LYAPUNOV EXPONENT, UNIVERSALITY AND PHASE TRANSITION FOR PRODUCTS OF RANDOM MATRICES

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ABSTRACT. We are devoted to solving one problem, which is proposed by Akemann, Burda, Kieburg [2] and Deift [19], on local statistics of finite Lyapunov exponents for M products of $N \times N$ Gaussian random matrices as both M and N go to infinity. When the ratio $(M+1)/N$ changes from 0 to ∞, we prove that the local statistics undergoes a transition from GUE to Gaussian. Especially at the critical scaling $(M+1)/N \to \gamma \in (0, \infty)$, we observe a phase transition phenomenon.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

1.1. **Lyapunov exponents.** In his famous 1892 monograph [39], Lyapunov introduced the concept of Lyapunov exponent, which originated from the problem of the stability of solutions of differential equations. For a linearized differential equation

$$\dot{v}(t) = X_t v, v(0) = v_0 \in \mathbb{R}^N, \tag{1.1}$$

where $X_{(\cdot)}$ is a continuous and bounded function from $\mathbb{R}_+$ to the space of $N \times N$ real matrices, the Lyapunov exponent of a solution $v(t; v_0)$ of (1.1) is defined in the following manner

$$\lambda(v_0) := \limsup_{t\to\infty} \frac{1}{t} \log \|v(t)\|. \tag{1.2}$$

Moreover, Lyapunov proved that $\lambda(v_0)$ is finite for every solution with $v_0 \neq 0$. Later, through the works of Furstenberg, Kesten, Oseledets, Kingman, Ruelle, Margulis, Avila and other mathematicians, Lyapunov exponents have recently emerged as an important concept in various fields of Mathematics and Physics, such as linear stochastic systems and stability theory, products of random matrices and random maps, spectral theory of random Schrödinger operators, smooth dynamics and translate surfaces; see e.g. [10, 55, 56].

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Let's turn to consider a discrete-time evolution of an N-dimensional real or complex stochastic system which is described by linear difference equations

$$v(t+1) = X_{t+1}v(t), \quad t = 0, 1, 2, \ldots, \tag{1.3}$$

then the total evolution is effectively driven by the product of random matrices at time $t = M$

$$\Pi_M = X_M \cdots X_2 X_1. \tag{1.4}$$

The study on products of random matrices can be dated at least from the seminal articles by Bellman[13] in 1954 and further by Furstenberg and Kesten [28] in 1960, in which classical limit theorems in probability theory were obtained under certain assumptions when M goes to infinity. In particular, if $X_1, X_2, \cdots, X_M$ are i.i.d. $N \times N$ random matrices, each of which has independent (and identically distributed) entries with mean zero and variance one, then the seminal theorem of Furstenberg and Kesten [28, Theorem 2] shows that for any fixed N the largest Lyapunov exponent defined as

$$\lambda_{\max} := \lim_{M\to\infty} \frac{1}{M} \log \|\Pi_M\| \tag{1.5}$$

exists with probability 1. Furthermore, all Lyapunov exponents exist by celebrated multiplicative ergodic theorem of Oseledets [46, 49], say,

$$\lambda_k := \lim_{M\to\infty} \frac{1}{2M} \log \left(k^{\text{th}} \, \text{largest eigenvalue of} \, \Pi_M^* \Pi_M \right), \quad k = 1, 2, \ldots, N. \tag{1.6}$$

Here it is worth stressing that the M-dependent eigenvalues on the right-hand side are usually referred to finite Lyapunov exponents, which are equivalent to singular values of Π_M up to a one-to-one mapping.

However, usually it's very hard to find both explicit formulae and effective algorithms of accurate approximation for the Lyapunov exponents. This was been posed by Kingman [35] as an outstanding problem in the field. Some noteworthy exceptions occur in the case of $N = 2$, see e.g. [16, 40, 41]. For general N, when each X_j is randomly chosen from a finite set of matrices with positive entries, in a recent work [47] Pollicott solves this problem for the largest Lyapunov exponent. Another special case is when $\{X_j\}$ are independent real/complex Ginibre matrices (that is, with i.i.d. standard real/complex Gaussian entries), which have high interest in Random Matrix Theory. Then the classical results of Newman [44] (real case, $\beta = 1$) and Forrester [24, 25] (real and complex cases with $\beta = 1, 2$) show that the Lyapunov spectrum

$$\lambda_k = \frac{1}{2} \left(\log \frac{2}{\beta} + \psi\big(\frac{\beta}{2}(N - k + 1)\big) \right), \quad k = 1, \ldots, N, \tag{1.7}$$

where $\psi(x)$ denotes the digamma function. Forrester [24, 25] also studied Gaussian random matrices with correlated entries; for more relevant works, see e.g. [2, 29, 33, 50] and references therein.

Up to now, the fundamental result about asymptotic behavior for products of random matrices discovered by Furstenberg and Kesten [28] have led to much of great interest in the topic over the last sixty years, see [15, 10] for the early articles. Recently, significant progresses have been achieved in the study of products of random matrices, which have important applications in Schrödinger operator theory [14], in statistical physics relating to disordered and chaotic dynamical systems [17], in wireless communication like MIMO (multiple-input and multiple-output) networks [54] and in free probability theory [43].

1.2. **Universality.** Historically, the pioneering work of Furstenberg and Kesten [28] and lots of subsequent works focused on statistical behavior of singular values for the products such as Lyapunov exponents, as the number of factors M tends to infinity. However, the more recent interest in products of random matrices lies in statistical properties of eigenvalues and singular values as the matrix size N goes to infinity, like a single random matrix. The study of one single random matrix originated from the seminal works from Wigner, Dyson, Mehta and others in 1950-60s, and has become a quite active research field named after Random Matrix

Theory (RMT), which relates to many important branches of Mathematics and Physics; see a handbook [1], monographs [9, 11, 18, 20, 22, 23, 42, 43, 48, 51, 54] and references therein.

Local statistical properties of eigenvalues in RMT are usually described by certain patterns, say, Sine, Bessel and Airy kernels. The secret hidden behind can be unveiled through the repulsion of eigenvalues, which is expected for many randomly disordered systems of the same symmetry class that have delocalized eigenfunctions. This is referred to as universality in RMT, which is different from classical Gaussian universality. Most of random matrix ensembles, like Wigner matrices and invariant ensembles, have been rigorously proved to exhibit universal phenomena, see e.g. [18, 20, 22] and references therein. As to finite products of large random matrices, statistical properties have been extensively studied in [6, 7, 8, 26, 30, 34, 36, 37]; see a recent survey [5] and references therein.

As a typical product like (1.4) with $X_1, \ldots, X_M$ being i.i.d. complex Ginibre matrices of size $N \times N$, the squared singular values $x_1, \ldots, x_N$ of Π_M, that is, eigenvalues of $\Pi_M^* \Pi_M$, are proved to form a determinantal point process with correlation kernel

$$K_N(x,y) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \oint_{\Sigma} \frac{dt}{2\pi i} \frac{x^t y^{-s-1}}{s-t} \frac{\Gamma(t)}{\Gamma(s)} \left(\frac{\Gamma(s+N)}{\Gamma(t+N)} \right)^{M+1}, \tag{1.8}$$

where $c > 0$ and Σ is a counter-clockwise contour encircling $0, -1, \ldots, -N+1$; see [7] for the derivation of the joint eigenvalue density and [37] for the correlation kernel. Note that (1.8) can be derived from the integral representation in [37] by deforming contours such that the s-contour lies on the RHS of the t-contour and then shifting variables by N. With the help of this structure, for any fixed M and as $N \to \infty$, with L. Zhang the two of the present authors proved the Sine and Airy kernels for singular values of Π_M in [38]. Reversing this order, for any fixed N and as $M \to \infty$, Akemann, Burda and Kieburg proved in [2] that N finite Lyapunov exponents for Π_M are asymptotically independent Gaussian random variables.

So a very natural question arises: What will happen when both the matrix size and the number of factors tend to infinity? Precisely, will the largest Lyapunov exponent undergo a crossover from Gausssian to Tracy-Widom distribution [53] at some proper scaling of M and N? Actually, at the end of [2, Sect.5] Akemann, Burda and Kieburg commented "*Since the two limits commute on the global scale while they do not commute on the local one, we claim that there should be a non-trivial double-scaling limit where new results should show up. In particular we expect a mesoscopic scale of the spectrum which may also show a new kind of universal statistics*". Also, in his 2017 list of open problems in random matrix theory and the theory of integrable systems, P. Deift ended in [19] with "*There are many other areas, closely related to the problems in the above list, where much progress has been made in recent years, and where much remains to be done. These include: ..., singular values of n products of $m \times m$ random matrices as $n, m \to \infty$, and many others*". It is our main goal in the present paper to solve this problem when Ginibre random matrices are involved.

1.3. **Main results.** When both M and N go to infinity, we investigate local statistical properties of singular values of the product. As to the global property, it was argued in [3] that the limiting eigenvalue density is a constant up to some proper scaling transform. Since the largest singular value (or finite Lyapunov exponent) is more important from aspect of RMT and dynamical systems, we place emphasis on local property of the largest one. For this, we need to divide three different regimes of the number of product factors and matrix sizes:

- I weakly correlated regime as $(M+1)/N \to \infty$;
- II intermediate regime as $(M+1)/N \to \gamma \in (0, \infty)$; and
- III strongly correlated regime of $(M+1)/N \to 0$.

We also assume that $M := M_N$ may depend on N, and $N \to \infty$. Besides, for notational simplicity, instead of K_N we consider the following transformed kernel

$$\widetilde{K}_N(x,y) = e^y K_N(e^x, e^y), \quad x, y \in \mathbb{R}, \tag{1.9}$$

which is more relevant to finite Lyapunov exponents.

Recall that for a determinantal point process with correlation kernel $\widetilde{K}_N(x,y)$, n-point correlation functions are given by

$$R_N^{(n)}(x_1,\dots,x_n) = \det[\widetilde{K}_N(x_i,x_j)]_{i,j=1}^n, \tag{1.10}$$

and the gap probability can also be expressed in terms of $\widetilde{K}_N(x,y)$; see e.g. [9, 23, 42]. So we just state our main results about correlation kernel.

