we can use Laplace's method to evaluate the integral as $n \to \infty$. The integral I_1 is localized close to $\theta = 0$, and Laplace's method immediately gives

$$\begin{aligned}\lim_{n\to\infty} I_1 &= \lim_{n\to\infty} \det(I - \mathbf{P}_t\tilde{\mathbf{K}}(\theta = 0)\mathbf{P}_t) + o(1)\\ &= \det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{cross}}(c;\theta = 0)\mathbf{P}_t).\end{aligned} \tag{139}$$

Noting that $\mathbf{K}_{\mathrm{cross}}(c;\theta = 0) \equiv \mathbf{K}_{\mathrm{cross}}(c)$ defined in (24), we find

$$\lim_{n\to\infty} I_1 = \det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{cross}}(c)\mathbf{P}_t). \tag{140}$$

It remains only to show that $\lim_{n\to\infty} I_2 = 0$. This follows immediately from (134) and (137), since the infinite sum in (134) is vanishing like the exponent of a power of n whereas the determinant is growing at most as the exponent of a power of $\log n$. This completes the proof of Theorem 2(b), provided that Lemma 3 is true. The remainder of this subsection is dedicated to the proof of this lemma.

Proof of Lemma 3(a) We use the expression for the kernel $\tilde{K}_n(x, y)$ in (113) and (114)–(116) for the pointwise approximation as x, y in a compact subset of $\mathbb{R}$. In the scaling $q = e^{-cn^{-1/3}}$, the estimate (119) becomes

$$|K_n^{(3,M)}(x, y)| \le Cc^{-1}e^{-cM}. \tag{141}$$

Combined with (121) we see that as M becomes large, $K_n^{(1,M)}(x, y)$ and $K_n^{(3,M)}(x, y)$ vanish, and the dominant contribution should come from $K_n^{(2,M)}(x, y)$. In the sum $K_n^{(2,M)}(x, y)$, we denote $r = n^{-1/3}(k - n)$ and write the sum as

$$\begin{aligned}&K_n^{(2,M)}(x, y) =\\ &n^{-1/6}\sum_{r\in\{n^{-1/3}\mathbb{Z}\cap[-M,M]\}} \frac{e^{\pi i\theta}q^{1/2+rn^{1/3}}}{1 + e^{\pi i\theta}q^{1/2+rn^{1/3}}}\varphi_{n+rn^{1/3}}(2\sqrt{n} + xn^{-1/6})\varphi_{n+rn^{1/3}}(2\sqrt{n} + yn^{-1/6})\\ &= \int_{-M}^{M} \frac{e^{\pi i\theta}q^{1/2+\lfloor rn^{1/3}\rfloor}}{1 + e^{\pi i\theta}q^{1/2+\lfloor rn^{1/3}\rfloor}} n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}(2\sqrt{n} + xn^{-1/6})n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}(2\sqrt{n} + yn^{-1/6})\, dr.\end{aligned} \tag{142}$$

From (109), we find that

$$\lim_{n\to\infty} n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}(2\sqrt{n} + n^{-1/6}x) = \mathrm{Ai}(x - r), \tag{143}$$

thus the integrand in (168) has the pointwise limit

$$\frac{e^{i\pi\theta}e^{-cr}}{1 + e^{i\pi\theta}e^{-cr}}\,\mathrm{Ai}(x - r)\,\mathrm{Ai}(y - r), \tag{144}$$

and the bounded convergence theorem gives

$$\lim_{n\to\infty} K_n^{(2,M)}(x, y) = \int_{-M}^{M} \frac{e^{i\pi\theta}e^{-cr}}{1 + e^{i\pi\theta}e^{-cr}}\,\mathrm{Ai}(x - r)\,\mathrm{Ai}(y - r)\, dr. \tag{145}$$

Since both $K_n^{(1,M)}(x,y)$ and $K_n^{(3,M)}(x,y)$ are bounded in n and vanish as $M\to\infty$, we now take $M\to\infty$ and obtain

$$\lim_{n\to\infty}\tilde{K}_n(x,y)=\lim_{M\to\infty}\lim_{n\to\infty}K_n^{(2,M)}(x,y)=\int_{-\infty}^{\infty}\frac{e^{i\pi\theta}e^{-cr}}{1+e^{i\pi\theta}e^{-cr}}\operatorname{Ai}(x-r)\operatorname{Ai}(y-r)\,dr, \tag{146}$$

which is the kernel of a trace-class operator for all $\theta\in(-1+\epsilon,1-\epsilon)$.

We have proved the pointwise convergence of the kernels in the determinant, and actually the convergence in (146) is uniform if x,y are in a compact subset of $\mathbb{R}$. To prove the determinant convergence (135), we will need estimates on the kernel $\tilde{K}_n(x,y)$ as $\max(x,y)\to\infty$. The estimates (125) and (126) imply that, if $q=e^{-cn^{-1/3}}$ and $y\geq t$, then for all $x>4\max(-t,1)$, we take $M=2\max(-t,1)$ and $N=x$, and have

$$|K_n^{(4,2\max(-t,1),x)}(x,y)|\leq Ce^{-\gamma x^{3/2}}\quad\text{and}\quad|K_n^{(5,x)}(x,y)|\leq Cc^{-1}e^{-cx}, \tag{147}$$

with constants C and γ independent of n. Using the method of estimating $K_n^{(4,M,N)}(x,y)$ in (126), we have a similar estimate for $K_n^{(2,2\max(-t,1))}(x,y)$, provided that $\theta\in(-1+\epsilon,1-\epsilon)$:

$$|K_n^{(2,2\max(-t,1))}(x,y)|\leq Ce^{-\gamma x^{3/2}}. \tag{148}$$

Combining (147) and (148) we obtain the uniform estimate for $x,y\geq t$

$$|\tilde{K}_n(x,y)|\leq\tilde{C}e^{-cx}, \tag{149}$$

where the constant $\tilde{C}$ depends on t and c, but independent of n.

The Fredholm determinant $\det(I-\mathbf{P}_s\mathbf{K}(q^{-n+1/2}e^{i\pi\theta};q)\mathbf{P}_s)=\det(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)$ is given by the series

$$\det(I-\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int_t^{\infty}dx_1\cdots\int_t^{\infty}dx_k\det(\tilde{K}_n(x_i,x_j))_{i,j=1}^k. \tag{150}$$

Each of the determinants in this series can be estimated using (149) along with Hadamard's inequality, giving

$$|\det(\tilde{K}(x_i,x_j))_{i,j=1}^k|\leq k^{k/2}\tilde{C}^k\prod_{i=1}^k e^{-cx_i}, \tag{151}$$

so each term in (150) is bounded by

$$\begin{aligned}\left|\frac{(-1)^k}{k!}\int_t^{\infty}dx_1\cdots\int_t^{\infty}dx_k\det(\tilde{K}(x_i,x_j))_{i,j=1}^k\right|&\leq\frac{k^{k/2}}{k!}\tilde{C}^k\int_t^{\infty}dx_1e^{-cx_1}\cdots\int_t^{\infty}dx_ke^{-cx_k}\\&\leq\frac{k^{-k/2}}{k!}\left(\tilde{C}c^{-1}e^{-ct}\right)^k.\end{aligned} \tag{152}$$

Thus the series (150) is dominated by an absolutely convergent series, and the dominated convergence theorem gives that the sum converges to the term-by-term limit. This is exactly $\det(I-\mathbf{P}_t\mathbf{K}_{\text{cross}}(c;\theta)\mathbf{P}_t)$, since the integrands are dominated by an absolutely integrable function according to (151), so the dominated convergence theorem implies that each term converges to the corresponding term in the series for $\det(I-\mathbf{P}_t\mathbf{K}_{\text{cross}}(c;\theta)\mathbf{P}_t)$. This completes the proof of Lemma 3(a).

Proof of Lemma 3(b) Our estimate of $\det(I - \mathbf{P}_t\tilde{\mathbf{K}}(\theta)\mathbf{P}_t)$ for θ close to 1 is based on the identity (see [32, Theorem 9.2(d)])

$$\det(I - \mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t) = \det{}_2(I - \mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)e^{\operatorname{Tr}\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t}, \tag{153}$$

where $\det_2$ is defined in [32, Chapter 9]. The $\det_2$ functional can be estimated using the Hilbert-Schmidt norm, see [32, Theorem 9.2(b)]. In particular we have

$$|\det{}_2(I - \mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)| \leq \exp(\|\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t\|_2^2), \tag{154}$$

where $\|\cdot\|_2$ represents the Hilbert-Schmidt norm. Combining this inequality with (153), we have

$$|\det(I - \mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t)| \leq \exp(\|\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t\|_2^2)e^{|\operatorname{Tr}\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t|}, \tag{155}$$

and we are left to estimate the trace and the Hilbert-Schmidt norms of $\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t$.

We begin by estimating the kernel $\tilde{K}_n(x, y)$ for $\theta \in (1-\epsilon, 1+\epsilon)$. Since (141), (121) and (147) still hold for $\theta \in (1-\epsilon, 1+\epsilon)$, we concentrate on $K_n^{(2,M)}(x, y)$. Let us estimate this sum. Using (109) we obtain the following estimate, which is uniform for x, y in compact sets and $n - Mn^{1/3} < k < n + Mn^{1/3}$:

$$\begin{aligned}\varphi_k(2\sqrt{n} + xn^{-1/6})\varphi_k(2\sqrt{n} + yn^{-1/6}) &= \\ n^{-1/6}\operatorname{Ai}(x - (k-n)/(2n^{1/3}))\operatorname{Ai}(y - (k-n)&/(2n^{1/3}))(1 + \mathcal{O}(n^{-2/3})).\end{aligned} \tag{156}$$

The kernel $K_n^{(2,M)}(x, y)$ is thus estimated as

$$|K_n^{(2,M)}(x, y)| \leq Cn^{-1/3} \sum_{k=n-Mn^{1/3}}^{n+Mn^{1/3}} \left| c_k(\theta)\operatorname{Ai}(x - (k-n)/(2n^{1/3}))\operatorname{Ai}(y - (k-n)/(2n^{1/3}))\right| \tag{157}$$

for some constant C which is independent of n, M and θ. We therefore need to estimate the coefficients c_k, and it is convenient to estimate the real and imaginary parts separately. They are

$$\Re c_{n+j}(\theta) = \frac{\cos(\pi\theta) + q^{j+1/2}}{2\cos(\pi\theta) + q^{j+1/2} + q^{-j-1/2}}, \qquad \Im c_{n+j}(\theta) = \frac{\sin(\pi\theta)}{2\cos(\pi\theta) + q^{j+1/2} + q^{-j-1/2}}. \tag{158}$$

