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A PDE for the multi-time joint probability of the Airy process

Dong Wang

Department of Mathematics, Brandeis University, Waltham, MA 02454, USA

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ABSTRACT

This paper gives a PDE for multi-time joint probability of the Airy process, which generalizes Adler and van Moerbeke's result on the 2-time case. As an intermediate step, the PDE for the multi-time joint probability of the Dyson Brownian motion is also given.

1. Introduction

The Airy process can be defined as the limit of the Dyson Brownian motion, as we are going to do later. However, it also appears in various statistical physical models, such as the polynuclear growth process [1,2] and the Domino tiling model [3].[1] Since the Airy process is stationary with continuous sample paths [1], we can pick any time t and consider the gap probability that all particles are in $(-\infty, u)$, denoted by $\mathbb{P}(u)$, and find that the probability is given by the GUE Tracy–Widom distribution [4]

$$\mathbb{P}(u) = \mathrm{e}^{-\int_u^\infty (s-u)q^2(s)\mathrm{d}s},$$

where $q(s)$ is the solution of the Painlevé II equation

$$q''(s) = sq(s) + 2q^2(s), \qquad q(s) \simeq \begin{cases} \dfrac{\mathrm{e}^{-(2/3)s^{3/2}}}{2\sqrt{\pi}s^{1/4}} & \text{for } s \to \infty, \\ \sqrt{-s/2} & \text{for } s \to -\infty. \end{cases}$$

In their study of the joint gap probability for several times of the Airy process, Prähofer and Spohn [1] posed the problem to find a PDE for the joint gap probability. Adler and van Moerbeke [5] solved the problem for the 2-time case, and assuming a plausible conjecture of the boundary condition, got the asymptotic expansion of the probability function $\mathbb{P}(t, u, v)$, which is the probability that all particles are in $(-\infty, u)$ initially and in $(-\infty, v)$ after a time t. Their solution was obtained by a previous result of theirs on the spectrum of coupled random matrices [6]. They regarded the joint distribution for the Dyson Brownian motion of 2-time as a τ function of the two-Toda lattice, and construct a PDE with variables in times and boundary points of the Dyson Brownian motion as a consequence of identities for τ functions and Virasoro identities specific to the situation. Then they got the PDE for the Airy process by taking the limit.

This paper generalizes their result to the multi-time case, and the technical heart is the same identity for τ functions, although in the generalized case we need more elaborate work to fit differential operators in times and boundary points of the Dyson Brownian motion into the structure of two-Toda τ functions.

After the description of the problem, we state the PDEs for both the Dyson Brownian motion with finite number of particles and its limit, the Airy process with infinitely many particles, and an example for the 3-time ($m = 2$) case for the Airy process. Section 2 derives the result for the Dyson process and Section 3 derives the result for the Airy process by taking a limit.

1.1. Description of the model

The free Brownian motion process is determined by the transition probability distribution

$$P(t, \bar{X}, X) = \frac{1}{\sqrt{(2\pi t)/\beta}} \mathrm{e}^{-\frac{(X-\bar{X})^2}{2t/\beta}},$$

E-mail address: wangdong@brandeis.edu.

[1] The definition of the Airy process in this paper is slightly different from the definition in these papers. See Remark 2.

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where $\bar{X}$ and X are initial and terminal coordinates of the particle, and β is the diffusion constant. The probability distribution $P(t,\bar{X},X)$, as a function of t and X, satisfies the diffusion equation

$$\frac{\partial P}{\partial t}=\frac{1}{2\beta}\frac{\partial^2}{\partial X^2}P.$$

If we add a harmonic potential $\rho X^2/2$ to the process, then the probability distribution $P(t,\bar{X},X)$ satisfies (see e.g. [7])

$$\frac{\partial P}{\partial t}=\left[\frac{1}{2\beta}\frac{\partial^2}{\partial X^2}-\frac{\partial}{\partial X}(-\rho X)\right]P,$$

and the process is determined by ($c=\mathrm{e}^{-\rho t}$)

$$P(t,\bar{X},X)=\frac{1}{\sqrt{\frac{\pi(1-c^2)}{\rho\beta}}}\mathrm{e}^{-\frac{(X-c\bar{X})^2}{(1-c^2)/\rho\beta}}.$$

While the free Brownian motion process is dispersive, the Brownian motion process in the harmonic potential well has a stationary distribution

$$P(X)=\frac{\mathrm{e}^{-\rho\beta X^2}}{\sqrt{\pi/\rho\beta}}.$$

Now we can define the Ornstein–Uhlenbeck process [8] of an $n\times n$ Hermitian matrix B, in which all the n^2 real variables – n for real diagonal entries, $n(n-1)/2$ for the real parts of off-diagonal entries, and the other $n(n-1)/2$ for the imaginary parts of them – are in independent Brownian motion in harmonic potential wells. The ρ for them is uniformly 1, and β is 1 for the n diagonal variables and 2 for the $n(n-1)$ off-diagonal variables. Therefore for i,j in $\{1,\dots,n\}$, ($c=\mathrm{e}^{-t}$)

$$\begin{cases}P_{ii}(t,\bar{B}_{ii},B_{ii})=\dfrac{1}{\sqrt{\pi(1-c^2)}}\mathrm{e}^{-\frac{(B_{ii}-c\bar{B}_{ii})^2}{1-c^2}},\\ P_{ij\Re}(t,\Re\bar{B}_{ij},\Re B_{ij})=\dfrac{1}{\sqrt{\pi(1-c^2)/2}}\mathrm{e}^{-\frac{(\Re B_{ij}-c\Re\bar{B}_{ij})^2}{(1-c^2)/2}},\\ P_{ij\Im}(t,\Im\bar{B}_{ij},\Im B_{ij})=\dfrac{1}{\sqrt{\pi(1-c^2)/2}}\mathrm{e}^{-\frac{(\Im B_{ij}-c\Im\bar{B}_{ij})^2}{(1-c^2)/2}},\end{cases}$$

and we can write the joint transition probability distribution as [2]

$$P(t,\bar{B},B)=\prod_{i=1}^{n}P_{ii}\prod_{1\le i<j\le n}\left(P_{ij\Re}P_{ij\Im}\right)=\frac{C^{-1}}{(1-c^2)^{n^2/2}}\mathrm{e}^{-\frac{\mathrm{Tr}(B-c\bar{B})^2}{1-c^2}}.$$

We consider the multi-time transition function with the initial state B_0 at $t_0=0$, the terminal state B_m and a series of intermediate states $B_1,\dots,B_{m-1}$, such that the time between state B_0 and B_i is t_i. If we denote

$$s_i=\begin{cases}0 & i=0,\\ t_1 & i=1,\\ t_i-t_{i-1} & i=2,\dots,m,\end{cases}$$

and

$$c_i=\mathrm{e}^{-s_i},$$

then

$$P(t_1,\dots,t_m;B_0,\dots,B_m)=C^{-1}\prod_{i=1}^{m}\mathrm{e}^{-\mathrm{Tr}\frac{(B_i-c_iB_{i-1})^2}{1-c_i^2}}.$$