Theorem 1.1 (Normality in case I)**.** *Suppose that* $\lim_{N\to\infty}(M+1)/N = \infty$*, and* $k \in \mathbb{N}$ *is fixed. Let* $x_N(k) = N(\psi(1-k+N) - \log N)$ *and*

$$g(k;\xi) = N\log N + x_N(k) + \xi\sqrt{\frac{N}{M+1}}. \tag{1.11}$$

(1) If ξ,η *are in a compact subset of* $\mathbb{R}$*, then with the function* $h_{N,M+1,k}$ *defined in* (2.15)*,*

$$\lim_{N\to\infty}\sqrt{\frac{M+1}{N}}\frac{h_{N,M+1,k}(\eta)}{h_{N,M+1,k}(\xi)}\widetilde{K}_N\left(\frac{M+1}{N}g(k;\xi),\frac{M+1}{N}g(k;\eta)\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\eta^2}. \tag{1.12}$$

(2) Let $\epsilon > 0$*. Then there exists* $C_0(\epsilon) > 0$ *such that*

$$\int_{\frac{M+1}{N}g(1;C_0(\epsilon))}^{+\infty}\widetilde{K}_N(,x)dx < \epsilon. \tag{1.13}$$

(3) Let $\epsilon > 0$*. Then for each* $k\in\mathbb{N}$ *there exists* $C_k(\epsilon) > 0$ *such that*

$$\int_{\frac{M+1}{N}g(k+1;C_k(\epsilon))}^{\frac{M+1}{N}g(k;-C_k(\epsilon))}\widetilde{K}_N(x,x)dx < \epsilon. \tag{1.14}$$

Remark 1.1. Part 1 of Theorem 1.1 means that in the $N\to\infty$ limit, there is a single eigenvalue of the matrix $\log(\Pi_M^*\Pi_M)$ that is at $\frac{M+1}{N}g(k;0)$, whose fluctuation is normal and higher point correlation functions vanish, with scaling $\sqrt{(M+1)/N}$. This Gaussian behaviour has been conjectured in [31]. Part 2 implies that the eigenvalue at $\frac{M+1}{N}g(1;0)$ is the largest one almost surely, and then part 3 further implies that for any fixed k, the eigenvalue at $\frac{M+1}{N}g(k;0)$ is the k-th largest one almost surely.

Theorem 1.2 (Criticality in case II)**.** *Suppose that* $\lim_{N\to\infty}(M+1)/N = \gamma \in (0,\infty)$*. Let*

$$g(\xi) = (M+1)\left(\log N - \frac{1}{2N}\right) + \xi. \tag{1.15}$$

(1) If ξ,η *are in a compact subset of* $\mathbb{R}$*, then*

$$\lim_{N\to\infty}\widetilde{K}_N\big(g(\xi),g(\eta)\big) = K_{\mathrm{crit}}(\xi,\eta;\gamma), \tag{1.16}$$

where

$$K_{\mathrm{crit}}(\xi,\eta;\gamma) = \int_{1-i\infty}^{1+i\infty}\frac{ds}{2\pi i}\oint_{\Sigma_{-\infty}}\frac{dt}{2\pi i}\frac{1}{s-t}\frac{\Gamma(t)}{\Gamma(s)}\frac{e^{\frac{\gamma s^2}{2}-\eta s}}{e^{\frac{\gamma t^2}{2}-\xi t}}, \tag{1.17}$$

with $\Sigma_{-\infty}$ *being a contour starting from* $-\infty - i\epsilon$*, looping around* $\{0,-1,-2,\dots\}$ *positively, and then going to* $-\infty + i\epsilon$*.*

(2) Let $\epsilon > 0$*. Then there exists* $C(\epsilon) > 0$ *such that*

$$\int_{g(C(\epsilon))}^{\infty}\widetilde{K}_N(x,x)dx < \epsilon. \tag{1.18}$$

Remark 1.2. Part 1 of Theorem 1.2 can be stated alternatively as that, with t_0 being the unique positive solution of $\psi'(t_0) = \gamma$, after a translation by γt_0 we have

$$e^{(\eta-\xi)t_0}\widetilde{K}_{\mathrm{crit}}(\xi-\gamma t_0,\eta-\gamma t_0;\gamma) = \widehat{K}_{\mathrm{crit}}(\xi,\eta;\gamma), \tag{1.19}$$

where

$$\widehat{K}_{\mathrm{crit}}(\xi,\eta;\gamma)=\int_{1-i\infty}^{1+i\infty}\frac{ds}{2\pi i}\oint_{\hat{\Sigma}_{-\infty}}\frac{dt}{2\pi i}\frac{1}{s-t}\frac{\Gamma(t+t_0)}{\Gamma(s+t_0)}\frac{e^{\frac{\gamma s^2}{2}-\eta s}}{e^{\frac{\gamma t^2}{2}-\xi t}}. \tag{1.20}$$

with $\hat{\Sigma}_{\infty}$ starting from $-\infty-i\epsilon$, looping around $\{-t_0,-t_0-1,-t_0-2,\dots\}$ positively, and then going to $-\infty+i\epsilon$. With this choice, noting that the Taylor expansion of $\log\Gamma(t+t_0)-\frac{\gamma t^2}{2}+\xi t$ at zero has a vanishing quadratic term, so as $\gamma\to 0$ its behaviour is better than that of K_{crit}, see Theorem 3.2 below.

Remark 1.3. Part 2 of Theorem 1.2 implies that the largest eigenvalue of $\log(\Pi_M^*\Pi_M)$ is not too far to the right of 0.

Theorem 1.3 (GUE statistics in case III)**.** *Suppose that* $\lim_{N\to\infty}(M+1)/N=0$. *Let*

$$g(\xi)=M\log N+\log(M+1)+v(\theta)+\frac{\xi}{\rho_N} \tag{1.21}$$

where ρ_N will be determined and the parametrization representation

$$v(\theta)=\theta\cot\theta+\log\frac{\theta}{\sin\theta},\quad \theta\in[0,\pi), \tag{1.22}$$

the following hold true for ξ,η in a compact subset of $\mathbb{R}$.

(1) When $\theta\in(0,\pi)$, let

$$\rho_N=\frac{N}{M+1}\frac{\theta}{\pi}, \tag{1.23}$$

then

$$\lim_{N\to\infty}e^{-\pi(\xi-\eta)\cot\theta}\frac{1}{\rho_N}\widetilde{K}_N\big(g(\xi),g(\eta)\big)=\frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)}. \tag{1.24}$$

(2) When $\theta=0$, let

$$\rho_N=2^{\frac{1}{3}}\Big(\frac{N}{M+1}\Big)^{\frac{2}{3}}, \tag{1.25}$$

then

$$\lim_{N\to\infty}e^{-\frac{N}{M+1}\frac{\xi-\eta}{\rho_N}}\frac{1}{\rho_N}\widetilde{K}_N\big(g(\xi),g(\eta)\big)=K_{\mathrm{Ai}}(\xi,\eta), \tag{1.26}$$

where K_{Ai} denotes the standard Airy kernel.

Remark 1.4. For the largest eigenvalue of the deformed GUE ensemble, Johansson has found a different transition from Tracy-Widom to Gaussian; see [32]. Our phase transition result is also different from the famous BBP transition introduced in the spiked Wishart matrix by Baik, Ben Arous and Péché [12]. The soft edge limit in Theorem 1.2 was independently obtained by Akemann, Burda and Kieburg [3]. Though different in form, our integral representation (1.17) is believed to be in essence the same as that in [3, Eq(15)]. Besides, we are inspired by [3, Eq (10)] to study the bulk critical limit in Sect. 3.2.

The rest of this article is organised as follows. In the next Section 2 we prove the main theorems stated above. In Section 3 we discuss a few relevant questions.

2. Proof of main theorems

Since many occurrences of notation $M+1$ are in the proofs of Theorems 1.1-1.3, throughout Section 2, symbol M stands for $M+1$.

2.1. **Proof of Theorems 1.1&1.2.** Before the proof of the theorems, we define several functions to be used later. Let N, M be fixed and w be a real parameter. We define function $F(t)$, depending on N, M, w as

$$F(t; w) = (\log N + w/N)t - \log \Gamma(t + N), \tag{2.1}$$

where log takes the principal branch. It is easy to see that

$$F'(t; w) = (\log N + w/N) - \psi(t + N), \quad F''(t; w) = -\psi'(t + N). \tag{2.2}$$

where $\psi(t) = \Gamma'(z)/\Gamma(z)$ is the digamma function, such that for $z \neq 0, -1, -2, \dots$

$$\psi(z) = -\gamma_0 + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right), \quad \psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}, \tag{2.3}$$

where γ_0 is the Euler constant.

We easily see that $\operatorname{Re} \psi'(z) > 0$ if $\arg z \in (-\pi/4, \pi/4)$. As $w \in (-\infty, +\infty)$, there is a unique t_w such that

$$F'(t_w; w) = 0 \quad \text{and} \quad t_w \in (-N, +\infty), \tag{2.4}$$

and we have that

$$t_w \text{ depends on } w \text{ monotonically, } t_w \to \infty \text{ as } w \to +\infty \text{ and } t_w \to -N \text{ as } w \to -\infty. \tag{2.5}$$

As $\operatorname{dist}(z, \mathbb{Z}_{\leq 0} = \{0, -1, -2, \dots\}) > 0$, by Stirling's formula (see [45, 5.11.1]), as $z \to \infty$ in the sector $\operatorname{ph}(z) \leq \pi - \phi < \pi$,

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \log \sqrt{2\pi} + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{2.6}$$

we have that

$$\psi(z) - \left(\log z - \frac{1}{2z}\right) = \mathcal{O}\left(\frac{1}{\operatorname{dist}(z, \mathbb{Z}_{\leq 0})^2}\right). \tag{2.7}$$