To estimate the imaginary part, note that $\Im c_{n+j}(1) = 0$, but c_{n+j} becomes large in a neighborhood of $\theta = 1$ when q is close to 1. In this neighborhood the critical points of $\Im c_{n+j}(\theta)$ are found to be at

$$\theta = 1 \pm \arcsin\left(\frac{q^{-j-1/2} - q^{j+1/2}}{q^{-j-1/2} + q^{j+1/2}}\right), \tag{159}$$

where $|\Im c_{n+j}(\theta)|$ attains the maximum. Plugging these critical points into $\Im c_{n+j}(\theta)$ we find the maximum value of $|\Im c_{n+j}(\theta)|$, obtaining

$$|\Im c_{n+j}(\theta)| = \frac{1}{|1 - q^{-2j-1}|}. \tag{160}$$

Now consider the real part of c_k. The maximum value of $|\Re c_k|$ is attained at $\theta = 1$. At this point we have

$$|\Re c_{n+j}| = \frac{1}{|1 - q^{-j-1/2}|}. \tag{161}$$

Combining (160) and (161) with (157) we obtain the estimate for x, y in a compact set and n large enough

$$\left|K_n^{(2,M)}(x,y)\right| \le C \sum_{k=n-Mn^{1/3}}^{n+Mn^{1/3}} \frac{|\mathrm{Ai}(x-(k-n)/(2n^{1/3}))\,\mathrm{Ai}(y-(k-n)/(2n^{1/3})))|}{2c|k-n|+1} = \tilde{C}\log n, \tag{162}$$

where $\tilde{C}$ is a positive constant depending on M, c but not n, θ. Now consider the behavior of $\tilde{K}_n(x,y)$ as $x \to \infty$ when $\theta \in (1-\epsilon, 1+\epsilon)$. The estimates (147) still hold here. The estimate (148) needs to be modified slightly for $\theta \in (1-\epsilon, 1+\epsilon)$. Since the dependence of $K_n^{(2,M)}(x,y)$ on θ comes entirely from the coefficients c_k, and the dependence on x and y comes entirely from the Hermite functions, we can combine the analysis leading to (162) with (148) to obtain the estimate

$$|K_n^{(2,2\max(-t,1))}(x,y)| \le Ce^{-\gamma x^{3/2}}\log n, \tag{163}$$

for $\theta \in (1-\epsilon, 1+\epsilon)$, where once again C and γ are constants independent of n. Analogous to (149), we therefore have the uniform estimate for all $x, y \ge t$

$$|\tilde{K}_n(x,y)| \le \tilde{C}e^{-cx}\log n, \tag{164}$$

where $\tilde{C}$ depends on t and c, but not n, θ. The trace of $\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t$ can therefore be estimated as

$$|\mathrm{Tr}\,\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t| \le \int_t^\infty |\tilde{K}_n(x,x)|\,dx \le \tilde{C}\log n \int_t^\infty e^{-cx},dx = \tilde{C}c^{-1}e^{-ct}\log n, \tag{165}$$

and the Hilbert-Schmidt norm is estimated as

$$\begin{aligned}\|\mathbf{P}_t\tilde{\mathbf{K}}\mathbf{P}_t\|_2^2 &= \int_t^\infty \int_t^\infty |\tilde{K}_n(x,y)|^2\,dy\,dx \\ &\le \tilde{C}^2(\log n)^2 \int_t^\infty \int_t^\infty e^{-cx}e^{-cy},dy\,dx = (\tilde{C}c^{-1}e^{-ct}\log n)^2.\end{aligned} \tag{166}$$

Combining this with (155), (165), and (166) we obtain (137). This completes the proof of Lemma 3(b).

3.3 Gap probability for the rightmost particle: fixed $q \in (0,1)$

Let $q \in (0,1)$ be fixed. Then we have the following lemma.

Lemma 4. *Let $s \equiv s_n = 2\sqrt{n} + tn^{-1/6}$. The following holds uniformly for all $\theta \in [-1,1]$.*

$$\det(I - \mathbf{P}_s\mathbf{K}(q^{-n+1/2}e^{i\pi\theta};q)\mathbf{P}_s) = \det(I - \mathbf{P}_t\mathbf{K}_{\mathrm{Airy}}\mathbf{P}_t) + o(1). \tag{167}$$

Sketch of proof. In the sum $K_n^{(2,M)}(x,y)$, we denote $r = n^{-1/3}(k-n)$ and write the sum as

$$\begin{aligned}&K_n^{(2,M)}(x,y) = \\ &n^{-1/6} \sum_{r\in\{n^{-1/3}\mathbb{Z}\cap[-M,M]\}} \frac{e^{\pi i\theta}q^{1/2+rn^{1/3}}}{1+e^{\pi i\theta}q^{1/2+rn^{1/3}}}\varphi_{n+rn^{1/3}}(2\sqrt{n}+xn^{-1/6})\varphi_{n+rn^{1/3}}(2\sqrt{n}+yn^{-1/6}) \\ &= \int_{-M}^{M} \frac{e^{\pi i\theta}q^{1/2+\lfloor rn^{1/3}\rfloor}}{1+e^{\pi i\theta}q^{1/2+\lfloor rn^{1/3}\rfloor}} n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}(2\sqrt{n}+xn^{-1/6})n^{1/12}\varphi_{n+\lfloor rn^{1/3}\rfloor}(2\sqrt{n}+yn^{-1/6})\,dr.\end{aligned} \tag{168}$$

Formula (143) implies that the step function in the integrand of (168) has the pointwise limit

$$\operatorname{Ai}(x-r)\operatorname{Ai}(y-r)\chi_{[-M,0]}(r). \tag{169}$$

The bounded convergence theorem then implies that

$$\begin{aligned}\lim_{n\to\infty} K_n^{(2,M)}(x,y) &= \int_{-M}^{0} \operatorname{Ai}(x-r)\operatorname{Ai}(y-r)\,dr \\ &= \int_0^M \operatorname{Ai}(x+r)\operatorname{Ai}(y+r)\,dr.\end{aligned} \tag{170}$$

Since M was arbitrary we can take it to infinity, in which case $K_n^{(1,M)}(x,y)$ and $K_n^{(3,M)}(x,y)$ vanish by (119) and (121), leaving

$$\lim_{n\to\infty} K_n(x,y) = \lim_{M\to\infty}\lim_{n\to\infty} K_n^{(2,M)}(x,y) = \int_0^\infty \operatorname{Ai}(x+r)\operatorname{Ai}(y+r)\,dr, \tag{171}$$

which is the kernel of $\mathbf{K}_{\text{Airy}}$.

To prove the convergence of the Fredholm determinant, we need to control the vanishing of $\tilde{K}_n(x,y)$ as $\max(x,y)\to\infty$. Since the procedure is the same as the proof of Lemma 3(a), we omit the detailed verification. We only note that the proof works for all $\theta \in [-1,1]$, since the coefficients $c_k(\theta)$ is uniformly bounded even if θ is around ± 1. □

As $n\to\infty$, we have the very fast convergence analogous to Lemma 2

$$\frac{(-q/w;q)_n}{(-q/w;q)_\infty} = 1+\mathcal{O}(q^n), \quad \frac{(q;q)_n}{(q;q)_\infty} = 1+\mathcal{O}(q^n). \tag{172}$$

Combining this fact with Lemma 4, we see that the integral (104) is

$$\mathbb{P}(\max(x_1,\dots,x_n)\le s) = \frac{1}{2}\int_{-1}^{1}\left(\sum_{k=-\infty}^{\infty} q^{k^2/2}e^{ik\pi\theta}\right)\det(I-\mathbf{P}_t\mathbf{K}_{\text{Airy}}\mathbf{P}_t)(1+o(1))d\theta. \tag{173}$$

After integrating, the only nonvanishing term in the infinite sum is $k=0$, thus we find

$$\begin{aligned}\mathbb{P}(\max(x_1,\dots,x_n)\le s) &= \frac{1}{2}\int_{-1}^{1}\det(I-\mathbf{P}_s\mathbf{K}_{\text{Airy}}\mathbf{P}_s)(1+o(1))d\theta \\ &= \det(I-\mathbf{P}_t\mathbf{K}_{\text{Airy}}\mathbf{P}_t)(1+o(1)).\end{aligned} \tag{174}$$

This proves part (a) of Theorem 2.

4 Proof of Theorem 3

As in the proof of Theorem 2, we give the detail in part (b), and then show that a simplified argument works for part (a). Also for notational simplicity we only consider the 2-correlation function, and the generalization to m-correlation function is straightforward.

4.1 Correlation functions for the bulk particles: $q = e^{-c/n}$

We assume the contour in (21) is

$$\Gamma = \left\{ |z| = q^{-n} - 1 + \frac{\delta_n}{n} = e^c - 1 + \frac{\delta_n}{n} \right\}, \tag{175}$$

such that $|\delta_n| < 1$ and there exists $\epsilon(q) > 0$ independent of n and

$$|1 - q^k(q^{-n} - 1 + \delta_n/n)| > \epsilon(q)/n \tag{176}$$

for all $k \geq 0$. For notational simplicity, we assume $\delta_n = 0$ later in this section.

We compute the asymptotics of $F(z)$ and $K(x, y; z; q)$ separately, and then prove Theorem 2(b).

For the asymptotics of $F(z)$, we have the following estimate:

Lemma 5. *Let $\epsilon > 0$ be a small constant independent of n.*

(a) If $z \in \Gamma$ and $|z - (e^c - 1)| < \epsilon$, then there exist $\delta > 0$ and $C > 0$ such that

$$|F(z)| < C\frac{\sqrt{2\pi n}}{\sqrt{ce^c(e^c - 1)}} \exp(-n\delta|z - (e^c - 1)|^2), \tag{177}$$

and if $|z - (e^c - 1)| < n^{-2/5}$, then

$$\frac{F(z)}{e^c - 1} = \frac{\sqrt{2\pi n}}{\sqrt{ce^c(e^c - 1)}} \exp\left(\frac{n(z - (e^c - 1))^2}{2ce^c(e^c - 1)}\right)(1 + \mathcal{O}(n^{-1/5})). \tag{178}$$

(b) If $z \in \Gamma$ and $|z - (e^c - 1)| \geq \epsilon$, then there exists $\delta > 0$ such that for large enough n,

$$|F(z)| < e^{-\delta n}. \tag{179}$$

Proof. We write

$$\frac{1}{n}\log F(z) = \frac{1}{n}\log\left(q^{-n(n-1)/2}(q;q)_n\right) - \log z + \int_0^\infty \log(1 + e^{-c\lfloor nx \rfloor/n}z)dx. \tag{180}$$

Unless z is very close to the negative real line, $n^{-1}\log F(z)$ is approximated by

$$\frac{1}{n}\log F(z) = G_n(z) + \mathcal{O}(n^{-1}) \quad \text{if } \arg z \in (-\pi + \epsilon', \pi - \epsilon'), \tag{181}$$

where ϵ' is any positive constant and

$$G_n(z) = \frac{1}{n}\log\left(q^{-n(n-1)/2}(q;q)_n\right) - \log z + \int_0^\infty \log(1 + e^{-cx}z)dx. \tag{182}$$