The Ornstein–Uhlenbeck process has a stationary distribution

$$P(B)=C^{-1}\mathrm{e}^{-\mathrm{Tr}B^2}. \tag{1}$$

Since the Ornstein–Uhlenbeck process is invariant under the unitary transformation, we define the process of the eigenvalues as the Dyson Brownian motion process [9], whose multi-time transition probability distribution is ($0=t_0<t_1<,\cdots,<t_m$)

$P(t_1,\dots,t_m;\lambda^{(0)},\dots,\lambda^{(m)})=$ The transition probability of the $n\times n$ Hermitian matrix with eigenvalues initially $\lambda^{(0)}=(\lambda_1^{(0)},\dots,\lambda_n^{(0)})$ and $\lambda^{(1)}$ after time t_1, $\lambda^{(2)}$ after time $t_2,\dots,$ and $\lambda^{(m)}$ after the total time t_m.

[2] Through out this paper, C stands for various constants, which we do not bother to write down explicitly.

If we change the coordinates of the $\mathbb{R}^{n^2}$ space of $n\times n$ Hermitian matrices in to the eigenvalue-angle coordinates $\lambda_1,\ldots,\lambda_n,\theta_1,\ldots,\theta_{n(n-1)}$, with the Jacobian identity (see e.g. [10])

$$\prod_{i=1}^{n}\mathrm{d}x_{ii}\prod_{1\le i<j\le n}\left(\mathrm{d}\Re(x_{ij})\mathrm{d}\Im(x_{ij})\right)=V(\lambda)^2\prod_{i=1}^{n}\mathrm{d}\lambda_i\prod_{i=1}^{n(n-1)}\mathrm{d}\theta_i,$$

where $V(\lambda)=\prod_{1\le i<j\le n}(\lambda_i-\lambda_j)$ is the Vandermonde, we find the explicit formula for $P(t_1,\ldots,t_m;\lambda^{(0)},\ldots,\lambda^{(m)})$:

$$P(t_1,\ldots,t_m;\lambda^{(0)},\ldots,\lambda^{(m)})=\frac{1}{C}\int\cdots\int\prod_{i=1}^{m}\mathrm{e}^{-\mathrm{Tr}\frac{\left(B(\lambda^{(i)},\theta^{(i)})-c_iB(\lambda^{(i-1)},\theta^{(i-1)})\right)^2}{1-c_i^2}}\prod_{i=1}^{m}V(\lambda^{(i)})^2\prod_{i=1}^{m}\mathrm{d}\theta^{(i)},$$

where $\theta^{(0)}$ appears in the integral but is not relevant to the result, since the transition probability is independent of $\theta^{(0)}$ for the unitary invariant property.

By the Harish–Chandra–Itzykson–Zuber (HCIZ) formula [11]

$$\int_{U(n)}\mathrm{e}^{\mathrm{Tr}(XUYU^{-1})}\mathrm{d}U=C\frac{\det(\mathrm{e}^{x_iy_j})}{V(x)V(y)},$$

where $X=\mathrm{diag}(x_1,\ldots,x_n)$ and $Y=\mathrm{diag}(y_1,\ldots,y_n)$ are diagonal matrices, we can evaluate the multi-time transition probability density as

$$\begin{aligned}P(t_1,\ldots,t_m;\lambda^{(0)},\ldots,\lambda^{(m)})&=\frac{1}{C}V(\lambda^{(0)})^{-1}V(\lambda^{(m)})\prod_{l=1}^{m}\det\left(\mathrm{e}^{\frac{2c_l}{1-c_l^2}\lambda_i^{(l-1)}\lambda_j^{(l)}}\right)\\&\quad\times\mathrm{e}^{-\frac{c_1^2}{1-c_1^2}\sum_{i=1}^{n}\lambda_i^{(0)2}}\prod_{l=1}^{m-1}\left(\mathrm{e}^{-\left(\frac{1}{1-c_l^2}+\frac{c_{l+1}^2}{1-c_{l+1}^2}\right)\sum_{i=1}^{n}\lambda_i^{(l)2}}\right)\mathrm{e}^{-\frac{1}{1-c_m^2}\sum_{i=1}^{n}\lambda_i^{(m)2}}.\end{aligned}$$

If we take the initial state with eigenvalues $\lambda^{(0)}$ from the stationary distribution (1), which is

$$\tilde{P}(\lambda^{(0)})=\frac{1}{C}V(\lambda^{(0)})^2\mathrm{e}^{-\sum_{i=1}^{n}\lambda_i^{(0)2}}.$$

We get the multi-time correlation function in the stationary Dyson process

$$\begin{aligned}\tilde{P}(t_1,\ldots,t_m;\lambda^{(0)},\ldots,\lambda^{(m)})&=\tilde{P}(\lambda^{(0)})P(t_1,\ldots,t_m;\lambda^{(0)},\ldots,\lambda^{(m)})\\&=\frac{1}{C}V(\lambda^{(0)})V(\lambda^{(m)})\prod_{l=1}^{m}\det\left(\mathrm{e}^{\frac{2c_l}{1-c_l^2}\lambda_i^{(l-1)}\lambda_j^{(l)}}\right)\mathrm{e}^{-\frac{1}{1-c_1^2}\sum_{i=1}^{n}\lambda_i^{(0)2}}\prod_{l=1}^{m-1}\left(\mathrm{e}^{-\left(\frac{1}{1-c_l^2}+\frac{c_{l+1}^2}{1-c_{l+1}^2}\right)\sum_{i=1}^{n}\lambda_i^{(l)2}}\right)\mathrm{e}^{-\frac{1}{1-c_m^2}\sum_{i=1}^{n}\lambda_i^{(m)2}}.\end{aligned}$$