Hence we have for w in a compact subset of $\mathbb{R}$,

$$t_w = \frac{1}{2} + w + \mathcal{O}(N^{-1}). \tag{2.8}$$

In the proofs of Theorems 1.1 and 1.2, we need to show that as $s = c + iy$, the function $\operatorname{Re} F(s; w)$ increases at a high enough speed as y goes upward from 0 to $+\infty$ or downward from 0 to $-\infty$, where c and w are in a compact subset of $\mathbb{R}$. To see it, we note that $-\operatorname{Im} F'(c+iy; w)$, and

$$\begin{aligned} \frac{d}{dy} \operatorname{Re} F(c + iy; w) = & -\operatorname{Im} F'(c + iy; w) = \operatorname{Im} \psi(c + iy + N) = -\sum_{n=0}^{\infty} \operatorname{Im} \frac{1}{n + N + c + iy} \\ = & \arctan(N^{-1}y)\big(1 + \mathcal{O}(N^{-1})\big). \end{aligned} \tag{2.9}$$

In the proofs of Theorems 1.1 and 1.2, we use the following positive oriented contours: For any $a \in (-N+1, 0)$, $\Sigma_-(a) = \Sigma_-^1(a) \cup \Sigma_-^2(a) \cup \Sigma_-^3(a) \cup \Sigma_-^4(a) \cup \Sigma_-^5$, where

$$\begin{gathered} \Sigma_-^1(a) = \{a - \frac{2-i}{4} t \mid t \in [0, 1]\}, \quad \Sigma_-^2(a) = \{a - \frac{2+i}{4} + \frac{2+i}{4} t \mid t \in [0, 1]\}, \\ \Sigma_-^3(a) = \{-t + \frac{i}{4} \mid t \in [\frac{1}{2} - a, N - \frac{1}{2}]\}, \quad \Sigma_-^4(a) = \{t - \frac{i}{4} \mid t \in [-N + \frac{1}{2}, a - \frac{1}{2}]\}, \\ \Sigma_-^5 = \{-N + \frac{1}{2} - it \mid t \in [-\frac{1}{4}, \frac{1}{4}]\}. \end{gathered} \tag{2.10}$$

Similarly, for any $b \in (-N+1, 1/2)$, $\Sigma_+(b) = \Sigma_+^1(b) \cup \Sigma_+^2(b) \cup \Sigma_+^3(b) \cup \Sigma_+^4(b) \cup \Sigma_+^5$, where

$$\begin{gathered} \Sigma_+^1(b) = \{b + \frac{2-i}{2}t \mid t \in [0,1]\}, \quad \Sigma_+^2(b) = \{b + \frac{2+i}{4} - \frac{2+i}{4}t \mid t \in [0,1]\}, \\ \Sigma_+^3(b) = \{t - \frac{i}{4} \mid t \in [b + \frac{1}{2} 1]\}, \quad \Sigma_+^4(b) = \{-t + \frac{i}{2} \mid t \in [-1, -b - \frac{1}{2}]\}, \\ \Sigma_+^5 = \{1 + it \mid t \in [-\frac{1}{4}, \frac{1}{4}]\}. \end{gathered} \tag{2.11}$$

Proof of Theorem 1.1. First we consider part 1. Define

$$x_N(k) = N(\psi(1-k+N) - \log N), \quad \text{such that } F'(t; x_N(k)) = 0 \text{ is solved by } t_{x_N(k)} = 1-k. \tag{2.12}$$

Then

$$\widetilde{K}\left(\frac{M}{N}g(k;\xi), \frac{M}{N}g(k;\eta)\right) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \oint_\Sigma \frac{dt}{2\pi i} \frac{\exp(MF(t; x_N(k) + \xi\sqrt{N/M})}{\exp(MF(s; x_N(k) + \eta\sqrt{N/M})} \frac{\Gamma(t)}{\Gamma(s)} \frac{1}{s-t}. \tag{2.13}$$

Later we apply the change of variables

$$t = (1-k) + \tau\sqrt{N/M}, \quad s = (1-k) + \sigma\sqrt{N/M}. \tag{2.14}$$

We define $\Sigma_0(1-k)$ to be the positive oriented circle centered at $1-k$ with radius $\sqrt{N/M}$, and then divide contour Σ into $\Sigma_-(1/2-k) \cup \Sigma_0(1-k)$ if $k=1$, and $\Sigma_-(1/2-k) \cup \Sigma_0(1-k) \cup \Sigma_+(3/2-k)$ if $k = 2, 3, \dots$. Then we have that for t on $\Sigma_0(1-k)$, under the change of variable in (2.14), with

$$h_{N,M,k}(\xi) = \exp(MF(1-k; x_N(k) + \xi\sqrt{N/M})), \tag{2.15}$$

$$MF(t; x_N(k) + \xi\sqrt{N/M}) = \log h_{N,M,k}(\xi) - \frac{N\psi'(1-k+N)}{2}\tau^2 + \xi\tau + \mathcal{O}(\tau^3/\sqrt{NM}), \tag{2.16}$$

which holds for all $\tau = \mathcal{O}(N^{1/8}M^{1/8})$. Note that for any fixed k, as $N \to \infty$, we have $N\psi'(1-k+N)/2 \to 1/2$.

We let the vertical contour for s be $\{1-k+2\sqrt{N/M} + iy \mid y \in \mathbb{R}\}$. (2.16) is a good estimation if $|\operatorname{Im}\sigma| \le N^{1/8}M^{1/8}$, or equivalently, $|\operatorname{Im} s| \le N^{\frac{5}{8}}M^{-\frac{3}{8}}$. Out of this window, we use the estimate (2.9) to find that for some $\epsilon > 0$

$$\begin{aligned} &\operatorname{Re} MF(1-k+2\sqrt{N/M}+iy; x_N(k)+\eta\sqrt{N/M}) \ge \\ &\begin{cases} \operatorname{Re} MF(1-k+2\sqrt{N/M}+iN^{\frac{5}{8}}M^{-\frac{3}{8}};\cdot) + \epsilon N^{-\frac{3}{8}}M^{\frac{5}{8}}(y - N^{\frac{5}{8}}M^{-\frac{3}{8}}) & y \ge N^{\frac{5}{8}}M^{-\frac{3}{8}}, \\ \operatorname{Re} MF(1-k+2\sqrt{N/M}-iN^{\frac{5}{8}}M^{-\frac{3}{8}};\cdot) + \epsilon N^{-\frac{3}{8}}M^{\frac{5}{8}}(-y - N^{\frac{5}{8}}M^{-\frac{3}{8}}) & y \le -N^{\frac{5}{8}}M^{-\frac{3}{8}}. \end{cases} \end{aligned} \tag{2.17}$$

We see that $\exp\left(MF(s; x_N(k) + \eta\sqrt{N/M}\right)$ grows fast and dominates the $\Gamma(s)$ term as $s \to \infty$ on the vertical contour.

On the other hand, for t on $\Sigma_-(1/2-k)$ and $\Sigma_+(3/2-k)$, we can check that

$$|\exp(MF(t; x_N(k) + \xi\sqrt{N/M}))| \le h_{N,M,k}(\xi)e^{-\epsilon\frac{M}{N}} \tag{2.18}$$

for some $\epsilon > 0$. We can find it by checking that the left-hand side of (2.18) attains its maximum on $\Sigma_-(1/2-k)$ at $1/2-k$, and attains its maximum on $\Sigma_+(3/2-k)$ at $3/2-k$. We omit the detail of calculation.

At last, we note that under the change of variables (2.14), we have

$$\frac{\Gamma(t)}{\Gamma(s)} = \frac{\sigma}{\tau}\left(1 + \mathcal{O}\left(\sqrt{\frac{N}{M}}\max(|\sigma|,|\tau|)\right)\right), \quad \frac{1}{s-t} = \frac{\sqrt{M/N}}{\tau-\sigma}, \tag{2.19}$$

and by the standard steepest-descent technique that the integral on the right-hand side of (2.13) concentrates on $s = t = 1 - k$, and under the change of variables (2.14), we have that as $N, M \to \infty$, the right-hand side of (2.13) becomes

$$\left(1 + \mathcal{O}(\sqrt{N/M})\right)\sqrt{\frac{N}{M}}\frac{h_{N,M,k}(\xi)}{h_{N,M,k}(\eta)}\int_{2-i\infty}^{2+i\infty}\frac{d\sigma}{2\pi i}\oint_{|\tau|=1}\frac{d\tau}{2\pi i}\frac{\exp(\sigma^2/2 - \eta\sigma)}{\exp(\tau^2/2 - \xi\tau)}\frac{1}{\sigma - \tau}\frac{\sigma}{\tau}. \tag{2.20}$$

Noting that

$$\int_{2-i\infty}^{2+i\infty}\frac{d\sigma}{2\pi i}\oint_{|\tau|=1}\frac{d\tau}{2\pi i}\frac{\exp(\sigma^2/2 - \eta\sigma)}{\exp(\tau^2/2 - \xi\tau)}\frac{1}{\sigma - \tau}\frac{\sigma}{\tau} = \frac{1}{\sqrt{2\pi}}e^{-\eta^2/2}, \tag{2.21}$$

we prove part 1.

To prove parts 3 and 2, we need to estimate $\widetilde{K}_N(x, x)$ where x is between $(M/N)g(k+1; 0)$ and $(M/N)g(k; 0)$ in part 3, and x is greater than $(M/N)g(1; 0)$ in part 2. We let

$$w = N(x/M - \log N). \tag{2.22}$$

Then we have that $t_w \in (-N, +\infty)$ defined by (2.4) is between $1 - k$ and $-k$ in part 3, and is to the right of 0 in part 2.

We deform contour Σ into $\Sigma_-(\sqrt{N/M})$ in part 2 and $\Sigma_-(-k+\sqrt{N/M}) \cup \Sigma_+(-k+1-\sqrt{N/M})$, and in both parts we let the contour for s be the vertical line through t_w, where w is related to x by (2.22). Below we consider part 2 and give a sketch of proof. We omit the argument for part 3 since it is similar.