Hence by differentiation, we have that for $|z| = e^c - 1$ and $\arg z \in (-\pi + \epsilon', \pi - \epsilon')$,

$$\frac{1}{n}\frac{d}{dz}\log F(z) = G_n'(z) + \mathcal{O}(n^{-1}) = -\frac{1}{z} + \int_0^\infty \frac{e^{-cx}}{1 + ze^{-cx}}dx + \mathcal{O}(n^{-1}), \tag{183}$$

$$\frac{1}{n}\frac{d^2}{dz^2}\log F(z) = G_n''(z) + \mathcal{O}(n^{-1}) = \frac{1}{z^2} - \int_0^\infty \frac{e^{-2cx}}{(1 + ze^{-cx})^2}dx + \mathcal{O}(n^{-1}), \tag{184}$$

and furthermore

$$\frac{1}{n}\frac{d}{dz}\log F(z)\bigg|_{z=e^c-1} = \mathcal{O}(n^{-1}), \qquad \frac{1}{n}\frac{d^2}{dz^2}\log F(z)\bigg|_{z=e^c-1} = \frac{1}{c}\frac{1}{e^c(e^c-1)} + \mathcal{O}(n^{-1}). \tag{185}$$

Hence $z = e^c - 1$ is a saddle point for $F(z)$, and as z moves away from the saddle point $e^c - 1$, $|F(z)|$ decreases rapidly, provided that z is in the vicinity of the saddle point. Actually, for z on Γ but not in the vicinity of $e^c - 1$, note that $|z^{-n-1}|$ is a constant for $z \in \Gamma$ while $|1 + q^k z|$ decreases as $\arg z$ changes from 0 to $\pm\pi$, $|F(z)|$ decreases as $\arg z$ changes from 0 to $\pm\pi$.

The remaining task is to evaluate $F(e^c - 1)$ as $n \to \infty$. Although a direct computation is possible, it is difficult due to the evaluation of $(q;q)_n$ with q close to 1. Instead, we take an indirect approach.

In the gap probability formula (18), if we take $A = \mathbb{R}$, we have that the probability on the left-hand side is 1, and Fredholm determinant on the right-hand side is trivially 1, so we have

$$\frac{1}{2\pi i}\oint_\Gamma F(z)\frac{dz}{z} = 1. \tag{186}$$

By the asymptotic properties of $F(z)$ discussed above, we apply the steepest-descent analysis, and have that

$$\frac{1}{2\pi i}\int_{-\infty\cdot i}^{\infty\cdot i} F(e^c-1)e^{\frac{w^2}{2ce^c(e^c-1)}}\frac{dw}{\sqrt{n}(e^c-1)} = 1 + \mathcal{O}(n^{-1}), \tag{187}$$

and then

$$\frac{F(e^c-1)}{e^c-1} = \frac{\sqrt{2\pi n}}{\sqrt{ce^c(e^c-1)}}(1 + \mathcal{O}(n^{-1})). \tag{188}$$

Hence the lemma is proved. □

We compute the asymptotics of $K(x_1, x_2; z; q)$ with the scaling

$$x_1 = 2x\sqrt{n} + \frac{\pi\xi}{\sqrt{n/c}}, \quad x_2 = 2x\sqrt{n} + \frac{\pi\eta}{\sqrt{n/c}}, \tag{189}$$

where $x \in \mathbb{R}$ is fixed and ξ, η in a compact subset of $\mathbb{R}$. The result we need is as follows.

Lemma 6. *Let $\epsilon > 0$ be a small constant independent of n. In both parts of the lemma we assume $q = e^{-c/n}$ and x_1, x_2 are as in* (189)*.*

(a) If $z \in \Gamma$ and $|z - (e^c - 1)| < \epsilon$, then

$$\lim_{n\to\infty}\frac{\sqrt{c}}{\sqrt{n}}K(x_1, x_2; z; q) = K_{\text{inter}}(\xi, \eta; x; c; z), \tag{190}$$

where

$$K_{\text{inter}}(\xi, \eta; x; c; z) = \frac{1}{\pi}\int_0^\infty \frac{z}{e^{u^2}e^{cx^2} + z}\cos(\pi u(\xi - \eta))\,du. \tag{191}$$

(b) If $z \in \Gamma$ and $|z - (e^c - 1)| \geq \epsilon$, then there exists $C > 0$ such that for large enough n,

$$|K(x_1, x_2; z; q)| < Cn^2. \tag{192}$$

Here we note that

$$K_{\text{inter}}(\xi, \eta; x; c; e^c - 1) = K_{\text{inter}}\left(\xi, \eta; \frac{e^{cx^2}}{e^c - 1}\right). \tag{193}$$

Proof of Lemma 6(a). We concentrate on the case $x > 0$. The argument for the $x < 0$ case is the same, since φ_k are even or odd functions, depending on the parity of k. The case $x = 0$ requires some modification, and we discuss it in Remark 5.

Recall that $K(x_1, x_2; z; q)$ is an infinite linear combination of $\varphi_k(x_1)\varphi_k(x_2)$ with $k \geq 0$. Let $\epsilon > 0$ be a small constant. Then we divide $K(x_1, x_2; z; q)$ into four parts as follows:

$$K^{\mathrm{sup}}(x_1; x_2; z; q) = \sum_{k>n(x^2+\epsilon)}^{\infty} \frac{q^k z}{1+q^k z}\varphi_k(x_1)\varphi_k(x_2), \tag{194}$$

$$K^{\mathrm{mid}}(x_1; x_2; z; q) = \sum_{n(x^2-\epsilon)<k\leq n(x^2+\epsilon)} \frac{q^k z}{1+q^k z}\varphi_k(x_1)\varphi_k(x_2), \tag{195}$$

$$K^{\mathrm{sub}}(x_1; x_2; z; q) = \sum_{n\epsilon<k\leq n(x^2-\epsilon)} \frac{q^k z}{1+q^k z}\varphi_k(x_1)\varphi_k(x_2), \tag{196}$$

$$K^{\mathrm{res}}(x_1; x_2; z; q) = \sum_{0\leq k\leq n\epsilon} \frac{q^k z}{1+q^k z}\varphi_k(x_1)\varphi_k(x_2). \tag{197}$$

Below we show that as $n \to \infty$, for all small enough $\epsilon > 0$, there exists $C > 0$ that is independent of ϵ, such that

$$\left| \frac{\sqrt{c}}{\sqrt{n}} K^{\mathrm{sup}}(x_1, x_2; z; q) - \frac{1}{\pi}\int_{\sqrt{\epsilon c}}^{\infty} \frac{z}{e^{u^2}e^{cx^2}+z}\cos\left(\pi u(\xi-\eta)\right) du \right| < C\sqrt{\epsilon}, \tag{198}$$

$$\left| \frac{\sqrt{c}}{\sqrt{n}} K^{\mathrm{mid}}(x_1, x_2; z; q) \right| < C\sqrt{\epsilon}, \quad \left| \frac{\sqrt{c}}{\sqrt{n}} K^{\mathrm{sub}}(x_1, x_2; z; q) \right| = o(1), \quad \left| \frac{\sqrt{c}}{\sqrt{n}} K^{\mathrm{res}}(x_1, x_2; z; q) \right| = o(1). \tag{199}$$

By taking $\epsilon \to 0$ in the inequalities above, we prove (190). Below we prove the four results. For notational simplicity, when we prove the three estimates in (199), we only consider the case that $x_1 = x_2 = 2x\sqrt{n}$.

First we prove (198). By [33, Formula 8.22.12], for $k > n(x^2 + \epsilon)$, we have

$$\begin{aligned} \sqrt{\pi}k^{1/4}\varphi_k(2x\sqrt{n}) &= \sin(\phi_k)^{-1/2}\sin\left[\frac{2k+1}{4}(\sin(2\phi_k)-2\phi_k)+\frac{3\pi}{4}\right]+\mathcal{O}(n^{-1}) \\ &= \left(1-\frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/4}\sin\left[-(2k+1)\theta_k\right]+\mathcal{O}(n^{-1}), \end{aligned} \tag{200}$$

where

$$\phi_k = \arccos\left(x\sqrt{\frac{n}{k+1/2}}\right), \quad \theta_k = -(2k+1)\int_{x\sqrt{\frac{n}{k+1/2}}}^{1}\sqrt{1-t^2}dt+\frac{3\pi}{4}. \tag{201}$$

If x_1 is as specified in (189), then

$$\sqrt{\pi}k^{1/4}\varphi_k(x_1) = \left(1-\frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/4}\sin\left(\theta_k + x\pi\xi\sqrt{c}\sqrt{\frac{k+1/2}{x^2 n}-1}\right)+\mathcal{O}(n^{-1}), \tag{202}$$

and also have an analogous formula for $\varphi_k(x_2)$ with x_2 specified in (189). Then we have

$$\pi k^{1/2}\varphi_k(x_1)\varphi_k(x_2) = \left(1-\frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/2} \times \frac{1}{2}\left[\cos\left(\pi\sqrt{c}(\xi+\eta)\sqrt{\frac{k+1/2}{n}-x^2}\right) - \cos\left(\theta_k + \pi\sqrt{c}(\xi+\eta)\sqrt{\frac{k+1/2}{n}-x^2}\right)\right] + \mathcal{O}(n^{-1}). \tag{203}$$

Now we define

$$K^{\sup,1}(x_1,x_2;z;q) = \sum_{k>n(x^2+\epsilon)}^{\infty} \frac{q^k z}{1+q^k z}\frac{k^{-1/2}}{2\pi}\left(1-\frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/2}\cos\left(\pi\sqrt{c}(\xi-\eta)\sqrt{\frac{k+1/2}{n}-x^2}\right), \tag{204}$$

and

$$\begin{aligned} K^{\sup,2}(x_1,x_2;z;q) &= K^{\sup}(x_1,x_2;z;q) - K^{\sup,1}(x_1,x_2;z;q) \\ &= \sum_{k>n(x+\epsilon)}^{\infty} \frac{q^k z}{1+q^k z}\frac{k^{-1/2}}{2\pi}\left(1-\frac{x^2 n}{k+\frac{1}{2}}\right)^{-1/2}\cos\left(\theta_k + \pi\sqrt{c}(\xi+\eta)\sqrt{\frac{k+1/2}{n}-x^2}\right) + \mathcal{O}(1). \end{aligned} \tag{205}$$