If we want to find the joint gap probability that all $\lambda_i^{(l)}$'s are in $U^{(l)}=(a_1^{(l)},a_2^{(l)})\cup\cdots\cup(a_{2r_l-1}^{(l)},a_{2r_l}^{(l)})$, with $-\infty\le a_1^{(l)}<a_2^{(l)}<a_3^{(l)}<\cdots<a_{2r_l}^{(l)}\le\infty$, for $l=0,1,\ldots,m$ and $i=1,\ldots,n$, which is

$$\mathbb{P}_n^{\mathrm{Dyson}}(t_1,\ldots,t_m;a_1^{(0)},\ldots,a_{2r_m}^{(m)})=\int\cdots\int_{U^{(0)n}\times\cdots\times U^{(m)n}}\tilde{P}(t_1,\ldots,t_m;\lambda^{(0)},\ldots,\lambda^{(m)})\prod_{l=0}^{m}\prod_{k=1}^{n}\mathrm{d}\lambda_k^{(l)},\tag{2}$$

we can simplify it by the fact that $\mathbb{P}_n^{\mathrm{Dyson}}(t_1,\ldots,t_m;a_1^{(0)},\ldots,a_{2r_m}^{(m)})$ is symmetric with respect to $\lambda_1^{(l)},\ldots,\lambda_n^{(l)}$ for any l, and get

$$\begin{aligned}\mathbb{P}_n^{\mathrm{Dyson}}(t_i;a_i^{(l)})&=\frac{1}{C}\int\cdots\int_{U^{(0)n}\times\cdots\times U^{(m)n}}V(\lambda^{(0)})V(\lambda^{(m)})\\&\quad\times\mathrm{e}^{-\frac{1}{1-c_1^2}\sum_{k=1}^{n}\lambda_k^{(0)2}}\prod_{l=1}^{m-1}\mathrm{e}^{-\frac{1-c_l^2c_{l+1}^2}{(1-c_l^2)(1-c_{l+1}^2)}\sum_{k=1}^{n}\lambda_k^{(l)2}}\mathrm{e}^{-\frac{1}{1-c_m^2}\sum_{k=1}^{n}\lambda_k^{(m)2}}\prod_{l=1}^{m}\mathrm{e}^{\frac{2c_l}{1-c_l^2}\sum_{k=1}^{n}\lambda_k^{(l-1)}\lambda_k^{(l)}}\prod_{l=0}^{m}\prod_{k=1}^{n}\mathrm{d}\lambda_k^{(l)}.\end{aligned}$$

We are going to give a PDE satisfied by $\log\mathbb{P}_n^{\mathrm{Dyson}}$ with variables t_i and $a_i^{(l)}$.

The Airy process can be defined as the limit of the Dyson process at the edge [5]. As $n\to\infty$, we can prove that the right-most particle in the Dyson process is almost surely around $\sqrt{2n}$ with the fluctuation scale $n^{1/6}$ [1,5]. If we take the rescaling

$$\bar{t}_l=n^{1/3}t_l,\tag{3}$$

$$\bar{\lambda}_k^{(l)}=\sqrt{2}n^{1/6}(\lambda_k^{(l)}-\sqrt{2n})\tag{4}$$

$$\bar{a}_i^{(l)}=\sqrt{2}n^{1/6}(a_i^{(l)}-\sqrt{2n}),\tag{5}$$

then for fixed $\bar{t}_l$ and $\bar{a}_i^{(l)}$, $\mathbb{P}_n^{\mathrm{Dyson}}$ converges to a function defined by the Fredholm determinant of a matrix integral operator [12,13,1,2]

$$\lim_{n\to\infty}\mathbb{P}_n^{\mathrm{Dyson}}\Big|_{\substack{t_l=\bar{t}_l/n^{1/3}\\ a_i^{(l)}=\sqrt{2n}+\bar{a}_i^{(l)}/(\sqrt{2}n^{1/6})}}=\mathbb{P}^{\mathrm{Airy}}(\bar{t}_1,\ldots,\bar{t}_m;\bar{a}_1^{(0)},\ldots,\bar{a}_{2r_m}^{(m)})=\det\left(I-(\chi_i^cK_{ij}^A\chi_j^c)_{1\le i,j\le m}\right),\tag{6}$$

where χ_l^c is the indicator function defined as

$$\chi_l^c(t) = \begin{cases} 0 & t \in \bigcup_{i=1}^{t_l}(a_{2i-1}, a_{2i}), \\ 1 & \text{otherwise}, \end{cases}$$

and (Ai stands for the Airy function)

$$K_{ij}^A(x,y) = \begin{cases} \int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\mathrm{d}z & \text{if } i=j, \\ \int_0^\infty \mathrm{e}^{-z(\bar{t}_i-\bar{t}_j)}\mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\mathrm{d}z & \text{if } i>j, \\ -\int_{-\infty}^0 \mathrm{e}^{z(\bar{t}_j-\bar{t}_i)}\mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\mathrm{d}z & \text{if } i<j. \end{cases}$$

Then we can define the Airy process, which contains infinitely many particles, by the multi-time joint gap probability (6). Furthermore, we are going to give a PDE satisfied by τ with variables $\bar{t}_i$ and $\bar{a}_i^{(l)}$.

Remark 1. To make the definition (6) meaningful, we need $\bar{a}_1^{(l)}$ to be $-\infty$ for all l. Otherwise the left-hand side of (6) is 0 and the right-hand side is not well defined.

Remark 2. We should emphasis a subtle difference between definitions. In this paper, we regard the Airy process as a limiting process of the n-particle Dyson Brownian motion as $n \to \infty$. Thus in the Airy process there are infinitely many particles. However, in the original definition of the Airy process in [1], it is defined as the process of the rightmost particle (whose existence is proved in [1]) among the infinitely many particles, and the ∞-particle process, which we call the Airy process, is called the ensemble of world lines of the Airy field. For details see Section 4 of [1], especially the definition 4.2. Most papers follow the definition in [1], e.g. [2,3], but in [5] our version of the definition of the Airy process is implicitly used, which is natural because in [5] the Sine process is studied in parallel, and the Sine process is another limiting process of the n-particle Dyson Brownian motion as $n \to \infty$, and has infinitely many particles. Although we follow the convention in [5], the PDE we get still solves the problem posed in [1,2].

1.2. Notational convenience

Throughout this paper, parentheses $(\ldots)$ always include numbers and functions; brackets $[\ldots]$ always include operators; braces $\{\ldots\}$ are always for Wronskians: $\{f,g\}_D = gDf - fDg$, where D is a differential operator.