$x > (M/N)g(1; 0)$ if and only if $x = (M/N)g(1; \xi)$ with $\xi > 0$. If $\xi > -2$, then $t_w > \sqrt{N/M}$, and the contour for s does not intersect the contour for t. Then for all $s \in \{t_w + iy \mid y \in \mathbb{R}\}$, and $t \in \Sigma_-(\sqrt{N/M})$ on the contour, the value of

$$\left|\frac{\exp(MF(t; x_N(1) + \xi\sqrt{N/M})}{\exp(MF(s; x_N(1) + \eta\sqrt{N/M})}\right| \tag{2.23}$$

attains its maximum at $s = t_w$ and $t = \sqrt{N/M}$, and it vanishes fast as s moves away from t_w or t moves away from $\sqrt{N/M}$. Moreover, as $\xi \to +\infty$, the maximum value also vanishes exponentially fast. Hence we conclude that $\widetilde{K}(x, x)$ vanishes exponentially fast as $x = (M/N)g(1; \xi)$ and $\xi \to +\infty$, by standard techniques of steepest-descent analysis. On the other hand, when $|\xi|$ is small, we have the approximation in part 1. The exponential decay and the estimate for small $|\xi|$ case imply part 2. □

Proof of Theorem 1.2. First we consider part 1. With w, w' in a compact subset of $\mathbb{R}$,

$$\widetilde{K}_N\left(M(\log N + w/N), M(\log N + w'/N)\right) = \int_{1-i\infty}^{1+i\infty}\frac{ds}{2\pi i}\oint_{\Sigma_-(1/2)}\frac{dt}{2\pi i}\frac{\exp(MF(t; w))}{\exp(MF(s; w'))}\frac{\Gamma(t)}{\Gamma(s)}\frac{1}{s - t}, \tag{2.24}$$

where the contour $\Sigma_-(1/2)$ is defined by (2.10). For a fixed $w \in \mathbb{R}$, if $|t| < N^{1/4}$, we have, by the Taylor expansion of $F(t; w)$ (cf. eqn (2.6)), that

$$MF(t; w) = MF(0; w) + M(\log N - \psi(N) + w/N)t - \frac{1}{2}M\psi'(N)t^2 + \mathcal{O}(N^{-1/4}), \tag{2.25}$$

and we have

$$\lim_{N\to\infty} M(\log N - \psi(N) + w/N) = \gamma(w + \frac{1}{2}), \quad \lim_{N\to\infty} M\psi'(N) = \gamma. \tag{2.26}$$

So as $s \in \{1 + iy\}$ and $t \in \Sigma_-(1/2)$ and $s, t = \mathcal{O}(N^{1/4})$, we have (here $o(1)$ is with respect to N)

$$\frac{\exp(MF(t; w))}{\exp(MF(s; w'))} = \frac{e^{\frac{\gamma}{2}s^2 - \gamma(w'+1/2)s}}{e^{\frac{\gamma}{2}t^2 - \gamma(w+1/2)t}}\left(1 + o(1)\right). \tag{2.27}$$

Next we estimate the integrand when either s or t is not $\mathcal{O}(N^{1/4})$. For $s = 1 + iy$, by (2.9) we have similar to (2.17) that there exists $\epsilon > 0$ such that

$$\operatorname{Re} MF(1+iy; w') \geq \begin{cases} \operatorname{Re} MF(1+iN^{\frac{1}{4}}) + \epsilon N^{\frac{1}{4}}(y - N^{\frac{1}{4}}) & y \geq N^{\frac{1}{4}}, \\ \operatorname{Re} MF(1-iN^{\frac{1}{4}}) + \epsilon N^{\frac{1}{4}}(-y - N^{\frac{1}{4}}) & y \leq -N^{\frac{1}{8}}. \end{cases} \tag{2.28}$$

Then it dominates $\Gamma(s)$ as $s \to \infty$ on the vertical contour.

On the other hand, as t moves to the left along $\Sigma^3_-(1/2) \setminus B(0, N^{1/4})$ or $\Sigma^4_-(1/2) \setminus B(0, N^{1/4})$, $\operatorname{Re} F(t, w)$ decreases monotonically. To see it, we check that on these horizontal contours, the second derivative

$$\frac{d^2}{dx^2} \operatorname{Re} F(x \pm i/4; w) = -\operatorname{Re} \psi'(x + N \pm i/4) = -\sum_{n=0}^{\infty} \frac{(n+N+x)^2 - \frac{1}{16}}{((n+N+x)^2 + \frac{1}{16})^2} < 0, \tag{2.29}$$

and at the rightmost ends of these horizontal contours, $\frac{d}{dx} \operatorname{Re} F(x \pm i/4; w) > 0$ by (2.25). So $|\exp(MF(t;w))|$ decreases monotonically on these horizontal contours to the left. At last, for $t \in \Sigma^5_-$, we have

$$\operatorname{Re} F(t; w) = -N \log N + \mathcal{O}(1) < -\log(N) - \epsilon N = F(0; w) - \epsilon N \tag{2.30}$$

for some $\epsilon > 0$. Hence the double contour integral (2.24) concentrates on the region $s, t \in B(0, N^{1/4})$, and by approximation (2.27), we have ($o(1)$ is with respect to N)

$$\widetilde{K}_N\big(M(\log N + w/N), M(\log N + w'/N)\big) = \\ \Big(1 + o(1)\Big) \int_{1-iN^{1/4}}^{1+iN^{1/4}} \frac{ds}{2\pi i} \oint_{\Sigma_-(1/2) \cap B(0,N^{1/4})} \frac{dt}{2\pi i} \frac{e^{\frac{\gamma}{2}s^2 - \gamma(w'+1/2)s}}{e^{\frac{\gamma}{2}t^2 - \gamma(w+1/2)t}} \frac{\Gamma(t)}{\Gamma(s)} \frac{1}{s-t}. \tag{2.31}$$

It is obvious that if we change $N^{1/4}$ to be ∞, (2.31) also holds. At last, by changing w, w' in the proof by $\gamma^{-1}\xi - 1/2$ and $\gamma^{-1}\eta - 1/2$ respectively, we prove part 1.

To prove part 2 we consider for $w \in \mathbb{R}$ that is not too small,

$$\widetilde{K}_N\big(M(\log N + w/N), M(\log N + w/N)\big) = \int_{t_w - i\infty}^{t_w + i\infty} \frac{ds}{2\pi i} \oint_{\Sigma_-(1/2)} \frac{dt}{2\pi i} \frac{\exp(MF(t;w))}{\exp(MF(s;w))} \frac{\Gamma(t)}{\Gamma(s)} \frac{1}{s-t}, \tag{2.32}$$

where t_w is the solution to the equation $F(t; w) = 0$ in (2.4). Below we assume that w is large enough, such that by (2.5), the contours for s and t do not intersect.

By argument similar to that in the proof of part 1, we have that as $s \in \{t_w + iy \mid y \in \mathbb{R}\}$ and $t \in \Sigma_-(1/2)$, the function $|\exp(MF(t;w))/\exp(MF(s;w))|$ attains it maximum when $s = t_w$ and $t = 1/2$. By (2.5), we have that this maximum vanishes as $w \to \infty$. Hence by standard method of steepest-descent analysis, we find that the integral vanishes fast as $w \to \infty$. Since in the small w case the integral is computed in part 1, we derive part 2 by the fast vanishing of $\widetilde{K}_N(M(\log N + w/N), M(\log N + w/N))$. □

2.2. **Proof of Theorem 1.3.**

Proof of Theorem 1.3. After change of variables $s, t \to sN/M, tN/M$, we obtain

$$\frac{1}{\rho_N} \widetilde{K}_N(g(\xi), g(\eta)) = \frac{1}{\rho_N} \frac{N}{M} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \oint_{\Sigma_N} \frac{dt}{2\pi i} \frac{1}{s-t} e^{\frac{N}{M}\left(f_{M,N}(s) - f_{M,N}(t)\right)} e^{\frac{N}{M} \frac{\xi t - \eta s}{\rho_N}} \tag{2.33}$$

where

$$f_{M,N}(t) = \frac{M^2}{N} \log \Gamma(N + \frac{N}{M}t) - \frac{M}{N} \log \Gamma(\frac{N}{M}t) - \big((M-1)\log N + \log M + v(\theta)\big)t \\ + M^2\Big(1 - (1 - \frac{1}{2N}) \log N - \frac{1}{2N} \log(2\pi)\Big) + \frac{M}{2N} \log \frac{M}{N}. \tag{2.34}$$

Here c and Σ_N will be properly chosen below.

In order to choose a proper t-contour, with the notations $\Sigma_\pm$ as in Lemma 2.1 below, for small $\delta > 0$ such that $\delta + \theta < \pi$, set

$$\Sigma^1_\pm = \Big\{ t \in \Sigma_\pm : \{\mathrm{Re}\{t\} \in [-C, 1]\} \cap \{t : |\arg t \pm \theta| < \delta\} \Big\}, \tag{2.35}$$

$$\Sigma^2_\pm = \Big\{ t \in \Sigma_\pm : \{\mathrm{Re}\{t\} \in [-C, 1]\} \cap \{t : |\arg t \pm \theta| \geq \delta\} \Big\}, \tag{2.36}$$

and

$$\Sigma^3_\pm = \Big\{ t = x \pm i h^{-1}(-C) : x \in [-M + \frac{M}{2N}, -C] \Big\}, \tag{2.37}$$

$$\Sigma^4_\pm = \Big\{ t = -M + \frac{M}{2N} \pm iy : y \in [0, h^{-1}(-C)] \Big\}. \tag{2.38}$$

Here h^{-1} denotes the inverse function of $h(\theta) = \theta \cot\theta, \theta \in (0, \pi)$, and the positive constant C is sufficiently large such that $h^{-1}(-C) > \max\{1, \delta + \theta\}$.

To prove part 1, according to Lemma 2.1 and Lemma 2.2 below, we deform the integral contours in (2.33) as follows. We choose the s-contour $\mathcal{C}$ to be a vertical line passing through $t_\pm = \theta\cot\theta \pm i\theta$ whenever $\theta\cot\theta$ is not a nonpositive integer; otherwise, this line routes around $(\theta\cot\theta, 0)$ with a very small circle. We choose the t-contour Σ_N as the union of two closed contours on opposite sides of $\mathcal{C}$, see Fig. 1. Let Σ_L (respectively, Σ_R) be the vertical bar connecting the two left (respectively, right) endpoints of Σ^1_+ and Σ^1_-. Thus, $\Sigma' := \Sigma^1_+ \cup \Sigma_L \cup \Sigma^1_- \cup \Sigma_R$ forms a closed contour and encircles the part of $\mathcal{C}$ whose imaginary part lies in $(-\theta_0, \theta_0)$, but does not encircle any pole for $f_{M,N}(t)$. When $\theta\cot\theta$ is an integer, we need to deform Σ_L or Σ_R such that it routes around $(\theta\cot\theta, 0)$with a very small circle. At last, let $\Sigma = \cup_{k=1}^4 \Sigma^k_\pm$ be an anticlockwise contour.