It is not hard to see that if $\arg(z) \in (-\pi+\epsilon', \pi-\epsilon')$ for $\epsilon' > 0$, then

$$\begin{aligned} \frac{\sqrt{c}}{\sqrt{n}}K^{\sup,1}(x_1,x_2;z;q) &= \frac{\sqrt{c}}{2\pi}\int_{x^2+\epsilon}^{\infty} \frac{e^{-c\kappa}z}{1+e^{-c\kappa}z}\frac{1}{\sqrt{\kappa}}\left(1-\frac{x^2}{\kappa}\right)^{-1/2}\cos\left(\pi\sqrt{c}\sqrt{\kappa-x^2}(\xi-\eta)\right)d\kappa + \mathcal{O}(n^{-1}) \\ &= \frac{\sqrt{c}}{2\pi}\int_{\epsilon}^{\infty} \frac{z}{e^{ct}e^{cx^2}+z}\cos\left(\pi\sqrt{tc}(\xi-\eta)\right)\frac{dt}{\sqrt{t}} + \mathcal{O}(n^{-1}) \\ &= \frac{1}{\pi}\int_{\sqrt{\epsilon c}}^{\infty} \frac{z}{e^{u^2}e^{cx^2}+z}\cos\left(\pi u(\xi-\eta)\right)du + \mathcal{O}(n^{-1}). \end{aligned} \tag{206}$$

On the other hand, since for any $k > n(x^2+\epsilon)$,

$$\theta_k - \theta_{k-1} = \arcsin(x\sqrt{n/k}) - \frac{\pi}{2} + \mathcal{O}(n^{-1}), \tag{207}$$

and $\theta_k - \theta_{k-1}$ is in a compact subset of $(-\pi, 0)$ that is independent of k, if n is large enough. Hence the neighboring terms in the series on the right-hand side of (205) has cancellations, so by comparing with $K^{\sup,1}$ that is defined in (204) by a series that is not oscillating, it is clear that as $n \to \infty$, if $\arg z \in (-\pi+\delta, \pi-\delta)$, then

$$\left|\frac{c}{\sqrt{n}}K^{\sup,2}(x_1,x_2;z;q)\right| = o(1). \tag{208}$$

The approximations (206) and (208) imply (198).

Next we prove the estimates (199) in the special case $x_1 = x_2 = 2x\sqrt{n}$. The analysis is nearly identical for general ξ and η.

To prove the first estimate, we use the approximation formula (109). For $n(c^2-\epsilon) < k \le n(c^2+\epsilon)$, and x in a compact subset of $(-1, +\infty)$,

$$k^{1/12}\varphi_k(2\sqrt{k+1/2}x) = 2^{1/6}\left(\frac{\zeta(x)}{x^2-1}\right)^{1/4}\mathrm{Ai}((2k+1)^{2/3}\zeta(x)) + \mathcal{O}(n^{-1}). \tag{209}$$

Hence we have

$$k^{1/12}\varphi_k(2x\sqrt{n}) = 2^{1/6}\left(\frac{\zeta(x_k)}{x_k^2-1}\right)^{1/4}\mathrm{Ai}((2k+1)^{2/3}\zeta(x_k)) + \mathcal{O}(n^{-1}), \quad \text{where} \quad x_k = \sqrt{\frac{x^2 n}{k+1/2}}. \tag{210}$$

Hence using the estimate (112) of Airy function, we have that if $\arg z \in (-\pi+\delta, \pi-\delta)$ and n is large enough, the first inequality of (199) is proved by the estimate

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\left|K^{\mathrm{mid}}(2x\sqrt{n}, 2x\sqrt{n}; z)\right| \\
&= \left(\frac{2}{n}\right)^{1/3}\left|\int_{x^2-\epsilon}^{x^2+\epsilon}\frac{e^{-c\kappa}z}{1+e^{-c\kappa}z}\left(\frac{\zeta(\frac{x}{\sqrt{\kappa}})}{\frac{x^2}{\kappa}-1}\right)^{1/2}\mathrm{Ai}\left((2n)^{2/3}\zeta\left(\frac{x}{\sqrt{\kappa}}\right)\right)^2 d\kappa + \mathcal{O}(n^{-1})\right| \\
&\le \left(\frac{2}{n}\right)^{1/3}\int_{x^2-\epsilon}^{x^2+\epsilon}\left|\frac{e^{-c\kappa}z}{1+e^{-c\kappa}z}\right| 2^{1/3}\left(\frac{\zeta(\frac{x}{\sqrt{\kappa}})}{\frac{x^2}{\kappa}-1}\right)^{1/2} f\left((2n)^{2/3}\zeta\left(\frac{x}{\sqrt{\kappa}}\right)\right)^2 d\kappa \\
&\le C\sqrt{\epsilon},
\end{aligned} \tag{211}$$

where $C > 0$ is independent of n and ϵ.

To prove the second estimate, By [33, Formula 8.22.13], for $n\epsilon < k \le n(x^2+\epsilon)$, we have

$$\begin{aligned}
\sqrt{\pi}k^{1/4}\varphi_k(2x\sqrt{n}) &= \frac{1}{2}\sinh(\phi_k)^{-1/2}\exp\left[\frac{2k+1}{4}(2\phi_k - \sinh(2\phi_k))\right](1+\mathcal{O}(n^{-1})) \\
&= \frac{1}{2}\left(\frac{x^2 n}{k+\frac{1}{2}} - 1\right)^{-1/4}\exp\left[-(2k+1)\int_1^{x\sqrt{\frac{n}{k+1/2}}}\sqrt{t^2-1}dt\right](1+\mathcal{O}(n^{-1})),
\end{aligned} \tag{212}$$

where

$$\phi_k = \operatorname{arccosh}\left(x\sqrt{\frac{n}{k+1/2}}\right). \tag{213}$$

It is clear that

$$|\varphi_k(2x\sqrt{n})| < e^{-\epsilon' n} \tag{214}$$

for all $n\epsilon < k \le n(c^2+\epsilon)$, where $\epsilon' > 0$ is a constant depending on ϵ and c. This estimate implies the second inequality of (199) with $x_1 = x_2 = 2x\sqrt{n}$.

Finally, by the estimate of Hermite polynomials provided in [27, Section 11.4, Exercises 4.2 and 4.3], we have that

$$|\varphi_k(2x\sqrt{n})| < e^{-\epsilon'' n} \tag{215}$$

for all $k \le n\epsilon$, where $\epsilon'' > 0$ depend on c only, providing that ϵ is small enough. This estimate implies the last inequality of (199) with $x = y = 2x\sqrt{n}$. Here we note that the result in [27, Section 11.4, Exercises 4.2 and 4.3] is valid even for very small k, like $k = 1, 2, \dots$, except for $k = 0$. But it is obvious that when $k = 0$, (215) holds. □

Remark 5. The case $x = 0$ is different, because K^{sub} is not longer meaningful, and K^{mid} and K^{res} need to be combined. The asymptotic analysis becomes easier, since $\varphi_k(\xi/\sqrt{n})$ has limiting formulas simpler than (200), (202), and (212), see [1, 22.15.3–4]. We omit the detail, because a similar computation is done in [19, Proof of Theorem 1.9].

Proof of Lemma 6(b). The difficulty is that when $\arg z$ is close to $\pm\pi$, the denominator $1+q^k z$ appearing in $K(x_1, x_2; z; q)$ can be close to zero. But since $|z| = q^{-n} - 1 + \delta_n/n = e^c - 1 + \delta_n/n$, and we assume that $\delta_n = 0$, for $k \geq n$

$$|1 + q^k z| \geq 1 - q^n(q^{-n} - 1) \geq q^n = e^{-c} > 0, \tag{216}$$

and the denominator is not close to zero. Then by the estimates that we use in the proof of part (a), we have for all $z \in \Gamma$,

$$\sum_{k\geq n}^{\infty} \frac{q^k z}{1+q^k z} \varphi_k(x_1)\varphi_k(x_2) = \mathcal{O}(n^{1/2}). \tag{217}$$

On the other hand, for $k < n$, by assumption (176) we have $|1+q^k z| \geq \epsilon(q)/n$, and then by the crude estimate (54) of Hermite polynomials, we have

$$\left| \sum_{k=0}^{n} \frac{q^k z}{1+q^k z} \varphi_k(x_1)\varphi_k(x_2) \right| < \epsilon(q) n^2. \tag{218}$$

The combination of (217) and (218) implies (192), and then finish the proof. □

Proof of Theorem 3(b) for 2-correlation function. Using the estimates in Lemmas 5 and 6, we have that the integral in (21) concentrates in the vicinity of the saddle point $z = e^c - 1$, and more precisely, in the region $|z - (e^c - 1)| = \mathcal{O}(n^{-1/2})$. A straightforward application of the Laplace method yields that if x_1, x_2 depend on ξ, η as in (189),

$$\begin{aligned}
&\lim_{n\to\infty} \left(\frac{\pi}{\sqrt{n/c}} \right)^2 R_n^{(2)}(x_1, x_2) \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \oint_0 F(z) \begin{vmatrix} \frac{\pi\sqrt{c}}{\sqrt{n}} K(x_1, x_2; z) & \frac{\pi\sqrt{c}}{\sqrt{n}} K(x_1, x_2; z) \\ \frac{\pi\sqrt{c}}{\sqrt{n}} K(x_2, x_1; z) & \frac{\pi\sqrt{c}}{\sqrt{n}} K(x_2, x_2; z) \end{vmatrix} \frac{dz}{z} \\
&= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{-\infty\cdot i}^{\infty\cdot i} \frac{F(e^c-1)}{e^c-1} e^{\frac{w^2}{2ce^c(e^c-1)}} \begin{vmatrix} K_{\text{inter}}(\xi, \xi; x; c; e^c-1) & K_{\text{inter}}(\xi, \eta; x; c; e^c-1) \\ K_{\text{inter}}(\eta, \xi; x; c; e^c-1) & K_{\text{inter}}(\eta, \eta; x; c; e^c-1) \end{vmatrix} \frac{dw}{\sqrt{n}} \\
&= \begin{vmatrix} K_{\text{inter}}(\xi, \xi; x; c; e^c-1) & K_{\text{inter}}(\xi, \eta; x; c; e^c-1) \\ K_{\text{inter}}(\eta, \xi; x; c; e^c-1) & K_{\text{inter}}(\eta, \eta; x; c; e^c-1) \end{vmatrix}.
\end{aligned} \tag{219}$$