1.3. Statement of main results

With notations defined in Section 1.1, we define differential operators ($l = 0, 1, \ldots, m$)

$$D^{l,1} = \sum_{i=1}^{2r_l} \frac{\partial}{\partial a_i^{(l)}}, \qquad D^{l,2} = \sum_{k=1}^{2r_l} a_k^{(l)} \frac{\partial}{\partial a_i^{(l)}},$$

if all $a_i^{(l)}$ are finite; otherwise we drop the $a_1^{(l)}$ (resp. $a_{2r_l}^{(l)}$) part if $a_1^{(l)} = -\infty$ (resp. $a_{2r_l}^{(l)} = \infty$). Then we denote

$$\mathcal{A}_1 = \sum_{l=0}^m \mathrm{e}^{-t_l} D^{l,1}, \tag{7}$$

$$\mathcal{B}_1 = \sum_{l=0}^m \mathrm{e}^{t_l - t_m} D^{l,1}, \tag{8}$$

$$\mathcal{A}_2 = \sum_{l=0}^m \mathrm{e}^{-2t_l} D^{l,2} + \sum_{l=1}^m (1 - \mathrm{e}^{-2t_l}) \frac{\partial}{\partial t_l} - \mathrm{e}^{-2t_m}, \tag{9}$$

$$\mathcal{B}_2 = \sum_{l=0}^m \mathrm{e}^{2(t_l - t_m)} D^{l,2} + \sum_{l=1}^m (\mathrm{e}^{2(t_l - t_m)} - \mathrm{e}^{-2t_m}) \frac{\partial}{\partial t_l} - \mathrm{e}^{-2t_m}. \tag{10}$$

Now we state

Theorem 1 (*Dyson Brownian Motion*). *Given $t_1, \ldots, t_m$, the logarithm of the joint gap distribution for the stationary Dyson Brownian motion $\mathbb{P}_n^{\text{Dyson}}$ defined in (2) (abbreviated as $\log \mathbb{P}_n$) satisfies a third order non-linear PDE in times and boundary points of $U^{(l)}$*

$$\mathcal{A}_1 \frac{\mathcal{B}_2 \mathcal{A}_1 \log \mathbb{P}_n}{\mathcal{B}_1 \mathcal{A}_1 \log \mathbb{P}_n + 2n\mathrm{e}^{-t_m}} = \mathcal{B}_1 \frac{\mathcal{A}_2 \mathcal{B}_1 \log \mathbb{P}_n}{\mathcal{A}_1 \mathcal{B}_1 \log \mathbb{P}_n + 2n\mathrm{e}^{-t_m}}. \tag{11}$$

Similarly with the notations

$$\mathcal{D} = \sum_{l=0}^{m} D^{l,1},$$

$$\mathcal{D}_{1L} = \sum_{l=0}^{m} (t_m - t_l) D^{l,1},$$

$$\mathcal{D}_{1R} = \sum_{l=0}^{m} t_l D^{l,1},$$

$$\mathcal{D}_1 = \mathcal{D}_{1L} - \mathcal{D}_{1R} = \sum_{l=0}^{m} (t_m - 2t_l) D^{l,1},$$

$$\mathcal{D}_2 = \sum_{l=0}^{m} ((t_m - t_l)^2 + t_l^2) D^{l,1},$$

$$\mathcal{D}_3 = \sum_{l=0}^{m} ((t_m - t_l)^3 - t_l^3) D^{l,1},$$

$$\mathcal{E} = \sum_{l=0}^{m} D^{l,2},$$

$$\mathcal{E}_1 = \sum_{l=0}^{m} (t_m - 2t_l) D^{l,2},$$

$$\mathcal{T}_1 = 2 \sum_{l=1}^{m} t_l \frac{\partial}{\partial t_l},$$

$$\mathcal{T}_2 = 2 \sum_{l=1}^{m} t_l (t_m - t_l) \frac{\partial}{\partial t_l},$$

we state the result for the Airy process

Theorem 2 (*Airy Process*). *Given $t_1, \ldots, t_m$, the logarithm of the joint gap probability for the Airy process $\mathbb{P}^{\text{Airy}}$ defined in (6) (abbreviated as $\log \mathbb{P}$) satisfies a third order non-linear PDE in times and boundary points of $U^{(l)}$*

$$\mathcal{D}^2[\mathcal{E}_1 + \mathcal{D}_3 + \mathcal{T}_2] \log \mathbb{P} - \mathcal{D}\mathcal{D}_1[\mathcal{E} + \mathcal{D}_2 + \mathcal{T}_1] \log \mathbb{P} - 2\mathcal{D}_{1L}\mathcal{D}_{1R}\mathcal{D}_1 \log \mathbb{P} = \{\mathcal{D}^2 \log \mathbb{P}, \mathcal{D}\mathcal{D}_1 \log \mathbb{P}\}_{\mathcal{D}}. \quad (12)$$

In the case of $m = 1$, our results agree with those in [5]. Especially, if $U^{(0)} = (-\infty, u)$, $U^{(1)} = (\infty, v)$ and denote $t_1 = t$, then the result for $\log \mathbb{P}^{\text{Airy}}(t; u, v)$ is

Corollary 1 ([5]). *The logarithm of the 2-time joint gap probability for the Airy process $\mathbb{P}^{\text{Airy}}(t; u, v)$ (abbreviated as $\log \mathbb{P}$) satisfies a third-order non-linear PDE in variables u, v and t*

$$\left[(v-u)\left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]\frac{\partial^2}{\partial u \partial v} + t\left[\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right]\frac{\partial}{\partial t} + t^2\left[\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right]\frac{\partial^2}{\partial u \partial v}\right]\log \mathbb{P}$$
$$= \frac{1}{2}\left\{\left[\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right]\log \mathbb{P}, \left[\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right]^2 \log \mathbb{P}\right\}_{\frac{\partial}{\partial u} + \frac{\partial}{\partial v}}. \quad (13)$$

In the $m = 2$ case, if $U^{(0)} = (-\infty, u)$, $U^{(1)} = (\infty, v)$, $U^{(2)} = (\infty, w)$, $t_1 = t$ and $t_2 = s$, the result for $\log \mathbb{P}^{\text{Airy}}(t, s; u, v, w)$ is

Corollary 2. *The logarithm of the 3-time joint gap probability for the Airy process $\mathbb{P}^{\text{Airy}}(t, s; u, v, w)$ (abbreviated as $\log \mathbb{P}$) satisfies a third-order non-linear PDE in variables u, v, w, t and t ($D = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} + \frac{\partial}{\partial u}$)*

$$\left[t(u-v)\frac{\partial^2}{\partial u \partial v} + s(u-w)\frac{\partial^2}{\partial u \partial w} + (s-t)(v-w)\frac{\partial^2}{\partial u \partial w} + \left[-s\frac{\partial}{\partial u} + (2t-s)\frac{\partial}{\partial v} + s\frac{\partial}{\partial w}\right]\left[t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s}\right] + t(s-t)D\frac{\partial}{\partial t}\right]$$
$$\times D \log \mathbb{P} + \left[-t^3\frac{\partial^3}{\partial u^2 \partial v} - s^3\frac{\partial^3}{\partial u^2 \partial w} + t^3\frac{\partial^3}{\partial u \partial v^2} + (2t-s)(2s-t)(s+t)\frac{\partial^3}{\partial u \partial v \partial w}\right.$$
$$\left. + s^3\frac{\partial^3}{\partial u \partial w^2} - (s-t)^3\frac{\partial^3}{\partial v^2 \partial w} + (s-t)^3\frac{\partial^3}{\partial v \partial w^2}\right]\log \mathbb{P}$$
$$= \frac{1}{2}\left\{\left[-s\frac{\partial}{\partial u} + (2t-s)\frac{\partial}{\partial v} + s\frac{\partial}{\partial w}\right]D \log \mathbb{P}, D^2 \log \mathbb{P}\right\}_D. \quad (14)$$