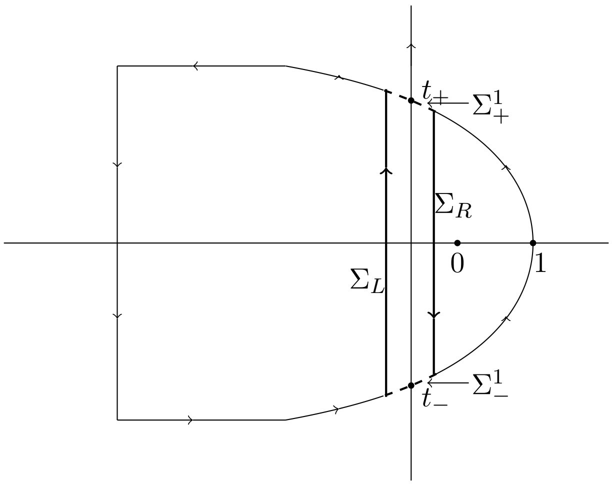

FIGURE 1. Contours of double integrals: Bulk

With these new contours we see from (2.33) that

$$\frac{1}{\rho_N}\widetilde{K}_N(x,y) = \frac{1}{\rho_N}\frac{N}{M}\,\mathrm{P.V.}\int_{\mathcal{C}}\frac{ds}{2\pi i}\oint_{\Sigma}\frac{dt}{2\pi i}\Big(\cdot\Big) - \frac{1}{\rho_N}\frac{N}{M}\int_{\mathcal{C}}\frac{ds}{2\pi i}\oint_{\Sigma'}\frac{dt}{2\pi i}\Big(\cdot\Big) := I_1 - I_2. \tag{2.39}$$

Noting that all poles of t-integral come from $s \in \mathcal{C}$ with $\mathrm{Im}\{s\} \in (-\theta, \theta)$, apply the residue theorem and we obtain

$$I_2 = -\frac{1}{\rho_N}\frac{N}{M}\int_{t_-}^{t_+}\frac{ds}{2\pi i}e^{\frac{N}{M}\frac{\xi-\eta}{\rho_N}s} = -\frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)}e^{\pi(\xi-\eta)\cot\theta}. \tag{2.40}$$

On the other hand, combining Lemma 2.1 and Lemma 2.2, we can use the argument of the steepest decent method to claim the following fact: The Cauchy principal integrals near two points $t_\pm$ give rise to the leading contribution for I_1. So, take a Taylor expansion and change variables, we obtain

$$I_1 \sim \text{P.V.}\frac{2}{\rho_N}\frac{N}{M}\sqrt{\frac{M}{N}}\int_{-i\sqrt{N/M}\delta}^{i\sqrt{N/M}\delta}\frac{ds}{2\pi i}\int_{-\sqrt{N/M}\delta}^{\sqrt{N/M}\delta}\frac{dt}{2\pi i}\frac{1}{s-t}e^{\frac{1}{2}f''(t_+)(s^2-t^2)} = \mathcal{O}\Big(\sqrt{\frac{M}{N}}\Big). \tag{2.41}$$

Combining (2.40) and (2.41) thus completes the proof of part 1.

Next, we are devoted to verifying the claimed fact. Take the derivative and use the reflection formula of the digamma function, we see from the asymptotic expansion (2.7) that

$$\begin{aligned} f'_{M,N}(t) &= M\psi(N+\frac{N}{M}t) - \psi(1-\frac{N}{M}t) + \pi\cot\frac{N}{M}\pi t - \big((M-1)\log N + \log M + v(\theta)\big) \\ &\sim M(\log(N+\frac{N}{M}t) - \log N) - \log(1-\frac{N}{M}t) - \log\frac{M}{N} + \pi\cot\frac{N}{M}\pi t - v(\theta). \end{aligned} \tag{2.42}$$

Thus, for any fixed $y_0 := h^{-1}(-C) > 1$ and for $t = x + iy_0$, when M, N are sufficiently large we have

$$\begin{aligned} \frac{\partial}{\partial x}\operatorname{Re}\{f_{M,N}(t)\} &\le M\Big(\operatorname{Re}\{\log(N+\frac{N}{M}t)\} - \log N\Big) - \operatorname{Re}\{\log(1-\frac{N}{M}t)\} - \log\frac{M}{N} \\ &= \frac{M}{2}\log\Big((1+\frac{x}{M})^2 + (\frac{y_0}{M})^2\Big) - \log\Big((\frac{M}{N}-x)^2 + y_0^2\Big) \\ &\le \frac{M}{2}\log\Big((1+\frac{x}{M})^2 + (\frac{y_0}{M})^2\Big). \end{aligned} \tag{2.43}$$

This shows that, whenever $x \le -C \le -\frac{y_0^2}{M}$, we see that

$$\frac{\partial}{\partial x}\operatorname{Re}\{f_{M,N}(x+iy_0)\} < 0. \tag{2.44}$$

Therefore, as t moves to the right endpoints along $\{t = x \pm iy_0 : x \in [-M + \frac{M}{2N}, -C]\}$ with fixed $y_0 > 1$, $\operatorname{Re}\{f_{M,N}(t)\}$ decreases monotonically.

While, for $t \in \Sigma_\pm$ with $\operatorname{Re}\{t\} \in [-C, 1]$, use the Stirling's formula (2.6) and we see

$$f_{M,N}(t) = f(t) + \mathcal{O}\Big(\max\big(\frac{M}{N}, \frac{1}{M}\big)\Big) \tag{2.45}$$

where

$$f(t) = \frac{1}{2}t^2 - t\log t - v(\theta)t + t. \tag{2.46}$$

According to Lemma 2.1 below, for any given $\theta \in [0, \pi)$, $\operatorname{Re}\{f(t)\}$ obtains its unique minimum at t_+ in the upper half plane (respectively, at t_- in the lower half plane).

Combine the two estimates above and we know that there exists $\epsilon_1 > 0$ such that

$$\operatorname{Re}\{f_{M,N}(t) - f_{M,N}(t_\pm)\} \ge \epsilon_1, \quad \forall t \in \Sigma_\pm^2 \cup \Sigma_\pm^3. \tag{2.47}$$

On the other hand, for large M and N and $t \in \Sigma_\pm^4$, by the Stirling formula we easily see that there exists $\epsilon_2 > 0$ such that

$$\operatorname{Re}\{f_{M,N}(t) - f_{M,N}(t_\pm)\} \ge \epsilon_2, \quad \forall t \in \Sigma_\pm^4. \tag{2.48}$$

Therefore, the claimed fact immediately follows from (2.47), (2.48) and the standard steepest decent argument.

Now we turn to prove part 2. For $\delta > 0$ small, let

$$\Sigma_{\text{local}} = \Big\{1 - \Big(\frac{M}{N}\Big)^{1/3} + re^{\pm i7\pi/12} : r \in [0, \delta)\Big\}. \tag{2.49}$$

and let Σ_{global} be the union of two vertical bars, connecting endpoints of Σ_{local} and $\Sigma_\pm$, and the rest parts of $\Sigma_\pm$, see illustration as in Figure 2. Similarly, we define

$$\mathcal{C}_{\text{local}} = \Big\{1 + \Big(\frac{M}{N}\Big)^{1/3} + re^{\pm i\pi/3} : r \in [0, \delta)\Big\}. \tag{2.50}$$

and $\mathcal{C}_{\text{global}}$ as the vertical line starting from the endpoints of $\mathcal{C}_{\text{local}}$ to infinity.

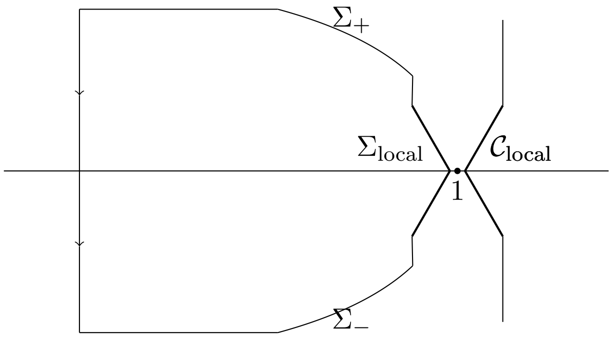

FIGURE 2. Contours of double integrals: Edge

Use similar estimates and the steepest decent method, we know from Lemma 2.1 and Lemma 2.2 that

$$\frac{1}{\rho_N}\widetilde{K}_N(x,y) \sim \frac{1}{\rho_N}\frac{N}{M}\int_{\mathcal{C}_{\text{local}}}\frac{ds}{2\pi i}\int_{\Sigma_{\text{local}}}\frac{dt}{2\pi i}\frac{1}{s-t}e^{\frac{N}{M}\left(f_{M,N}(s)-f_{M,N}(t)\right)}e^{\frac{N}{M}\frac{\xi t-\eta s}{\rho_N}}. \tag{2.51}$$

Using the Taylor expansion of $f_{M,N}$ at $s,t=1$, this reduces to

$$\frac{1}{\rho_N}\widetilde{K}_N(x,y) \sim \frac{1}{\rho_N}\frac{N}{M}\int_{\mathcal{C}_{\text{local}}}\frac{ds}{2\pi i}\int_{\Sigma_{\text{local}}}\frac{dt}{2\pi i}\frac{1}{s-t}e^{\frac{N}{6M}\left((s-1)^3-(t-1)^3\right)}e^{\frac{N}{M}\frac{\xi t-\eta s}{\rho_N}} \tag{2.52}$$

$$\sim e^{\frac{N}{M+1}\frac{\xi-\eta}{\rho_N}}K_{\text{Ai}}(\xi,\eta). \tag{2.53}$$

This thus completes part 2. □

Lemma 2.1. *Let* $f(t)=\frac{1}{2}t^2-t\log t-v(\theta_0)t+t$ *with* $\theta_0\in[0,\pi)$ *and*

$$\Sigma_\pm=\{t=g(\pm\theta):\theta\in[0,\pi)\},\quad g(\theta)=\frac{\theta}{\sin\theta}e^{i\theta}, \tag{2.54}$$

then $\mathrm{Re}\{f(t)\}$ *attains its unique minimum over* Σ_+/Σ_- *at the point* $g(\theta_0)/g(-\theta_0)$.