By (193) we prove the 2-correlation function formula in Theorem 3(b). □

4.2 Correlation functions for the bulk particles: fixed $q \in (0, 1)$

We let q be in a compact subset of $(0, 1)$. We assume that the contour in (21) is $|z| = q^{-n+1/2}$, and take the change of variable like in (98)

$$w = q^n z \quad \text{with} \quad |w| = \sqrt{q}. \tag{220}$$

Then analogous to (103), we write the $m=2$ case of (21) as

$$\begin{aligned}R^{(2)}(x_1,x_2) &= \frac{q^{n/2}}{Z_n(q)}q^{n^2}q^{-\frac{n(n+1)}{2}}\frac{1}{2\pi i}\oint_0 \frac{dw}{w}\left(\prod_{k=0}^{\infty}(1+q^k w)\right)\left(\prod_{k=1}^{n}(1+q^k w^{-1})\right)\\ &\quad\times\begin{vmatrix}K(x_1,x_1;q^{-n}w;q) & K(x_1,x_2;q^{-n}w;q)\\ K(x_2,x_1;q^{-n}w;q) & K(x_2,x_2;q^{-n}w;q)\end{vmatrix}\\ &= \frac{(q;q)_n}{(q;q)_\infty}\frac{1}{2\pi i}\oint_0 \frac{dw}{w}\left(\sum_{k=-\infty}^{\infty} q^{\frac{k(k-1)}{2}}w^k\right)\frac{(-q/w;q)_n}{(-q/w;q)_\infty}\\ &\quad\times\begin{vmatrix}K(x_1,x_1;q^{-n}w;q) & K(x_1,x_2;q^{-n}w;q)\\ K(x_2,x_1;q^{-n}w;q) & K(x_2,x_2;q^{-n}w;q)\end{vmatrix},\end{aligned} \tag{221}$$

where we make use of identity (102). Next we find the asymptotics of $K(x_i,x_j;q^{-n}w;q)$. We write

$$K(x_i,x_j;q^{-n}w;q) = K_n^{(0)}(x_i,x_j) + K^{(1)}(x_i,x_j;q^{-n}w;q) - K^{(2)}(x_i,x_j;q^{-n}w;q), \tag{222}$$

where

$$K^{(0)}(x_i,x_j) = \left(\sum_{k=0}^{n-1}\varphi_k(x_i)\varphi_k(x_j)\right), \tag{223}$$

$$K^{(1)}(x_i,x_j;q^{-n}w;q) = \left(\sum_{k=0}^{\infty}\frac{q^k w}{1+q^k w}\varphi_{n+k}(x_i)\varphi_{n+k}(x_j)\right), \tag{224}$$

$$K^{(2)}(x_i,x_j;q^{-n}w;q) = \left(\sum_{k=1}^{n}\frac{q^k w^{-1}}{1+q^k w^{-1}}\varphi_{n-k}(x_i)\varphi_{n-k}(x_j)\right). \tag{225}$$

It is well known that $K_n^{(0)}(x_i,x_j)$ is the correlation kernel of n-dimensional GUE random matrix, and for

$$x_i = 2\sqrt{n}x + \frac{\pi\xi_i}{(1-x^2)^{1/2}\sqrt{n}}, \quad x_j = 2\sqrt{n}x + \frac{\pi\xi_j}{(1-x^2)^{1/2}\sqrt{n}}, \quad \text{where} \quad x\in(-1,1), \tag{226}$$

we have [3, Chapter 3]

$$\lim_{n\to\infty}\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}K_n^{(0)}(x_i,x_j) = K_{\sin}(\xi_i,\xi_j) := \frac{\sin(\pi(\xi_i-\xi_j))}{\pi(\xi_i-\xi_j)}. \tag{227}$$

To estimate $K^{(1)}$ and $K^{(2)}$, it suffices to use the rough estimate (54) and have (with $k\approx 1.086435$)

$$\left|K^{(1)}(x_i,x_j;q^{-n}w;q)\right| \le \sum_{k=0}^{\infty}\left|\frac{q^k w}{1+q^k w}\right|\frac{k^2}{\sqrt{2\pi}} < \sum_{k=0}^{\infty}\frac{q^k}{1-\sqrt{q}}\frac{k^2}{\sqrt{2\pi}} < \frac{1}{(1-q)(1-\sqrt{q})}. \tag{228}$$

Similarly, we also have

$$\left|K^{(2)}(x_i,x_j;q^{-n}w;q)\right| < \frac{1}{(1-q)(1-\sqrt{q})}. \tag{229}$$

Hence we have that uniformly in w on the circle $|w|=\sqrt{q}$

$$\lim_{n\to\infty}\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}K(x_i,x_j;q^{-n}w;q) = K_{\sin}(\xi_i,\xi_j). \tag{230}$$

Using the very fast convergence (172), we have

$$\lim_{n\to\infty}\left(\frac{\pi}{(1-x^2)^{1/2}\sqrt{n}}\right)^2 R^{(2)}(x_1,x_2) = \frac{1}{2\pi i}\oint_0 \frac{dw}{w}\left(\sum_{k=-\infty}^{\infty} q^{\frac{k(k-1)}{2}} w^k\right)(1+o(1))\det(K_{\sin}(\xi_i,\xi_j))_{i,j=1}^2$$
$$= \det(K_{\sin}(\xi_i,\xi_j))_{i,j=1}^2. \tag{231}$$

Hence Theorem 3(b) is proved for the 2-correlation function case.

5 Relation to interacting particle systems

Theorem 1(a) for the gap probability in the MNS model has analogues in the study of several interacting particle systems that are related to the Kardar–Parisi–Zhang (KPZ) universality class. In this section we discuss three interacting particle systems. First we consider the q-Whittaker processes, which are obtained by a specialization of Macdonald processes [6, Section 3]. Next we consider the q-deformed Totally Asymmetric Simple Exclusion Process (q-TASEP), which is a continuous limit of the q-Whittaker process [6, Section 3.3], [7]. At last we consider the q-deformed Totally Asymmetric Zero Range Process (q-TAZRP), which is the dual process of q-TASEP [21], [24].

Identity of Fredholm determinants Let $f(\xi)$ be a meromorphic function on $\mathbb{C}$ with the finite set of poles $\mathbb{A} = \{a_1,\dots,a_m\} \not\ni 0$, and suppose that $f(0)=1$. Let $\Gamma_{0,\mathbb{A}}$ be a contour with positive orientation such that 0 and $\mathbb{A}$ are enclosed in $\Gamma_{0,\mathbb{A}}$. On the other hand, let $\Gamma_{\mathbb{A}}$ be a contour with positive orientation such that $\mathbb{A}$ is enclosed in $\Gamma_{\mathbb{A}}$ but 0 is outside of $\Gamma_{\mathbb{A}}$. We assume the condition

$$\Gamma_{0,\mathbb{A}} \cap q\cdot\Gamma_{0,\mathbb{A}} = \emptyset, \quad \text{and} \quad \Gamma_{\mathbb{A}} \cap q^k\cdot\Gamma_{\mathbb{A}} = \emptyset, \quad \text{for } k=1,2,\dots. \tag{232}$$

where for a contour C, $q^k\cdot C = \{q^k z \mid z\in C\}$. Furthermore, we define $\Gamma = \Gamma_{0,\mathbb{A}} \cup (-\Gamma_{\mathbb{A}})$, where $-\Gamma_{\mathbb{A}}$ is the contour $\Gamma_{\mathbb{A}}$ with negative orientation.

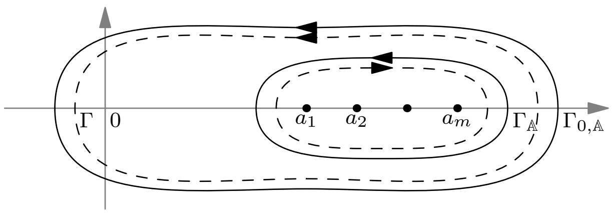

Figure 2: The shapes of $\Gamma_{\mathbb{A}}$ and $\Gamma_{0,\mathbb{A}}$ (solid) and the shape of Γ (dashed).

We define kernel functions

$$M(\xi,\eta;q) = \frac{f(\xi)}{\xi - q\eta}, \tag{233}$$

and

$$K(\xi,\eta;z;q) = \frac{1}{1+z}\frac{f(\xi)}{\xi}\,{}_2\phi_1\left[\begin{matrix}-z,q\\-qz\end{matrix};q,\frac{q\eta}{\xi}\right]$$
$$= \frac{f(\xi)}{\xi}\frac{-1}{2\pi i}\int_{-i\cdot\infty}^{i\cdot\infty}\frac{1}{1+zq^s}\frac{\pi}{\sin(\pi s)}\left(-\frac{q\eta}{\xi}\right)^s ds, \tag{234}$$

where we use the contour integral representation (3) of ${}_2\phi_1\left[\begin{smallmatrix}-z,q\\-qz\end{smallmatrix};q,q\eta\xi^{-1}\right]$. Note that the contour in (234) satisfies that the poles $0,1,2,\dots$ are on its right, and the poles $-1,-2,\dots$ and $(\log(-z^{-1})+2\pi ik)/\log q$ with $k\in\mathbb{Z}$ are on its left. Also that $K(\xi,\eta;z;q)$ has poles $-1,-q^{-1},-q^{-2},\dots$ as a meromorphic function in z.

These two kernels define integral operators on $L^2(\Gamma_{0,\mathbb{A}})$, $L^2(\Gamma_{\mathbb{A}})$ and $L^2(\Gamma)$, where the measure is $(2\pi i)^{-1}d\eta$ with the orientation positive on $\Gamma_{0,\mathbb{A}}$ and negative on $\Gamma_{\mathbb{A}}$. We denote these integral operators all by $\mathbf{M}(q)$ and $\mathbf{K}(z;q)$, and the domain is specified when they occur. We have the following technical lemma.

Lemma 7. *Let contours $\Gamma_{0,\mathbb{A}}$, $\Gamma_{\mathbb{A}}$ and meromorphic function $f(\xi)$ be given above. Suppose the meromorphic functions $M(\xi,\eta;q)$ and $K(\xi,\eta;z;q)$ are defined by* (233) *and* (234) *respectively, and $\mathbf{M}(q)$ and $\mathbf{K}(z;q)$ are integral operators with kernels $M(\xi,\eta;q)$ and $K(\xi,\eta;z;q)$. Then*

$$\det(I+z\mathbf{M}(q)_{L^2(\Gamma_{0,\mathbb{A}})})=(-z;q)_\infty\det(I+\mathbf{K}(z;q)_{L^2(\Gamma_{\mathbb{A}})}). \tag{235}$$

Before giving the proof, we note that the left-hand side of (235) is entire in z, and then the right-hand side has only apparent singularities on $z=-1,-q^{-1},-q^{-2},\dots$.