1.4. Relation to other results

For the 2-time joint gap probability of the Airy process, Adler and van Moerbeke not only found the PDE (13), but also computed the asymptotic expansion of $\mathbb{P}^{\text{Airy}}(t; u, v)$ with respect to t^{-1} as $t \to \infty$. Their method of computation cannot be analogously applied in an obvious way to the general multi-time situation. Even for the 3-time case, the author is unable to analyze the asymptotic behavior of $\log \mathbb{P}^{\text{Airy}}(t, s; u, v, w)$ when $t, s \to \infty$ by (14) and simple boundary conditions. Widom found another method for computing the asymptotic expansion of $\mathbb{P}^{\text{Airy}}(t; u, v)$ in [14], which is based on results in [4,15]. It is an interesting question whether one can get such asymptotic behavior for the multi-time situation by that method.

To find equations satisfied by the gap probabilities in various matrix models, one can get PDEs by Adler and van Moerbeke's method, or get systems of differential equations by Tracy and Widom's method. The two kinds of results appear to be quite different, albeit they describe the same model. For example. the reader may compare the results in our paper and the corresponding results in [16]. Nevertheless, these two approaches stem from the same integrable structure: the Toda lattice. For the Gaussian unitary ensemble, which is equivalent to the Dyson Brownian motion at a single snapshot, comparisons between the two approaches have been done in [17,18]. The exact relation between our results of the multi-time joint gap probability and those in [16] is still an open problem.

In this paper we utilize the fact that the joint gap probability of the Dyson Brownian motion is a specialization of the two-Toda τ function. Now mounting evidence suggests that it is also an isomonodromic τ function in Jimbo, Miwa and Ueno's sense [19]. For the 1-time gap probability, which is the same as the gap probability in the Gaussian unitary ensemble, it has been proved in [20]. For the 2-time joint gap probability, Bertola et al. have been making steady progress, e.g. [21]. The relation between isomonodromic τ functions and Toda τ functions is not very clearly understood. Our PDE may help clarify that relation.

2. The joint probability in the Dyson Brownian motion

To get the PDE, we need to consider a generalized integral in which for all $l = 1, \ldots, m$, $t_i^{(l)}$ $(i = 1, 2, \ldots)$ and $c_{i,j}^{(l)}$ $(i, j = 1, 2, \ldots)$ are formal variables,

$$\tau_n(t_i^{(l)}, c_{i,j}^{(l)}; a_i^{(l)}) = \frac{1}{C} \int \cdots \int_{{U^{(0)}}^n \times \cdots \times {U^{(m)}}^n} V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{l=0}^{m} \mathrm{e}^{\sum_{i=1}^{\infty} t_i^{(l)} \sum_{k=1}^{n} {\lambda_k^{(l)}}^i} \prod_{l=1}^{m} \mathrm{e}^{\sum_{i,j=1}^{\infty} c_{i,j}^{(l)} \sum_{k=1}^{n} {\lambda_k^{(l-1)}}^i {\lambda_k^{(l)}}^j} \prod_{l=0}^{m} \prod_{k=1}^{n} \mathrm{d}\lambda_k^{(l)}, \tag{15}$$

with C a normalization constant such that $\mathbb{P}_n^{\text{Dyson}} = \tau_n|_{\mathfrak{L}}$, where the locus $\mathfrak{L}$ is defined as $(l = 1, 2, \ldots, m-1, k = 1, 2, \ldots, m, c_k = \mathrm{e}^{-s_k})$

$$\mathfrak{L} = \begin{cases} t_2^{(0)} = -\dfrac{1}{1 - c_1^2}, \\ t_2^{(l)} = -\left(\dfrac{1}{1 - c_l^2} + \dfrac{c_{l+1}^2}{1 - c_{l+1}^2} \right), \\ t_2^{(m)} = -\dfrac{1}{1 - c_m^2}, \\ c_{1,1}^{(k)} = \dfrac{2c_k}{1 - c_k^2}, \\ \text{and all other coefficients 0.} \end{cases} \tag{16}$$

In the latter part of this paper, the phrase "variables are on the locus $\mathfrak{L}$" means that $t_i^{(l)}$ and $c_{i,j}^{(l)}$ are given by (16).

Remark 3. In the latter part of the paper, we often regard $\mathbb{P}_n^{\text{Dyson}}$ as a function with variables $t_i^{(l)}$ and $c_{i,j}^{(l)}$, and parameters $a_i^{(l)}$, though most of the variables are 0, according to (16). Therefore it is legitimate to consider $\frac{\partial}{\partial t_1^{(0)}} \mathbb{P}_n^{\text{Dyson}}$ etc.

Remark 4. Since we allow $s_1^{(l)}$ to be $-\infty$ and $a_{2r_l}^{(l)}$ to be $+\infty$, the integral in (15) may be divergent for general values of $t_i^{(l)}$ and $c_{i,j}^{(k)}$. However, if we assume $t_i^{(l)} = 0$ for $i > 2$, $c_{i,j}^{(k)} = 0$ for $\max(i, j) > 1$, and values of $t_1^{(l)}$, $t_2^{(l)}$ and $c_{1,1}^{(k)}$ are near to the locus $\mathfrak{L}$, then the integral is convergent, and all algebraic operations in latter part of the paper can be taken in this restricted setting, so they are legitimate.

Now we consider actions of $D^{l,1}$ on τ_n. Since $D^{l,1}$ acts on the integral domains of $\lambda_1^{(l)}, \ldots, \lambda_n^{(l)}$, by the formula

$$\left[\frac{\partial}{\partial a} + \frac{\partial}{\partial b} \right] \int_a^b f(x) \mathrm{d}x = f(b) - f(a) = \int_a^b f'(x) \mathrm{d}x,$$