Proof. Note that $v(\theta)$, defined in (1.22), is a strictly decreasing function of $\theta\in(0,\pi)$, we see from $\theta-\sin\theta>0$ that

$$\frac{d}{d\theta}\mathrm{Re}\{f(g(\theta))\}=\big(v(\theta)-v(\theta_0)\big)\frac{\cos\theta\sin\theta-\theta}{\sin^2\theta}\begin{cases}<0, & \theta\in(0,\theta_0),\\ >0, & \theta\in(\theta_0,\pi).\end{cases} \tag{2.55}$$

Moreover,

$$\begin{aligned}\frac{d^2}{d\theta^2}\mathrm{Re}\{f(g(\theta))\}\Big|_{\theta=\theta_0}&=\mathrm{Re}\left\{\Big(1-\frac{1}{g(\theta_0)}\Big)\Big(\frac{\cos\theta_0\sin\theta_0-\theta_0}{\sin^2\theta_0}+i\Big)^2\right\}\\ &=\Big(1-\frac{\cos\theta_0\sin\theta_0}{\theta_0}\Big)\Big(\Big(\frac{\cos\theta_0\sin\theta_0-\theta_0}{\sin^2\theta_0}\Big)^2+1\Big)>0\end{aligned} \tag{2.56}$$

Combining (2.55) and (2.56) implies that θ_0 is the unique minimum point of $\mathrm{Re}\{f(t)\}$ over Σ_+. Similarly, $-\theta_0$ is the unique minimum point of $\mathrm{Re}\{f(t)\}$ over Σ_-. □

Lemma 2.2. *With the same notation as in Lemma 2.1, let*

$$\mathcal{C}_{\theta_0}=\left\{\frac{\theta_0\cos\theta_0}{\sin\theta_0}+iy:\ y\in\mathbb{R}\right\}, \tag{2.57}$$

then $\mathrm{Re}\{f(t)\}$ *with* $t\in\mathcal{C}_{\theta_0}$ *has only two (one) maximum points* $\frac{\theta_0\cos\theta_0}{\sin\theta_0}\pm i\theta_0$ *for* $\theta_0>0$ *(*$\theta_0=0$*).*

Proof. Without loss of generality, we consider $y > 0$. Since

$$\frac{\partial}{\partial y}\operatorname{Re}\{f(x+iy)\} = \begin{cases} -y + \arctan\frac{y}{x}, & \text{if } x > 0, \\ -y + \frac{\pi}{2}, & \text{if } x = 0, \\ -y + \arctan\frac{y}{x} + \pi, & \text{if } x < 0, \end{cases} \tag{2.58}$$

we see know that it is a strictly decreasing function of y. The lemma immediately follows from

$$\frac{\partial}{\partial y}\operatorname{Re}\Big\{f(\frac{\theta_0\cos\theta_0}{\sin\theta_0} + iy)\Big\}\Big|_{y=\theta_0} = 0. \tag{2.59}$$

□

3. Further discussion

In this last section we discuss a few relevant questions and add some comments.

3.1. **Largest Lyapunov exponent.** The critical correlation kernel $\widehat{K}_{\mathrm{crit}}(x,y;\gamma)$ defined as in (1.20) admits other forms which may be more convenient for application. If we introduce two families of functions

$$f_{-1}(x) = \oint_{\hat{\Sigma}_{-\infty}} \frac{dt}{2\pi i}\Gamma(t+t_0)e^{-\frac{\gamma}{2}t^2+xt}, \quad g_{-1}(x) = \int_{1+i\mathbb{R}} \frac{ds}{2\pi i}\frac{1}{\Gamma(s+t_0)}e^{\frac{\gamma}{2}s^2-xs} \tag{3.1}$$

and for $k = 0, 1, \dots,$

$$f_k(x) = \oint_{\hat{\Sigma}_{-\infty}} \frac{dt}{2\pi i}\frac{\Gamma(t+t_0)}{t+t_0+k}e^{-\frac{\gamma}{2}t^2+xt}, \quad g_k(x) = \int_{1+i\mathbb{R}} \frac{ds}{2\pi i}\frac{1}{(s+t_0+k)\Gamma(s+t_0)}e^{\frac{\gamma}{2}s^2-xs}, \tag{3.2}$$

then the critical kernel can be rewritten as

$$\widehat{K}_{\mathrm{crit}}(x,y;\gamma) = \int_0^\infty f_{-1}(u+x)g_{-1}(u+y)du, \tag{3.3}$$

or an integrable form

$$\widehat{K}_{\mathrm{crit}}(x,y;\gamma) = \frac{1}{x-y}\left(-\gamma f_{-1}(x)g_{-1}(y) + \sum_{k=0}^{\infty} f_k(x)g_k(y)\right). \tag{3.4}$$

These can be derived from the simple facts

$$\frac{1}{s-t} = \int_0^\infty e^{-(s-t)u}du, \quad (x-y)e^{xt-ys} = \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\Big)e^{xt-ys}, \tag{3.5}$$

and the series expansion of the digamma function(cf. [45, 5.7.6]).

If we strengthen the result in Theorem 1.2 from uniform convergence to the trace norm convergence of the integral operator with respect to the critical kernel $\widehat{K}_{\mathrm{crit}}(x,y;\gamma)$ in (1.20), then as a direct consequence the limiting distribution of the largest Lyapunov exponent (or the largest singular value), after rescaling, converges to a new limit distribution which admits a Fredholm determinant expression

$$F_{\mathrm{crit}}(x;\gamma) = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int_x^\infty \cdots \int_x^\infty \det[\widehat{K}_{\mathrm{crit}}(y_i,y_j;\gamma)]_{i,j=1}^k dy_1\cdots dy_k, \tag{3.6}$$

see [9, Sect.3.4] for more details about Fredholm determinants. Since the proof of trace norm convergence is only a technical elaboration that confirms a well-expected result, we do not give the detail.

3.2. **Criticality in the bulk.** Except for the critical scaling limit at the soft edge, Akemann, Burda and Kieburg investigated local bulk statistics in the critical regime and obtained a new interpolating kernel in [3]. Inspired by their result [3, Eq (10)](see [4] for more details), we consider the bulk critical limit and give a different derivation.

In order to state the main theorem in this subsection, we need some notations. Recall that the Jacobi theta function, defined as

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}, \quad \mathrm{Im}\{\tau\} > 0, \ z \in \mathbb{C}, \tag{3.7}$$

admits an integral representation

$$\vartheta(z;\tau) = i \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} du\, e^{\pi i \tau u^2} \frac{\cos(2\pi u z + \pi u)}{\sin \pi u}, \tag{3.8}$$

which can be derived by applying the Cauchy residue theorem. Besides, let $[x]$ be the greatest integer less than or equal to x.

Theorem 3.1 (Bulk criticality in case II)**.** *Suppose that* $\lim_{N\to\infty}(M+1)/N = \gamma \in (0,\infty)$*. For any given* $u \in (0,1)$*, let* $\gamma' = \gamma/(1-u)$ *and*

$$g(\xi) = M \log N(1-u) + \log \frac{1-u}{u} + \frac{M+1}{N(1-u)} \left(Nu - [Nu] + \frac{1}{2} \right) + \xi. \tag{3.9}$$

With the kernel $\widetilde{K}_N$ *in* (1.9)*, if* ξ, η *are in a compact subset of* $\mathbb{R}$*, then*

$$\lim_{N\to\infty} e^{(g(\xi)-g(\eta))[Nu]} \widetilde{K}_N\big(g(\xi), g(\eta)\big) = K_{\mathrm{crit}}^{(\mathrm{bulk})}(\xi,\eta;\gamma'), \tag{3.10}$$

where

$$K_{\mathrm{crit}}^{(\mathrm{bulk})}(\xi,\eta;\gamma') = \frac{1}{\sqrt{8\pi\gamma'}} \int_{-1}^{1} dw\, e^{\frac{1}{2\gamma'}(\pi w - i\eta)^2} \vartheta\Big(\frac{1}{2\pi}(\pi w - i\xi); \frac{i}{2\pi}\gamma' \Big). \tag{3.11}$$

Proof. Use the Euler's reflection formula for the gamma function and the identity

$$\frac{\sin \pi s}{\sin \pi t} = \frac{\sin \pi(s-t)}{\sin \pi t} e^{i\pi t} + e^{-i\pi(s-t)}, \tag{3.12}$$

we rewrite $\widetilde{K}_N$ in (1.9) as

$$\widetilde{K}_N(x,y) = \int \frac{ds}{2\pi i} \oint \frac{dt}{2\pi i} \frac{e^{xt-ys}}{s-t} \frac{\Gamma(1-s)}{\Gamma(1-t)} \left(\frac{\Gamma(s+N)}{\Gamma(t+N)} \right)^{M+1} \left(\frac{\sin \pi(s-t)}{\sin \pi t} e^{i\pi t} + \frac{e^{i\pi t}}{e^{i\pi s}} \right). \tag{3.13}$$

Note that the second integral in the above summation vanishes since there is no pole for t-function, we thus arrive at

$$\widetilde{K}_N(x,y) = \int \frac{ds}{2\pi i} \oint \frac{dt}{2\pi i} \frac{\sin \pi(s-t)}{s-t} \frac{e^{i\pi t}}{\sin \pi t} \frac{e^{-ys}\Gamma(1-s)}{e^{-xt}\Gamma(1-t)} \left(\frac{\Gamma(s+N)}{\Gamma(t+N)} \right)^{M+1}. \tag{3.14}$$

Make change of variables $s \to s - [Nu]$, $t \to t - [Nu]$ and we obtain

$$\widetilde{K}_N(g(\xi), g(\eta)) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \oint_{\Sigma_N} \frac{dt}{2\pi i} \frac{\sin \pi(s-t)}{s-t} \frac{e^{i\pi t}}{\sin \pi t} \frac{e^{f_N(\eta,s)}}{e^{f_N(\xi,t)}}, \tag{3.15}$$

where

$$f_N(\xi,t) = -t g(\xi) + \log \Gamma(1 + [Nu] - t) + (M+1) \log \Gamma(N - [Nu] + t), \tag{3.16}$$

and Σ_N is a rectangular contour with four vertexes $\frac{1}{2} + [Nu] \pm \frac{i}{2}$ and $-N + [Nu] + \frac{1}{2} \pm \frac{i}{2}$.