Proof of Lemma 7. The proof is similar to that of Lemma 1. We first prove the result under the restriction that

$$q<\left|\frac{\xi}{\eta}\right|<q^{-1}\quad\text{for all }\xi,\eta\in\Gamma. \tag{236}$$

Define two sequences of L^2 functions on Γ: $\{\varphi_n(\xi)=f(\xi)\xi^{-n-1}\}_{n=0}^\infty$ and $\{\psi_n(\eta)=\eta^n\}_{n=0}^\infty$. They satisfy the orthogonality condition

$$\frac{1}{2\pi i}\int_\Gamma\varphi_j(\xi)\psi_k(\xi)d\xi=\delta_{jk},\quad j,k=0,1,2,\dots. \tag{237}$$

Under the additional condition (236), we have that for $\xi,\eta\in\Gamma$

$$M(\xi,\eta;q)=\sum_{k=0}^\infty q^k\varphi_k(\xi)\psi_k(\eta),\quad K(\xi,\eta;z;q)=\sum_{k=0}^\infty\frac{q^kz}{1+q^kz}\varphi_k(\xi)\varphi_k(\eta). \tag{238}$$

We find that our integral operator $\mathbf{M}(q)_{L^2(\Gamma)}$ satisfies $\mathbf{M}(q^k)_{L^2(\Gamma)}=(\mathbf{M}(q)_{L^2(\Gamma)})^k$ as in (65). Then analogous to (65), we define $\mathbf{R}(z;q)$ by

$$I-\mathbf{R}(z;q)=(I+z\mathbf{M}(q)_{L^2(\Gamma)})^{-1}, \tag{239}$$

and find that if $|z|<1$, then $\mathbf{K}(z;q)_{L^2(\Gamma)}=\mathbf{R}(z;q)_{L^2(\Gamma)}$ like (69). Hence similar to (70), we have the identity of operators on $L^2(\Gamma)$

$$(I+z\mathbf{M}(q)_{L^2(\Gamma)})(I-\mathbf{K}(z;q)_{L^2(\Gamma)}\chi_{-\Gamma_{\mathbb{A}}})=I+z\mathbf{M}(q)_{L^2(\Gamma)}\chi_{\Gamma_{0,\mathbb{A}}}. \tag{240}$$

Then similar to (64), we have that

$$\begin{aligned}\det(I+z\mathbf{M}(q)_{L^2(\Gamma_{0,\mathbb{A}})})&=\det(I+z\mathbf{M}(q)_{L^2(\Gamma)}\chi_{\Gamma_{0,\mathbb{A}}})\\&=\det(I+z\mathbf{M}(q)_{L^2(\Gamma)})\det(I-\mathbf{K}(z;q)_{L^2(\Gamma)}\chi_{(-\Gamma_{\mathbb{A}})})\\&=\det(I+z\mathbf{M}(q)_{L^2(\Gamma)})\det(I-\mathbf{K}(z;q)_{L^2(-\Gamma_{\mathbb{A}})})\\&=\det(I+z\mathbf{M}(q)_{L^2(\Gamma)})\det(I+\mathbf{K}(z;q)_{L^2(\Gamma_{\mathbb{A}})}),\end{aligned} \tag{241}$$

where in the last line the orientation of the integral contour is changed, and so does the sign for the operator $\mathbf{K}(z;q)$. Similar to (74), we have

$$\det(I + z\mathbf{M}(q))_{L^2(\Gamma)} = (-z;q)_\infty, \tag{242}$$

and prove (235) under technical restrictions $|z| < 1$ and (236).

We note that for fixed q and $a_1, \dots, a_m$, the contour $\Gamma = \Gamma_{0,\mathbb{A}} \cup (-\Gamma_{\mathbb{A}})$ that satisfies both (232) and (236) may not exist. However, if q is fixed and $a_1, \dots, a_m$ are regarded as movable parameters, then the contour Γ exists given that $a_1, \dots, a_m$ cluster tightly enough. Although (235) is proved under the additional condition (236), since the kernels $M(\xi, \eta; q)$ and $K(\xi, \eta; z; q)$ are meromorphic functions, by deforming the contours $\Gamma_{0,\mathbb{A}}$ and $\Gamma_{\mathbb{A}}$ and moving the poles $a_1, \dots, a_m$ if necessary, we can remove the technical restriction (236). Then by analytic continuation we can remove the technical restriction $|z| < 1$. □

Below we consider applications of Lemma 7 on three interacting particle systems.

q-Whittaker processes The q-Whittaker processes are interacting particle systems defined in [6, Section 3]. Since the definition is relatively involved, we refer the reader to the original paper, and only remark that in the N-particle model, (i) the speeds of particles depend on parameters $a_1, \dots, a_N \in (0, \infty)$, and (ii) the transition probabilities depend on parameters $\alpha_1, \dots, \alpha_N; \beta_1, \dots, \beta_N; \gamma$.

[6, Theorem 3.23] gives a moment generating formula for the q-Whittaker processes

$$\left\langle \frac{1}{(-zq^{\lambda_N};q)_\infty} \right\rangle_{\mathbf{MM}_{t=0}(a_1,\dots,a_N;\rho)} = \frac{1}{(-z;q)_\infty} \det(1 + z\mathbf{M}(q)_{L^2(\Gamma_{0,\mathbb{A}})}), \tag{243}$$

where the integral operator $\mathbf{M}(q)$ has the kernel $M(\xi, \eta; q)$ defined by (233), such that

$$f(\xi) = \left(\prod_{m=1}^{N} \frac{a_m}{a_m - \xi} \right) \left(\prod_{i \geq 1} (1 - \alpha_i \xi) \frac{1 + q\beta_i \xi}{1 + \beta_i \xi} \right) \exp((q-1)\gamma\xi), \tag{244}$$

and $\Gamma_{0,\mathbb{A}}$ (denoted by $\tilde{C}_{a,\rho}$ in [6]) is a star-shaped contour centered at 0 and containing $\mathbb{A} = \{a_1, \dots, a_N\}$ but no other singularities of $f(\xi)$. Then [6, Corollary 3.24] gives the probability distribution

$$\mathbb{P}_{\mathbf{MM}_{t=0}(a_1,\dots,a_N;\rho)}(\lambda_N = n) = \frac{q^n}{2\pi i} \oint_C \frac{\det(1 + z\mathbf{M}(q)_{L^2(\Gamma_{0,\mathbb{A}})})}{(-z;q)_{n+1}} dz, \tag{245}$$

where the contour C encloses poles $-1, -q^{-1}, \dots, -q^n$. It is obvious that this $\Gamma_{0,\mathbb{A}}$ satisfies (232). For the meaning of the notations and technical conditions, see the paper [6].

Suppose there also exists a contour $\Gamma_{\mathbb{A}}$ that encloses $\mathbb{A}$ but not 0 and satisfies condition (232). Lemma 7 immediately implies that

$$\left\langle \frac{1}{(-zq^{\lambda_N};q)_\infty} \right\rangle_{\mathbf{MM}_{t=0}(a_1,\dots,a_N;\rho)} = \det(I + \mathbf{K}(z;q)_{L^2(\Gamma_{\mathbb{A}})}), \tag{246}$$

$$\mathbb{P}_{\mathbf{MM}_{t=0}(a_1,\dots,a_N;\rho)}(\lambda_N = n) = \frac{q^n}{2\pi i} \oint_C (zq^{n+1};q)_\infty \det(I + \mathbf{K}(z;q)_{L^2(\Gamma_{\mathbb{A}})}), \tag{247}$$

where and the integral operator $\mathbf{K}(z;q)$ has the kernel $K(\xi, \eta; z; q)$ defined in (234) with f specified in (244).

We note that [6, Corollary 3.17] expresses the moment generating formula on the left-hand side of (246) by a Fredholm determinant formula, where the domain consists of infinitely many copies of $\Gamma_{\mathbb{A}}$ (denoted by $C_{a,\rho}$ there), and the integral kernel is considerably more complicated than our $K(\xi, \eta; z; q)$. Moreover, if all the parameters α_i and β_i are 0 and $\gamma > 0$, then in [6, Theorem 3.18] a simplified Fredholm determinant formula is provided for the moment generating function, such that the domain is $\Gamma_{\mathbb{A}}$, while the integral kernel is still no less, if not more, complicated than our $K(\xi, \eta; z; q)$.

q-TASEP The q-deformed Totally Asymmetric Simple Exclusion Process (q-TASEP) is a well-studied model in the KPZ universality class [7], [13], [5], [18]. It is also a continuous limit of the q-Whittaker processes [6]. We refer to [7] for the definition of q-TASEP and for the meaning of the notations, and only remark that the speeds of the particles $x_1, \dots, x_n$ depend on parameters $a_1, \dots, a_n$.

In [7, Theorem 3.13], with the so-called step initial condition, a moment generating function for the position of the n-th particle at time t is provided as

$$\mathbb{E}\left[\frac{1}{(-zq^{x_n(t)+n}; q)_\infty}\right] = \frac{1}{(-z; q)_\infty} \det(I + z\mathbf{M}(q)_{L^2(\Gamma_{0,\mathbb{A}})}), \tag{248}$$

where the integral operator $\mathbf{M}(q)$ has the kernel $M(\xi, \eta; q)$ given by (233) with

$$f(\xi) = \left(\prod_{m=1}^{n} \frac{a_m}{a_m - \xi}\right) \exp((q-1)t\xi), \tag{249}$$

and the contour $\Gamma_{0,\mathbb{A}}$ (denoted by $\tilde{C}_a$ in [7]) is a star-shaped contour centered at 0 and enclosing $\mathbb{A} = \{a_1, \dots, a_n\}$. Hence suppose there exists a contour $\Gamma_{\mathbb{A}}$ that encloses $\mathbb{A}$ but not 0, then by Lemma 7, we have the alternative moment generating function

$$\mathbb{E}\left[\frac{1}{(-zq^{x_n(t)+n}; q)_\infty}\right] = \det(I + \mathbf{K}(z; q)_{L^2(\Gamma_{\mathbb{A}})}), \tag{250}$$

where $\mathbf{K}(z; q)$ is the integral operator with kernel (234)and $\mathbb{A}$ is a contour containing $\mathbb{A}$ but not 0. We note that a similar Fredholm determinant expression of the moment generating function is given in [7, Theorem 3.12], which is also on the domain $\Gamma_{\mathbb{A}}$, but the kernel is different, and not less complicated than our $K(\xi, \eta; z; q)$.

q-TAZRP The q-deformed Totally Asymmetric Zero Range process (q-TAZRP) is a dual process to the q-TASEP, see [21], [35] and [24] for a detailed definition of the model and the duality. The q-TAZRP was originally defined in [31] with the name q-boson process.