we get

$$\begin{aligned} D^{0,1}\tau_n &= \frac{1}{C} \sum_{i=1}^{2r_0} \left[\frac{\partial}{\partial a_i^{(0)}} \right] \int \cdots \int_{{U^{(0)}}^n \times \cdots \times {U^{(m)}}^n} V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{l=0}^{m} \mathrm{e}^{\sum_{i=1}^{\infty} t_i^{(l)} \sum_{k=1}^{n} {\lambda_k^{(l)}}^i} \prod_{l=1}^{m} \mathrm{e}^{\sum_{i,j=1}^{\infty} c_{i,j}^{(l)} \sum_{k=1}^{n} {\lambda_k^{(l-1)}}^i {\lambda_k^{(l)}}^j} \prod_{l=0}^{m} \prod_{k=1}^{n} \mathrm{d}\lambda_k^{(l)} \\ &= \frac{1}{C} \int \cdots \int_{{U^{(0)}}^n \times \cdots \times {U^{(m)}}^n} \left[\sum_{k=1}^{n} \frac{\partial}{\partial \lambda_k^{(0)}} \right] \left(V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{l=0}^{m} \mathrm{e}^{\sum_{i=1}^{\infty} t_i^{(l)} \sum_{k=1}^{n} {\lambda_k^{(l)}}^i} \prod_{l=1}^{m} \mathrm{e}^{\sum_{i,j=1}^{\infty} c_{i,j}^{(l)} \sum_{k=1}^{n} {\lambda_k^{(l-1)}}^i {\lambda_k^{(l)}}^j} \right) \prod_{l=0}^{m} \prod_{k=1}^{n} \mathrm{d}\lambda_k^{(l)} \\ &= \frac{1}{C} \int \cdots \int_{{U^{(0)}}^n \times \cdots \times {U^{(m)}}^n} \left[\sum_{i=1}^{\infty} i t_i^{(0)} \sum_{k=1}^{n} {\lambda_k^{(0)}}^{i-1} + \sum_{i,j=0}^{\infty} i c_{i,j}^{(1)} \sum_{k=1}^{n} {\lambda_k^{(0)}}^{i-1} {\lambda_k^{(1)}}^j \right] \end{aligned}$$

$$\times\left(V(\lambda^{(0)})V(\lambda^{(m)})\prod_{l=0}^{m}\mathrm{e}^{\sum_{i=1}^{\infty}t_i^{(l)}\sum_{k=1}^{n}\lambda_k^{(l)i}}\prod_{l=1}^{m}\mathrm{e}^{\sum_{i,j=1}^{\infty}c_{i,j}^{(l)}\sum_{k=1}^{n}\lambda_k^{(l-1)i}\lambda_k^{(l)j}}\right)\prod_{l=0}^{m}\prod_{k=1}^{n}\mathrm{d}\lambda_k^{(l)}$$

$$=\frac{1}{C}\int\cdots\int_{U^{(0)n}\times\cdots\times U^{(m)n}}\left[nt_1^{(0)}+\sum_{i=2}^{\infty}it_i^{(0)}\frac{\partial}{\partial t_{i-1}^{(0)}}+\sum_{i=1}^{\infty}c_{1,i}^{(1)}\frac{\partial}{\partial t_i^{(1)}}+\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}ic_{i,j}^{(1)}\frac{\partial}{\partial c_{i-1,j}^{(1)}}\right]$$

$$\times\left(V(\lambda^{(0)})V(\lambda^{(m)})\prod_{l=0}^{m}\mathrm{e}^{\sum_{i=1}^{\infty}t_i^{(l)}\sum_{k=1}^{n}\lambda_k^{(l)i}}\prod_{l=1}^{m}\mathrm{e}^{\sum_{i,j=1}^{\infty}c_{i,j}^{(l)}\sum_{k=1}^{n}\lambda_k^{(l-1)i}\lambda_k^{(l)j}}\right)\prod_{l=0}^{m}\prod_{k=1}^{n}\mathrm{d}\lambda_k^{(l)}$$

$$=\left[nt_1^{(0)}+\sum_{i=2}^{\infty}it_i^{(0)}\frac{\partial}{\partial t_{i-1}^{(0)}}+\sum_{i=1}^{\infty}c_{1,i}^{(1)}\frac{\partial}{\partial t_i^{(1)}}+\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}ic_{i,j}^{(1)}\frac{\partial}{\partial c_{i-1,j}^{(1)}}\right]\tau_n, \tag{17}$$

and similarly ($l = 1, \dots, m-1$)

$$D^{l,1}\tau_n=\left[nt_1^{(l)}+\sum_{i=2}^{\infty}it_i^{(l)}\frac{\partial}{\partial t_{i-1}^{(l)}}+\sum_{i=1}^{\infty}c_{i,1}^{(l)}\frac{\partial}{\partial t_i^{(l-1)}}+\sum_{j=2}^{\infty}\sum_{i=1}^{\infty}jc_{i,j}^{(l)}\frac{\partial}{\partial c_{i,j-1}^{(l)}}+\sum_{i=1}^{\infty}c_{1,i}^{(l+1)}\frac{\partial}{\partial t_i^{(l+1)}}+\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}ic_{i,j}^{(l+1)}\frac{\partial}{\partial c_{i-1,j}^{(l+1)}}\right]\tau_n,$$

$$D^{m,1}\tau_n=\left[nt_1^{(m)}+\sum_{i=2}^{\infty}it_i^{(m)}\frac{\partial}{\partial t_{i-1}^{(m)}}+\sum_{i=1}^{\infty}c_{i,1}^{(m)}\frac{\partial}{\partial t_i^{(m-1)}}+\sum_{j=2}^{\infty}\sum_{i=1}^{\infty}jc_{i,j}^{(m)}\frac{\partial}{\partial c_{i,j-1}^{(m)}}\right]\tau_n. \tag{18}$$

As explained in Remark 3, we regard $\mathbb{P}_n^{\mathrm{Dyson}}$ as a specialization of τ_n, such that the values of $t_i^{(l)}$ and $c_{i,j}^{(l)}$ are given by the locus $\mathfrak{L}$ as in (16), and we get (We abbreviate $\mathbb{P}_n^{\mathrm{Dyson}}$ as $\mathbb{P}_n$ here and later.)

$$D^{0,1}\mathbb{P}_n=\left[-\frac{2}{1-c_1^2}\frac{\partial}{\partial t_1^{(0)}}+\frac{2c_1}{1-c_1^2}\frac{\partial}{\partial t_1^{(1)}}\right]\mathbb{P}_n, \tag{19}$$

$$D^{l,1}\mathbb{P}_n=\left[\frac{2c_l}{1-c_l^2}\frac{\partial}{\partial t_1^{(l-1)}}-\left(\frac{2}{1-c_l^2}+\frac{2c_{l+1}^2}{1-c_{l+1}^2}\right)\frac{\partial}{\partial t_1^{(l)}}+\frac{2c_{l+1}}{1-c_{l+1}^2}\frac{\partial}{\partial t_1^{(l+1)}}\right]\mathbb{P}_n, \tag{20}$$

$$D^{m,1}\mathbb{P}_n=\left[\frac{2c_m}{1-c_m^2}\frac{\partial}{\partial t_1^{(m-1)}}-\frac{2}{1-c_m^2}\frac{\partial}{\partial t_1^{(m)}}\right]\mathbb{P}_n. \tag{21}$$