We will make use of the asymptotic expansion (see [45, 5.11.8] and cf. (2.6) for the uniform convergence condition) for large z and slowly growing h, say, $h = \mathcal{O}(z^{1/4})$,

$$\log \Gamma(z+h) \sim (z + h - \frac{1}{2}) \log z - z + \log \sqrt{2\pi} + \sum_{k=2}^{\infty} \frac{(-1)^k B_k(h)}{k(k-1) z^{k-1}}, \tag{3.17}$$

where $B_k(h)$ is the Bernoulli polynomial of degree k and in particular $B_2(h) = h^2 - h + 1/6$. Whenever $t = \mathcal{O}(N^{1/4})$, noting the assumptions and the cancelation of the coefficient of t one gets

$$\begin{aligned} f_N(\xi,t) - f_N(\xi,0) &= -tg(\xi) - t\log[Nu] + \mathcal{O}\big(N^{-1/2}\big) \\ &\quad + (M+1)\Big(t\log(N-[Nu]) + \frac{B_2(t)-B_2(0)}{2(N-[Nu])} + \mathcal{O}\big(N^{-5/4}\big)\Big) \\ &= \frac{1}{2}\gamma' t^2 - \xi t + \mathcal{O}\big(N^{-1/4}\big). \end{aligned} \tag{3.18}$$

Similarly, one has for $|s| \leq N^{1/4}$

$$f_N(\eta,s) - f_N(\eta,0) = \frac{1}{2}\gamma' s^2 - \eta s + \mathcal{O}\big(N^{-1/4}\big). \tag{3.19}$$

So as $s \in i\mathbb{R}$ and $t \in \Sigma_N$ and $s,t = \mathcal{O}(N^{1/4})$, we see from $f_N(\xi,0) = f_N(\eta,0)$ that

$$\frac{e^{f_N(\eta,s)}}{e^{f_N(\xi,t)}} = \frac{e^{\frac{\gamma'}{2}s^2-\eta s}}{e^{\frac{\gamma'}{2}t^2-\xi t}}\Big(1+o(1)\Big). \tag{3.20}$$

Next, we estimate the integrand when either s or t is not $\mathcal{O}(N^{1/4})$. For $s = iy$,

$$\begin{aligned} \frac{d}{dy}\operatorname{Re} f_N(\eta,iy) &= \operatorname{Im}\psi(1+[Nu]-iy) - \operatorname{Im}\big\{(M+1)\psi(N-[Nu]+iy)\big\} \\ &= -\Big(\arctan\frac{y}{1+[Nu]} + (M+1)\arctan\frac{y}{N-[Nu]}\Big)\Big(1+\mathcal{O}(N^{-1})\Big). \end{aligned} \tag{3.21}$$

Thus, there exists $\epsilon > 0$ such that

$$\operatorname{Re}\{f_N(\eta,iy) - f_N(\eta,0)\} \leq -\epsilon N^{1/4}(|y| - N^{1/4}), \quad |y| \geq N^{1/4}. \tag{3.22}$$

Then it dominates the factor $\sin\pi(s-t)$ as $s \to \infty$ on the vertical contour.

On the other hand, as t moves to the right endpoint along $\{x \pm \frac{i}{2} : x \in [-N+[Nu]+\frac{1}{2}, -N^{1/4}]\}$ or to the left endpoint along $\{x \pm \frac{i}{2} : x \in [N^{1/4}, [Nu]+\frac{1}{2}]\}$, $\operatorname{Re} f_N(\eta,t)$ decreases monotonically. To see it, we check that on these horizontal contours, the second derivative

$$\begin{aligned} \frac{d^2}{dx^2}\operatorname{Re} f_N(\xi, x \pm \frac{i}{2}) &= \operatorname{Re}\big\{\psi'(1+[Nu]-(x\pm\frac{i}{2})) + (M+1)\psi'(N-[Nu]+(x\pm\frac{i}{2}))\big\} \\ &= \sum_{n=0}^{\infty}\bigg(\frac{(n+1+[Nu]-x)^2-\frac{1}{4}}{((n+1+[Nu]-x)^2+\frac{1}{4})^2} + (M+1)\frac{(n+N-[Nu]+x)^2-\frac{1}{4}}{((n+N-[Nu]+x)^2+\frac{1}{4})^2}\bigg) > 0. \end{aligned} \tag{3.23}$$

Take the first derivative, we know from (3.18) that for large N

$$\frac{d}{dx}\operatorname{Re} f_N\big(\xi, N^{1/4} \pm \frac{i}{2}\big) > 0, \quad \frac{d}{dx}\operatorname{Re} f_N\big(\xi, -N^{1/4} \pm \frac{i}{2}\big) < 0. \tag{3.24}$$

So $\operatorname{Re} f_N(\xi, x \pm \frac{i}{2})$ decreases monotonically on these horizontal contours to the endpoints $\pm N^{1/4} \pm \frac{i}{2}$.

At last, on the two vertical lines of Σ_N, applying the Stirling formula leads to for any $-1/2 \leq y \leq 1/2$

$$\operatorname{Re}\big\{f_N(\xi, -N+[Nu]+\frac{1}{2}+iy) - f_N(\xi,0)\big\} = (M+1)N(1-u) + \mathcal{O}\big(N\log N\big), \tag{3.25}$$

and

$$\operatorname{Re}\big\{f_N(\xi, [Nu]+\frac{1}{2}+iy) - f_N(\xi,0)\big\} = (M+1)N(-u-\log(1-u)) + \mathcal{O}\big(N\log N\big). \tag{3.26}$$

The integrals over them are thus negligible.

Combine these estimates and we know that the double integral (3.15) concentrates on the region of $s,t=\mathcal{O}(N^{1/4})$, that is,

$$e^{(g(\xi)-g(\eta))[Nu]}\widetilde{K}_N(g(\xi),g(\eta))\sim$$
$$\left(\int_{iN^{1/4}}^{iN^{1/4}}\frac{ds}{2\pi i}\int_{N^{1/4}+\frac{i}{2}}^{-N^{1/4}+\frac{i}{2}}\frac{dt}{2\pi i}+\int_{iN^{1/4}}^{iN^{1/4}}\frac{ds}{2\pi i}\int_{-N^{1/4}-\frac{i}{2}}^{N^{1/4}-\frac{i}{2}}\frac{dt}{2\pi i}\right)\frac{\sin\pi(s-t)}{s-t}\frac{e^{i\pi t}}{\sin\pi t}\frac{e^{\frac{\gamma'}{2}s^2-\eta s}}{e^{\frac{\gamma'}{2}t^2-\xi t}}. \tag{3.27}$$

This further gives us

$$\lim_{N\to\infty}e^{(g(\xi)-g(\eta))[Nu]}\widetilde{K}_N\big(g(\xi),g(\eta)\big)=K_{\rm crit}^{(\rm bulk)}(\xi,\eta;\gamma'):=$$
$$\left(\int_{i\infty}^{i\infty}\frac{ds}{2\pi i}\int_{\frac{i}{2}+\infty}^{\frac{i}{2}-\infty}\frac{dt}{2\pi i}+\int_{i\infty}^{i\infty}\frac{ds}{2\pi i}\int_{-\frac{i}{2}-\infty}^{-\frac{i}{2}+\infty}\frac{dt}{2\pi i}\right)\frac{\sin\pi(s-t)}{s-t}\frac{e^{i\pi t}}{\sin\pi t}\frac{e^{\frac{\gamma'}{2}s^2-\eta s}}{e^{\frac{\gamma'}{2}t^2-\xi t}}. \tag{3.28}$$

Use first the simple fact that

$$\frac{\sin\pi(s-t)}{s-t}=\frac{1}{2\pi}\int_{-1}^{1}dw\,e^{-i\pi(s-t)w}, \tag{3.29}$$

and then integrate out variable s, we have

$$K_{\rm crit}^{(\rm bulk)}(\xi,\eta;\gamma')=\sqrt{\frac{\pi}{8}}\int_{-1}^{1}dw\,e^{\frac{(\pi w-i\eta)^2}{2\gamma'}}\left(\int_{-\frac{i}{2}-\infty}^{-\frac{i}{2}+\infty}\frac{dt}{2\pi i}-\int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty}\frac{dt}{2\pi i}\right)\frac{e^{i\pi t}}{\sin\pi t}e^{-\frac{\gamma'}{2}t^2+\xi t}. \tag{3.30}$$

Finally, changing t to $-t$ in the t-integral over the line $-\frac{i}{2}+\mathbb{R}$ and using (3.8), we arrive at the desired formula (3.11).

We thus complete the proof. □

Remark 3.1. Our expression form for the critical limit in the bulk (3.11) is actually the same as that in [3, Eq (10)], just by noting that the summation in [3, Eq (10)] can simplify to an integral in terms of the Jacobi theta function. Besides, as $\gamma'\to 0$ one can recover the Sine kernel; see [3, Eq (12)].

3.3. **Transition from critical kernel.** The parameter γ is defined as the limit ratio of $M+1$ and N, so as $\gamma\to 0,\infty$, we expect the Tracy-Widom (Airy kernel) and Gaussian phenomenon respectively. We also discuss a transition from the soft edge to bulk critical limit.