Let the (inhomogeneous) q-TAZRP be defined as in [24], with the conductance of the sites given by b_k ($k \in \mathbb{Z}$), and assume that the particle number is N and the initial condition is the so-called step initial condition that $x_1(0) = \cdots = x_N(0) = 0$. Then the distribution of the leftmost particle x_N at time $t > 0$ is by [24, Proposition 2.1, Formulas (117) and (118)]

$$\begin{aligned}
&\mathbb{P}_{0^N}(x_N(t) > M) \\
&= \frac{1}{(2\pi i)^N} \int_{\Gamma_{0,\mathbb{A}}} \frac{dw_1}{dw_1} \cdots \int_{\Gamma_{0,\mathbb{A}}} \frac{dw_N}{dw_N} \prod_{1 \le i < j \le N} \frac{w_i - w_j}{q w_i - w_j} \prod_{j=1}^{N} \left[\prod_{k=0}^{M} \left(\frac{b_k}{b_k - w_j} \right) e^{-w_j t} \right] \\
&= \frac{[N]_q!}{N!} \frac{(q-1)^N q^{-N(N-1)/2}}{(2\pi i)^N} \int_{\Gamma_{0,\mathbb{A}}} \frac{dw_1}{dw_1} \cdots \int_{\Gamma_{0,\mathbb{A}}} \frac{dw_N}{dw_N} \det \left(\frac{1}{q w_k - w_j} \right)_{j,k=1}^{N} \\
&\qquad \times \prod_{j=1}^{N} \left[\prod_{k=0}^{M} \left(\frac{b_k}{b_k - w_j} \right) e^{-w_j t} \right] \\
&= [N]_q! \frac{(q-1)^N}{q^{N(N-1)/2}} \frac{1}{2\pi i} \oint_0 \det(1 + z\mathbf{M}(q)_{L^2(\Gamma_{0,\mathbb{A}})}) \frac{dz}{z^{N+1}},
\end{aligned} \tag{251}$$

where $\mathbf{M}(q)$ is the integral operator on $L^2(\Gamma_{0,\mathbb{A}})$ with kernel $M(\xi, \eta; q)$ given in (233) with the function f specified as

$$f(\xi) = \prod_{k=0}^{M} \left(\frac{b_k}{b_k - \xi} \right) e^{-\xi t}, \tag{252}$$

and the contour $\Gamma_{0,\mathbb{A}}$ is the same as the contour C in [24, Proposition 2.1] that is a large enough circle. Then applying Lemma 7, we have that

$$\mathbb{P}_{0^N}(x_N(t) > M) = [N]_q! \frac{(q-1)^N}{q^{N(N-1)/2}} \frac{1}{2\pi i} \oint_0 \frac{(-z;q)_\infty}{z^{N+1}} \det(1 + \mathbf{K}(z;q)_{L^2(\Gamma_{\mathbb{A}})}) dz, \tag{253}$$

where $\mathbf{K}(z;q)$ is the integral operator on $L^2(\Gamma_{\mathbb{A}})$ with kernel $K(\xi, \eta; z; q)$ given in (234) with the function f specified as in (252), and the contour $\Gamma_{\mathbb{A}}$ encloses $b_1, \dots, b_N$ counterclockwise and satisfies (232), given that such $\Gamma_{\mathbb{A}}$ exists.

6 Multi-time correlation functions and gap probabilities

In this section, we prove Theorem 4. Our derivation of the multi-time correlation functions is based on [23, Formulas (50), (51) and (52)], while our derivation of the multi-time gap probabilities is based on [23, Formulas (60) and (61)].

Since the model of free fermions at finite temperature and that of time-periodic nonintersecting OU processes are equivalent, we prove Theorem 4 for free fermions at finite temperature.

6.1 Multi-time correlation functions

Let $\tau_1, \tau_2, \dots, \tau_m \in [0, \beta)$ be imaginary times in Proposition 2. Since the free fermions at finite temperature are distributed over eigenstates with respect to the Boltzmann distribution as in [23, Formula (78)], The multi-time m-correlation function at x_i and time τ_i with $i = 1, \dots, m$ is expressed as

$$R^{(m)}(x_1, \dots, x_m; \tau_1, \dots, \tau_m) = \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots < k_n} q^{k_1 + \cdots + k_n} R^{(m)}_{k_1, \dots, k_n}(x_1, \dots, x_m; \tau_1, \dots, \tau_m), \tag{254}$$

where $R^{(m)}_{k_1, \dots, k_n}(x_1, \dots, x_m; \tau_1, \dots, \tau_m)$ is the multi-time m-correlation function for the n-particle model with eigenstate $(k_1, \dots, k_n)$. By [23, Formulas (50)–(52)], We have

$$R^{(m)}_{k_1, \dots, k_n}(x_1, \dots, x_m; \tau_1, \dots, \tau_m) = \det \left(K_{k_1, \dots, k_n}(x_i, x_j; \tau_i, \tau_j) \right)_{i,j=1}^{m}, \tag{255}$$

where

$$K_{k_1,\ldots,k_n}(x,y;\tau,\sigma) = \tilde{K}_{k_1,\ldots,k_n}(x,y;\tau,\sigma) + E(x,y;\tau,\sigma), \tag{256}$$

such that $E(x,y;\tau,\sigma)$ is defined in (43), and

$$\tilde{K}_{k_1,\ldots,k_n}(x,y;\tau,\sigma) = \sum_{i=1}^{n} \varphi_{k_i}(x)\varphi_{k_i}(y)e^{k_i(\tau-\sigma)}. \tag{257}$$

Note that if $\tau = \sigma$, $K_{k_1,\ldots,k_n}(x,y;\tau,\sigma)$ degenerates into $K_{k_1,\ldots,k_n}(x,y)$ in (80), and if $\tau_1,\ldots,\tau_m$ are identical, then $R^{(m)}_{k_1,\ldots,k_n}(x_1,\ldots,x_m;\tau_1,\ldots,\tau_m)$ degenerates into $R^{(m)}_{k_1,\ldots,k_n}(x_1,\ldots,x_m)$ in (80).

Then analogous to (81), it is straightforward to write

$$\begin{aligned} R^{(m)}_{k_1,\ldots,k_n}(x_1,\ldots,x_m;\tau_1,\ldots,\tau_m) &= \sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} \det\left(\hat{K}_{i_l}(x_j,x_l;\tau_j,\tau_l;n)\right)_{j,l=1}^{m} \\ &= \sum_{\substack{j_1<j_2<\cdots<j_m \\ \{j_1,\ldots,j_m\}\subseteq\{k_1,\ldots,k_n\}}} \hat{R}^{(m)}_{j_1,\ldots,j_m}(x_1,\ldots,x_m;\tau_1,\ldots,\tau_m;n), \end{aligned} \tag{258}$$

where

$$\hat{K}_k(x_j,x_l;\tau_j,\tau_l;n) = \varphi_k(x_j)\varphi_k(x_l)e^{k(\tau_j-\tau_l)} - \frac{1}{n}E(x_j,x_l;\tau_j,\tau_l) \tag{259}$$

is a generalization of $\varphi_k(x_j)\varphi_k(x_l)$ occurring in the entries of the determinant in (81), and

$$\begin{aligned} \hat{R}^{(m)}_{j_1,\ldots,j_m}(x_1,\ldots,x_m;\tau_1,\ldots,\tau_m;n) &= \det\left(\sum_{i=1}^{m}\hat{K}_{j_i}(x_k,x_l;\tau_k,\tau_l;n)\right)_{k,l=1}^{m} \\ &= \sum_{\kappa,\lambda\in S_n} \operatorname{sgn}(\lambda)\prod_{i=1}^{m}\hat{K}_{j_{\kappa(i)}}(x_i,x_{\lambda(i)};\tau_i,\tau_{\lambda(i)};n) \end{aligned} \tag{260}$$

is a generalization of $\hat{R}^{(m)}_{k_1,\ldots,k_n}(x_1,\ldots,x_m)$ in (82). Hence analogous to (83), we have

$$R^{(m)}(x_1,\ldots,x_m;\tau_1,\ldots,\tau_m) = \frac{q^{n/2}}{Z_n(q)}\sum_{0\le j_1<\cdots<j_m} C_{j_1,\ldots,j_m}\hat{R}^{(m)}_{j_1,\ldots,j_m}(x_1,\ldots,x_m;\tau_1,\ldots,\tau_m), \tag{261}$$

where $C_{j_1,\ldots,j_m}$ is defined in (84). On the other hand, we have

$$\begin{aligned} &\det\left(K_n(x_i,x_j;\tau_i,\tau_j;z;q)\right)_{i,j=1}^{m} \\ &\quad= \sum_{0\le j_1<\cdots<j_m} q^{j_1+\cdots+j_m}z^m\left[\prod_{i=1}^{m}(1+q^{j_i}z)^{-1}\right]\hat{R}^{(m)}_{j_1,\ldots,j_m}(x_1,\ldots,x_m;\tau_1,\ldots,\tau_m;n). \end{aligned} \tag{262}$$

where $K_n(x,y;\tau,\sigma;z;q)$ is defined in (44). Hence analogous to (21), we prove (47) by comparing (258), (85) and (262).

6.2 Multi-time gap probability

Next we assume the times $\tau_1,\ldots,\tau_m$ are distinct, and compute the gap probability for free fermions at finite temperature such that at times $\tau_1,\ldots,\tau_m$, all particles are in the measurable sets $A_1, \ldots, A_m$ respectively. We denote this probability by $\mathbb{P}(A_1, \ldots, A_m; \tau_1, \ldots, \tau_m)$. According to the Boltzmann distribution of eigenstates, we have that [23, Formula (78)]

$$\mathbb{P}(A_1, \ldots, A_m; \tau_1, \ldots, \tau_m) = \frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots k_n} q^{k_1 + \cdots + k_n} \mathbb{P}_{k_1, \ldots, k_n}(A_1, \ldots, A_m; \tau_1, \ldots, \tau_m), \tag{263}$$

where $\mathbb{P}_{k_1,\ldots,k_n}(A_1, \ldots, A_m; \tau_1, \ldots, \tau_m)$ is the gap probability that all particles ate in $A_1, \ldots, A_m$ at times $\tau_1, \ldots, \tau_m$ respectively for the determinantal point process characterized by the multi-time correlation kernel $K_{k_1,\ldots,k_n}(x, y; \tau, \sigma)$ in (256). By [23, Formulas (60) and (61)], we have

$$\mathbb{P}_{k_1,\ldots,k_n}(A_1, \ldots, A_m; \tau_1, \ldots, \tau_m) = \det(I - \mathbf{K}_{k_1,\ldots,k_n}(\tau_1, \ldots, \tau_m)\chi_{A_1^c,\ldots,A_m^c}), \tag{264}$$

where $\mathbf{K}_{k_1,\ldots,k_n}(\tau_1, \ldots, \tau_m)$ is, analogous to $\mathbf{K}(\tau_1, \ldots, \tau_m; z; q)$ in (45), an integral operator on $L^2(\mathbb{R}\times\{1, 2, \ldots, m\})$ whose kernel is represented by an $m\times m$ matrix $(K_{k_1,\ldots,k_n}(x_i, x_j; \tau_i, \tau_j))_{i,j=1}^m$, and

$$(\mathbf{K}_{k_1,\ldots,k_n}(\tau_1, \ldots, \tau_m)f)(x; k) = \sum_{j=1}^m \int_{\mathbb{R}} K_{k_1,\ldots,k_n}(x, y; \tau_k, \tau_j) f(y; j) dy. \tag{265}$$