Now we define an $(m+1)\times(m+1)$ matrix

$$J=\begin{pmatrix}2t_2^{(0)} & c_{1,1}^{(1)} & & & \\ c_{1,1}^{(1)} & 2t_2^{(1)} & c_{1,1}^{(2)} & & \\ & c_{1,1}^{(2)} & \ddots & \ddots & \\ & & \ddots & \ddots & c_{1,1}^{(m)} \\ & & & c_{1,1}^{(m)} & 2t_2^{(m)}\end{pmatrix}^{-1},$$

whose rows and columns are indexed from 0 to m. When $t_i^{(l)}$ and $c_{i,j}^{(l)}$ are on the locus $\mathfrak{L}$ as in (16), J becomes

$$J|_{\mathfrak{L}}=\begin{pmatrix}-\frac{2}{1-c_1^2} & \frac{2c_1}{1-c_1^2} & & & \\ \frac{2c_1}{1-c_1^2} & -\frac{2}{1-c_1^2}-\frac{2c_2^2}{1-c_2^2} & \frac{2c_2}{1-c_2^2} & & \\ & \frac{2c_2}{1-c_2^2} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{2c_m}{1-c_m^2} \\ & & & \frac{2c_m}{1-c_m^2} & -\frac{2}{1-c_m^2}\end{pmatrix}^{-1}.$$

We can find the entries of the first and the last row of J explicitly when the variables are on the locus $\mathfrak{L}$:

$$J_{0,l}|_{\mathfrak{L}}=-\frac{1}{2}\prod_{i=1}^{l}c_i=-\frac{1}{2}\mathrm{e}^{-t_l}, \tag{22}$$

$$J_{m,l}|_{\mathfrak{L}}=-\frac{1}{2}\prod_{i=1}^{m-l}c_{m-i+1}=-\frac{1}{2}\mathrm{e}^{t_l-t_m}, \tag{23}$$

and especially

$$J_{0,m}|_{\mathfrak{L}} = J_{m,0}|_{\mathfrak{L}} = -\frac{1}{2}\prod_{i=1}^{m} c_i = -\frac{1}{2}\mathrm{e}^{-t_m}.$$

Then let

$$\begin{pmatrix} E^{0,1} \\ E^{1,1} \\ \vdots \\ E^{m,1} \end{pmatrix} = J \begin{pmatrix} D^{0,1} \\ D^{1,1} \\ \vdots \\ D^{m,1} \end{pmatrix}, \tag{24}$$

and we have

Lemma 1. *When the variables $t_i^{(l)}$ and $c_{i,j}^{(l)}$ are on the locus $\mathfrak{L}$ as in (16), we have*

$$E^{0,1}E^{m,1}\log\mathbb{P}_n = E^{m,1}E^{0,1}\log\mathbb{P}_n = \frac{\partial^2 \log\mathbb{P}_n}{\partial t_1^{(0)}\partial t_1^{(m)}} - \frac{n}{2}\mathrm{e}^{-t_m}. \tag{25}$$

Proof. First, since $E^{0,1}$ and $E^{m,1}$ are linear combinations of $D^{l,1}$'s, they are differential operators of order 1, and we have

$$E^{0,1}E^{m,1}\log\mathbb{P}_n = -\frac{E^{0,1}\mathbb{P}_n E^{m,1}\mathbb{P}_n}{\mathbb{P}_n^2} + \frac{E^{0,1}E^{m,1}\mathbb{P}_n}{\mathbb{P}_n}. \tag{26}$$

By (19)–(23), when variables are on the locus $\mathfrak{L}$, we get for $l = 0, \ldots, m$

$$E^{l,1}\mathbb{P}_n = \left[\sum_{i=0}^{m} J_{l,i}|_{\mathfrak{L}} D^{i,1}\right]\mathbb{P}_n = \frac{\partial\mathbb{P}_n}{\partial t_1^{(0)}}. \tag{27}$$

Therefore when variables are on the locus $\mathfrak{L}$,

$$E^{0,1}E^{m,1}\log\mathbb{P}_n = -\frac{\frac{\partial\mathbb{P}_n}{\partial t_1^{(0)}}\frac{\partial\mathbb{P}_n}{\partial t_1^{(m)}}}{\mathbb{P}_n^2} + \frac{E^{0,1}\frac{\partial}{\partial t_1^{(m)}}\mathbb{P}_n}{\mathbb{P}_n}. \tag{28}$$

Here we need to be careful about the term $E^{0,1}\frac{\partial}{\partial t_1^{(m)}}\mathbb{P}_n$. By (17) and (18), the action of $E^{m,1}$ on τ_n is equivalent to that of a differential operator of the form $\frac{\partial}{\partial t_1^{(m)}} + \cdots$, which does not contain $a_i^{(l)}$ explicitly. When variables are on the locus $\mathfrak{L}$, all terms of the differential operator vanish except for $\frac{\partial}{\partial t_1^{(m)}}$, so we can ignore them and replace $E^{m,1}$ by $\frac{\partial}{\partial t_1^{(m)}}$ between $E^{0,1}$ and $\mathbb{P}_n$.

Since $E^{0,1}$ and $\frac{\partial}{\partial t_1^{(m)}}$ commute,

$$E^{0,1}\frac{\partial}{\partial t_1^{(m)}}\mathbb{P}_n = \frac{\partial}{\partial t_1^{(m)}}E^{0,1}\mathbb{P}_n. \tag{29}$$

By (17), (18), (22) and (24), we have the identity for the action of $E^{0,1}$ on τ_n

$$\begin{aligned} E^{0,1}\tau_n &= J_{0,0}\left[nt_1^{(0)} + \sum_{i=2}^{\infty} it_i^{(0)}\frac{\partial}{\partial t_{i-1}^{(0)}} + \sum_{i=1}^{\infty} c_{1,i}^{(1)}\frac{\partial}{\partial t_i^{(1)}} + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty} ic_{i,j}^{(1)}\frac{\partial}{\partial c_{i-1,j}^{(1)}}\right]\tau_n \\ &\quad + \sum_{l=1}^{m-1} J_{0,l}\left[nt_1^{(l)} + \sum_{i=2}^{\infty} it_i^{(l)}\frac{\partial}{\partial t_{i-1}^{(l)}} + \sum_{i=1}^{\infty} c_{i,1}^{(l)}\frac{\partial}{\partial t_i^{(l-1)}} + \sum_{j=2}^{\infty}\sum_{i=1}^{\infty} jc_{i,j}^{(l)}\frac{\partial}{\partial c_{i,j-1}^{(l)}}\right. \\ &\quad \left. + \sum_{i=1}^{\infty} c_{1,i}^{(l+1)}\frac{\partial}{\partial t_i^{(l+1)}} + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty} ic_{i,j}^{(l+1)}\frac{\partial}{\partial c_{i-1,j}^{(l+1)}}\right]\tau_n \\ &\quad + J_{0,m}\mathrm{e}^{-t_m}\left[nt_1^{(m)} + \sum_{i=2}^{\infty} it_i^{(m)}\frac{\partial}{\partial t_{i-1}^{(m)}} + \sum_{i=1}^{\infty} c_{i,1}^{(m)}\frac{\partial}{\partial t_i^{(m-1)}} + \sum_{j=2}^{\infty}\sum_{i=1}^{\infty} jc_{i,j}^{(m)}\frac{\partial}{\partial c_{i,j-1}^{(m)}}\right]\tau_n \\ &= \left[\frac{\partial}{\partial t_1^{(m)}} + nJ_{0,m}t_1^{(m)} + \cdots\right]\tau_n, \end{aligned} \tag{30}$$