Theorem 3.2. *Let $t_0>0$ be the unique solution of $\psi'(t_0)=\gamma$, the following hold true uniformly for x,y in a compact subset of $\mathbb{R}$:*

$$\lim_{\gamma\to\infty}e^{x-y}\sqrt{\gamma}\widehat{K}_{\rm crit}\big(\sqrt{\gamma}(x-1),\sqrt{\gamma}(y-1);\gamma\big)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \tag{3.31}$$

and

$$\lim_{\gamma\to 0}2^{-\frac{1}{3}}\gamma^{\frac{2}{3}}\widehat{K}_{\rm crit}\big(-\log t_0+2^{-\frac{1}{3}}\gamma^{\frac{2}{3}}x,-\log t_0+2^{-\frac{1}{3}}\gamma^{\frac{2}{3}}y;\gamma\big)=K_{\rm Airy}(x,y). \tag{3.32}$$

Proof. As to (3.31), after change of variables $s\mapsto(s-1)/\sqrt{\gamma}$ and $(t-1)\mapsto t/\sqrt{\gamma}$ the kernel (1.20) becomes

$$e^{x-y}\sqrt{\gamma}\widehat{K}_{\rm crit}\big(\sqrt{\gamma}(x-1),\sqrt{\gamma}(y-1);\gamma\big)$$
$$=\int_{\frac{1}{2}+i\mathbb{R}}\frac{ds}{2\pi i}\oint_{\mathcal{C}_{-\infty}\cup\mathcal{C}_0}\frac{dt}{2\pi i}\frac{1}{s-t}\frac{\Gamma(\frac{t-1}{\sqrt{\gamma}}+t_0+1)}{\Gamma(\frac{s-1}{\sqrt{\gamma}}+t_0+1)}\frac{s-1+\sqrt{\gamma}t_0}{t-1+\sqrt{\gamma}t_0}\frac{e^{\frac{1}{2}s^2-ys}}{e^{\frac{1}{2}t^2-st}}. \tag{3.33}$$

Here $\mathcal{C}_{-\infty}$ denotes a counterclockwise contour encircling the interval $(-\infty,-1/2)$, starting at and returning to $(-1/2,0)$, while $\mathcal{C}_0$ is a small circle around the origin.

As $\gamma\to\infty$, one sees from $\psi'(t_0)=\gamma$ that

$$t_0=\frac{1}{\sqrt{\gamma}}\Big(1+\mathcal{O}(\frac{1}{\sqrt{\gamma}})\Big),$$

cf. [45, 5.7.4]. Thus, by taking limit in (3.33) only the integral over $\mathcal{C}_0$ gives us a leading term, which further results in the desired formula.

Next, we turn to (3.32). Change s, t to $2^{\frac{1}{3}}\gamma^{-\frac{2}{3}}s, 2^{\frac{1}{3}}\gamma^{-\frac{2}{3}}t$ and we rewrite (1.20) as

$$2^{-\frac{1}{3}}\gamma^{\frac{2}{3}}\widehat{K}_{\mathrm{crit}}\big(-\log t_0 + 2^{-\frac{1}{3}}\gamma^{\frac{2}{3}}x, -\log t_0 + 2^{-\frac{1}{3}}\gamma^{\frac{2}{3}}y;\gamma\big) \\ = \int_{\Sigma_L}\frac{ds}{2\pi}\oint_{\Sigma_R}\frac{dt}{2\pi i}\frac{1}{s-t}e^{xt-ys}e^{f_\gamma(s)-f_\gamma(t)}, \tag{3.34}$$

where

$$f_\gamma(t) = -\log\Gamma(\log t_0 + 2^{\frac{1}{3}}\gamma^{-\frac{2}{3}}x) + (2\gamma)^{-\frac{1}{3}}t^2 + 2^{\frac{1}{3}}\gamma^{-\frac{2}{3}}t\log t_0. \tag{3.35}$$

Here we have chosen a clockwise contour $\Sigma_R := \{ye^{\pm i\frac{\pi}{3}} : y \geq 0\}$ and a counterclockwise contour $\Sigma_L := \{ye^{\pm i\frac{2\pi}{3}} : y \geq 0\}$.

As $\gamma \to 0$, it's easy to see from the relation $\psi'(t_0) = \gamma$ that $t_0 \to \infty$ and further

$$\gamma = \frac{1}{t_0} + \frac{1}{2t_0^2} + \mathcal{O}\Big(\frac{1}{t_0^3}\Big), \tag{3.36}$$

cf. [45, 5.15.1]. The Taylor expansion for the gamma function (2.6) gives us

$$f_\gamma(t) = t_0 - (t_0 - \frac{1}{2})\log t_0 - \log\sqrt{2\pi} + 2^{\frac{1}{3}}\gamma^{-\frac{2}{3}}t + (2\gamma)^{-\frac{1}{3}}t^2 \\ - (t_0 - \frac{1}{2} + 2^{\frac{1}{3}}\gamma^{-\frac{2}{3}})\log\Big(1 + 2^{\frac{1}{3}}\gamma^{-\frac{2}{3}}\frac{t}{t_0}\Big) + \mathcal{O}\Big(\frac{1}{t_0 + 2^{\frac{1}{3}}\gamma^{-\frac{2}{3}}t}\Big) \\ = t_0 - (t_0 - \frac{1}{2})\log t_0 - \log\sqrt{2\pi} + \frac{1}{3}t^3 + o(1). \tag{3.37}$$

Combine the similar result for $f_\gamma(s)$ over Σ_L and we thus complete the proof of (3.32). $\square$

We next turn to the transition from the kernels K_{crit} in (1.17) to $K_{\mathrm{crit}}^{(\mathrm{bulk})}$ in (3.11).

Theorem 3.3. *Let k be a positive integer k, we have for any x, y in a compact subset of $\mathbb{R}$*

$$\lim_{k\to\infty} e^{k(x-y)}K_{\mathrm{crit}}\big(-\gamma k - \log k + x, -\gamma k - \log k + y;\gamma\big) = K_{\mathrm{crit}}^{(\mathrm{bulk})}(x, y;\gamma). \tag{3.38}$$

Proof. We proceed in the similar procedure as the proof of Theorem 3.1.

Let $g(x) = -\gamma k - \log k + x$, for simplicity. Make change of variables $s \to s - k$, $t \to t - k$, as in Theorem 3.1 we obtain

$$e^{k(x-y)}K_{\mathrm{crit}}(g(x), g(y)) = \int_{-i\infty}^{i\infty}\frac{ds}{2\pi i}\oint_{\Sigma_k}\frac{dt}{2\pi i}\frac{\sin\pi(s-t)}{s-t}\frac{e^{i\pi t}}{\sin\pi t}\frac{e^{f_k(y,s)}}{e^{f_k(x,t)}}, \tag{3.39}$$

where

$$f_k(x,t) = \log\Gamma(1 + k - t) + \frac{1}{2}\gamma t^2 - (x - \log k)t, \tag{3.40}$$

and Σ_k is a contour starting from $-\infty - i\epsilon$, looping around $\{k, k-1, k-2, \dots\}$ positively, and then going to $-\infty + i\epsilon$.

Whenever $t = \mathcal{O}(k^{1/4})$, applying the Stirling formula gives rise to

$$f_k(x,t) - f_k(x,0) = \frac{1}{2}\gamma t^2 - xt + \mathcal{O}\big(k^{-1/4}\big). \tag{3.41}$$

As in the proof of Theorem 3.1, we can prove that the double integral concentrates on the region of $s, t = \mathcal{O}(N^{1/4})$ and have a limiting kernel defined in (3.28). $\square$

3.4. **With different sizes.** When each X_j is a complex Ginibre matrix of size $(\nu_j + N) \times (\nu_{j-1} + N)$ with $\nu_0 = 0$ and $\nu_1, \ldots, \nu_M \geq 0$, the eigenvalues of $\log\big(\Pi_M^* \Pi_M\big)$ with the product Π_M in (1.4) also form a determinantal point process with correlation kernel

$$\widetilde{K}'_N(x,y) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \oint_{\Sigma} \frac{dt}{2\pi i} \frac{e^{xt-ys}}{s-t} \frac{\Gamma(t)}{\Gamma(s)} \prod_{j=0}^{M} \frac{\Gamma(s+\nu_j+N)}{\Gamma(t+\nu_j+N)}, \tag{3.42}$$

see [37]. In this general case, all Lyapunov exponents can be expressed as certain time average

$$\lambda_k = \lim_{M\to\infty} \frac{1}{2(M+1)} \sum_{j=0}^{M} \psi\big(\nu_j + N - k + 1\big), \quad k = 1, \ldots, N, \tag{3.43}$$

whenever the limits exist; see [29].

We believe that our main results hold true in this rectangular case, at least in the critical regime.

Theorem 3.4. *Suppose that*

$$\lim_{N\to\infty} \sum_{j=0}^{M} \frac{1}{\nu_j + N} = \gamma \in (0, \infty), \tag{3.44}$$

let t_0 be the unique positive solution of $\psi'(t_0) = \gamma$ and $L_N = \sum_{j=0}^{M} \log(\nu_j + N)$, then for any ξ, η in a compact subset of $\mathbb{R}$

$$\lim_{N\to\infty} \frac{e^{\eta t_0}}{e^{\xi t_0}} \widetilde{K}'_N \left(L_N + \gamma(t_0 - \frac{1}{2}) + \xi, L_N + \gamma(t_0 - \frac{1}{2}) + \eta \right) = \widehat{K}_{\mathrm{crit}}(\xi, \eta; \gamma). \tag{3.45}$$

Proof. Following the same argument as that in Theorem 1.2, we can complete the proof. □

3.5. **Open questions.** As discussed in the introduction, the product of M random matrices of size $N \times N$ relates classical law of large numbers and central limit theorems, and Lyapunov exponents when $M \to \infty$, to RMT statistics when $N \to \infty$. As both M and N go to infinity such that $(M+1)/N \to \gamma \in (0, \infty)$, there is a phase transition phenomenon as observed in Theorem 1.2 and [3]. These draw us to conclude this last section with a few questions which are well worth considering.

Question 1. Find an explicit form for the distribution $F_{\mathrm{crit}}(\gamma; x)$ defined by (3.6) as in the Tracy-Widom distribution; cf. [53].

Question 2. Consider the product of real Gaussian random matrices and prove a phase transition from GOE statistics to Gaussian. Furthermore, find an explicit interpolating process associated with the largest Lyapunov exponent.

Question 3. Prove the phase transition phenomenon for the product of truncated unitary/orthogonal matrices; see [25] and [34].

Question 4. Verify Theorems 1.1-1.3 for singular values of products of non-Hermitian random matrices with i.i.d. entries under certain moment assumptions. This is one of the most challenging and difficult problems related to infinite products of large random matrices; see [21, 52] or [22] for a significant breakthrough on Wigner matrices.

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