By [32, Formula (3.5)], we have the expansion

$$\det(I - \mathbf{K}_{k_1,\ldots,k_n}(\tau_1, \ldots, \tau_m)\chi_{A_1^c,\ldots,A_m^c}) = 1 + \sum_{l=1}^\infty \frac{(-1)^l}{l!} \operatorname{Tr}\left[\Lambda^l \left(\mathbf{K}_{k_1,\ldots,k_n}(\tau_1, \ldots, \tau_m)\chi_{A_1^c,\ldots,A_m^c}\right)\right], \tag{266}$$

where the trace of the l-th exterior power of $\mathbf{K}_{k_1,\ldots,k_n}(\tau_1, \ldots, \tau_m)\chi_{A_1^c,\ldots,A_m^c}$ is computed as

$$\begin{aligned} &\operatorname{Tr}\left[\Lambda^l \left(\mathbf{K}_{k_1,\ldots,k_n}(\tau_1, \ldots, \tau_m)\chi_{A_1^c,\ldots,A_m^c}\right)\right] \\ &= \sum_{s_1=1}^m \sum_{s_2=1}^m \cdots \sum_{s_l=1}^m \int_{A_{s_1}^c} dx_1 \int_{A_{s_2}^c} dx_2 \cdots \int_{A_{s_l}^c} dx_l \det(K_{k_1,\ldots,k_n}(x_i, x_j; \tau_{s_i}, \tau_{s_j}))_{i,j=1}^l \\ &= \sum_{s_1=1}^m \sum_{s_2=1}^m \cdots \sum_{s_l=1}^m \int_{A_{s_1}^c} dx_1 \int_{A_{s_2}^c} dx_2 \cdots \int_{A_{s_l}^c} dx_l R^{(m)}_{k_1,\ldots,k_n}(x_1, \ldots, x_l; \tau_{s_1}, \ldots, \tau_{s_l}). \end{aligned} \tag{267}$$

The proof of (267) is analogous to that of [32, Theorem 3.10].

By (263), (264), (266) and (267), we have

$$\begin{aligned} \mathbb{P}(A_1, \ldots, A_m; \tau_1, \ldots, \tau_m) = \frac{q^{n/2}}{Z_n(q)} &\sum_{0 \le k_1 < k_2 < \cdots k_n} q^{k_1 + \cdots + k_n} \cdot 1 \\ &+ \sum_{l=1}^\infty \frac{(-1)^l}{l!} \sum_{s_1=1}^m \cdots \sum_{s_l=1}^m \int_{A_{s_1}^c} dx_1 \cdots \int_{A_{s_l}^c} dx_l \\ &\frac{q^{n/2}}{Z_n(q)} \sum_{0 \le k_1 < k_2 < \cdots k_n} q^{k_1 + \cdots + k_n} R^{(m)}_{k_1,\ldots,k_n}(x_1, \ldots, x_l; \tau_{s_1}, \ldots, \tau_{s_l}). \end{aligned} \tag{268}$$

By (11), (254) and (47), we have

$$
\begin{aligned}
&\mathbb{P}(A_1,\dots,A_m;\tau_1,\dots,\tau_m)\\
&=1+\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\sum_{s_1=1}^{m}\cdots\sum_{s_l=1}^{m}\int_{A_{s_1}^c}dx_1\cdots\int_{A_{s_l}^c}dx_l R^{(m)}(x_1,\dots,x_m;\tau_1,\dots,\tau_m)\\
&=1+\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\sum_{s_1=1}^{m}\cdots\sum_{s_l=1}^{m}\int_{A_{s_1}^c}dx_1\cdots\int_{A_{s_l}^c}dx_l\frac{1}{2\pi i}\oint_0 F(z)\det(K(x_i,x_j;\tau_{s_i},\tau_{s_j};z;q))_{i,j=1}^l\frac{dz}{z}\\
&=\frac{1}{2\pi i}\oint_0 F(z)\left[1+\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\sum_{s_1=1}^{m}\cdots\sum_{s_l=1}^{m}\int_{A_{s_1}^c}dx_1\cdots\int_{A_{s_l}^c}dx_l\det(K(x_i,x_j;\tau_{s_i},\tau_{s_j};z;q))_{i,j=1}^l\right]\frac{dz}{z}\\
&=\frac{1}{2\pi i}\oint_0 F(z)\det(I-\mathbf{K}(\tau_1,\dots,\tau_m;z;q)\chi_{A_1^c,\dots,A_m^c})\frac{dz}{z},
\end{aligned}
\tag{269}
$$

and prove Theorem 4(b).

A Equivalence to MNS random matrix model

With the help of (51) and (52), the density function $P_n(x_1,\dots,x_n)$ in (12) is expressed as

$$
\begin{aligned}
P_n(x_1,\dots,x_n)&=\frac{q^{n/2}}{n!Z_n(q)}\sum_{k_1,\dots,k_n=0}^{\infty}\begin{vmatrix}\varphi_{k_1}(x_1)&\dots&\varphi_{k_1}(x_n)\\ \vdots&&\vdots\\ \varphi_{k_n}(x_1)&\dots&\varphi_{k_n}(x_n)\end{vmatrix}^2 q^{k_1+\dots+k_n}\\
&=\frac{q^{n/2}}{n!Z_n(q)}\sum_{k_1,\dots,k_n=0}^{\infty}q^{k_1+\dots+k_n}\begin{vmatrix}H_{k_1}(x_1)&\dots&H_{k_1}(x_n)\\ \vdots&&\vdots\\ H_{k_n}(x_1)&\dots&H_{k_n}(x_n)\end{vmatrix}\\
&\quad\times\begin{vmatrix}\frac{1}{\sqrt{2\pi}k_1!}H_{k_1}(x_1)e^{-x_1^2/2}&\dots&\frac{1}{\sqrt{2\pi}k_1!}H_{k_1}(x_n)e^{-x_n^2/2}\\ \vdots&&\vdots\\ \frac{1}{\sqrt{2\pi}k_n!}H_{k_n}(x_1)e^{-x_1^2/2}&\dots&\frac{1}{\sqrt{2\pi}k_n!}H_{k_n}(x_n)e^{-x_n^2/2}\end{vmatrix}\\
&=\frac{q^{n/2}}{n!Z_n(q)}\sum_{k_1,\dots,k_n=0}^{\infty}\frac{1}{(\sqrt{2\pi}i)^n}\int_{-i\infty}^{i\infty}ds_1\cdots\int_{-i\infty}^{i\infty}ds_n\prod_{j=1}^{n}e^{\frac12(s_j-x_j)^2}\det\left((qs)_j^{k_l}\right)_{j,l=1}^n\\
&\quad\times\frac{1}{((2\pi)^{3/2}i)^n}\oint_\Gamma\frac{dt_1}{t_1}\cdots\oint_\Gamma\frac{dt_n}{t_n}\prod_{j=1}^{n}e^{-\frac12(t_j-x_j)^2}\det\left(t_j^{-k_l}\right)_{j,l=1}^n\\
&=\frac{q^{n/2}}{Z_n(q)}\frac{1}{(2\pi i)^{2n}}\int_{-i\infty}^{i\infty}ds_1\cdots\int_{-i\infty}^{i\infty}ds_n\oint_\Gamma\frac{dt_1}{t_1}\cdots\oint_\Gamma\frac{dt_n}{t_n}\prod_{j=1}^{n}\frac{e^{\frac12(s_j-x_j)^2}}{e^{\frac12(t_j-x_j)^2}}\\
&\quad\times\sum_{k_1,\dots,k_n=0}^{\infty}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\left(\frac{qs_{\sigma(j)}}{t_j}\right)^{k_j},
\end{aligned}
\tag{270}
$$

where in the first step we symmetrize the indices $k_1, \ldots, k_n$, and in the last step we use the symmetry among $k_1, \ldots, k_n$. Under the assumption that $|t_j| > q|s_k|$ for all j, k, We have

$$\begin{aligned}\sum_{k_1,\ldots,k_n=0}^{\infty} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\frac{q s_{\sigma(j)}}{t_j} \right)^{k_j} &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \frac{1}{1 - q s_{\sigma(j)}/t_j} \\ &= \prod_{j=1}^{n} t_j \det \left(\frac{1}{t_j - q s_l} \right)_{j,l=1}^{n}.\end{aligned} \tag{271}$$

Hence we deform the contour Γ in (270) into $\Gamma_s = \{z \in \mathbb{C} \mid |z| = \max(|s_k|)$ that depends on $s_1, \ldots, s_n$, and plug (271) into (270). Using the residue theorem, we have

$$\begin{aligned}&P_n(x_1, \ldots, x_n) \\ =& \frac{q^{n/2}}{Z_n(q)} \frac{1}{(2\pi i)^{2n}} \int_{-i\infty}^{i\infty} ds_1 \cdots \int_{-i\infty}^{i\infty} ds_n \oint_{\Gamma_s} dt_1 \cdots \oint_{\Gamma_s} dt_n \prod_{j=1}^{n} \frac{e^{\frac{1}{2}(s_j - x_j)^2}}{e^{\frac{1}{2}(t_j - x_j)^2}} \det \left(\frac{1}{t_j - q s_l} \right)_{j,l=1}^{n} \\ =& \frac{q^{n/2}}{Z_n(q)} \frac{1}{(2\pi i)^{n}} \int_{-i\infty}^{i\infty} ds_1 \cdots \int_{-i\infty}^{i\infty} ds_n \prod_{j=1}^{n} e^{\frac{1}{2}(s_j - x_j)^2} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} e^{-\frac{1}{2}(q s_j - x_{\sigma(j)})^2} \\ =& \frac{q^{n/2}}{Z_n(q)} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{1}{(2\pi i)^{n}} \int_{-i\infty}^{i\infty} ds_1 \cdots \int_{-i\infty}^{i\infty} ds_n \prod_{j=1}^{n} \exp \left(\frac{1}{2} [(s_j - x_j)^2 - (q s_j - x_{\sigma(j)})^2] \right) \\ =& \frac{q^{n/2}}{Z_n(q)} (2\pi(1 - q^2))^{-n/2} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e^{-\frac{1}{2(1-q^2)}(x_{\sigma(j)} - q x_j)^2} \\ =& \frac{q^{n/2}}{Z_n(q)} (2\pi(1 - q^2))^{-n/2} \prod_{j=1}^{n} e^{-\frac{1}{2}\frac{1+q^2}{1-q^2} x_j^2} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e^{\frac{q}{1-q^2} x_{\sigma(j)} x_j} \\ =& \frac{q^{n/2}}{Z_n(q)} (2\pi(1 - q^2))^{-n/2} \prod_{j=1}^{n} e^{-\frac{1}{2}\frac{1+q^2}{1-q^2} x_j^2} \det \left(e^{\frac{q}{1-q^2} x_j x_k} \right)_{j,k=1}^{n}.\end{aligned} \tag{272}$$

Comparing the right-hand side of (272) with [26, Formula (3)], we prove Proposition 1.

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