such that all terms except for $\frac{\partial}{\partial t_1^{(m)}} + nJ_{0,m}t_1^{(m)}$ in the operator on the right-hand side of (30) do not contain $t_1^{(m)}$ and vanish when variables are on the locus $\mathfrak{L}$. So when variables are on the locus $\mathfrak{L}$,

$$\frac{\partial}{\partial t_1^{(m)}}E^{0,1}\mathbb{P}_n = \frac{\partial}{\partial t_1^{(m)}}\left[\frac{\partial}{\partial t_1^{(m)}} + nJ_{0,m}t_1^{(m)} + \cdots\right]\mathbb{P}_n = \frac{\partial^2\mathbb{P}_n}{\partial t_1^{(0)}\partial t_1^{(m)}} + nJ_{0,m}|_{\mathfrak{L}}\mathbb{P}_n, \tag{31}$$

and

$$E^{0,1}E^{m,1}\log\mathbb{P}_n = -\frac{\frac{\partial\mathbb{P}_n}{\partial t_1^{(0)}}\frac{\partial\mathbb{P}_n}{\partial t_1^{(m)}}}{\mathbb{P}_n^2} + \frac{\frac{\partial^2\mathbb{P}_n}{\partial t_1^{(0)}\partial t_1^{(m)}} + nJ_{0,m}|_{\mathfrak{L}}\mathbb{P}_n}{\mathbb{P}_n} = \frac{\partial^2\log\mathbb{P}_n}{\partial t_1^{(0)}\partial t_1^{(m)}} - \frac{n}{2}\mathrm{e}^{-t_m}. \quad \square$$

Similarly to (17) and (18), with the help of the formula

$$\left[a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}\right]\int_a^b f(x)\mathrm{d}x = bf(b) - af(a) = \int_a^b (xf(x))'\mathrm{d}x,$$

we get $\left(\left[\sum_{k=1}^n \frac{\partial}{\partial\lambda_k^{(0)}}\lambda_k^{(0)}\right] \text{ is regarded as an operator}\right)$

$$\begin{aligned}
D^{0,2}\tau_n &= \frac{1}{C}\sum_{i=1}^{2r_0}\left[a_i^{(0)}\frac{\partial}{\partial a_i^{(0)}}\right]\int\cdots\int_{U^{(0)n}\times\cdots\times U^{(m)n}} V(\lambda^{(0)})V(\lambda^{(m)})\prod_{l=0}^m \mathrm{e}^{\sum_{i=1}^\infty t_i^{(l)}\sum_{k=1}^n \lambda_k^{(l)i}}\prod_{l=1}^m \mathrm{e}^{\sum_{i,j=1}^\infty c_{i,j}^{(l)}\sum_{k=1}^n \lambda_k^{(l-1)i}\lambda_k^{(l)j}}\prod_{l=0}^m\prod_{k=1}^n \mathrm{d}\lambda_k^{(l)} \\
&= \frac{1}{C}\int\cdots\int_{U^{(0)n}\times\cdots\times U^{(m)n}}\left[\sum_{k=1}^n\frac{\partial}{\partial\lambda_k^{(0)}}\lambda_k^{(0)}\right]\left(V(\lambda^{(0)})V(\lambda^{(m)})\prod_{l=0}^m \mathrm{e}^{\sum_{i=1}^\infty t_i^{(l)}\sum_{k=1}^n \lambda_k^{(l)i}}\prod_{l=1}^m \mathrm{e}^{\sum_{i,j=1}^\infty c_{i,j}^{(l)}\sum_{k=1}^n \lambda_k^{(l-1)i}\lambda_k^{(l)j}}\right)\prod_{l=0}^m\prod_{k=1}^n \mathrm{d}\lambda_k^{(l)} \\
&= \frac{1}{C}\int\cdots\int_{U^{(0)n}\times\cdots\times U^{(m)n}}\left[\sum_{i=1}^\infty it_i^{(0)}\sum_{k=1}^n\lambda_k^{(0)i} + \sum_{i,j=0}^\infty ic_{i,j}^{(1)}\sum_{k=1}^n\lambda_k^{(0)i}\lambda_k^{(1)j} + \frac{n(n+1)}{2}\right] \\
&\quad\times\left(V(\lambda^{(0)})V(\lambda^{(m)})\prod_{l=0}^m \mathrm{e}^{\sum_{i=1}^\infty t_i^{(l)}\sum_{k=1}^n \lambda_k^{(l)i}}\prod_{l=1}^m \mathrm{e}^{\sum_{i,j=1}^\infty c_{i,j}^{(l)}\sum_{k=1}^n \lambda_k^{(l-1)i}\lambda_k^{(l)j}}\right)\prod_{l=0}^m\prod_{k=1}^n \mathrm{d}\lambda_k^{(l)} \\
&= \frac{1}{C}\int\cdots\int_{U^{(0)n}\times\cdots\times U^{(m)n}}\left[\sum_{i=2}^\infty it_i^{(0)}\frac{\partial}{\partial t_i^{(0)}} + \sum_{i,j=1}^\infty ic_{i,j}^{(1)}\frac{\partial}{\partial c_{i,j}^{(1)}} + \frac{n(n+1)}{2}\right] \\
&\quad\times\left(V(\lambda^{(0)})V(\lambda^{(m)})\prod_{l=0}^m \mathrm{e}^{\sum_{i=1}^\infty t_i^{(l)}\sum_{k=1}^n \lambda_k^{(l)i}}\prod_{l=1}^m \mathrm{e}^{\sum_{i,j=1}^\infty c_{i,j}^{(l)}\sum_{k=1}^n \lambda_k^{(l-1)i}\lambda_k^{(l)j}}\right)\prod_{l=0}^m\prod_{k=1}^n \mathrm{d}\lambda_k^{(l)} \\
&= \left[\sum_{i=1}^\infty it_i^{(0)}\frac{\partial}{\partial t_i^{(0)}} + \sum_{i=1}^\infty\sum_{j=1}^\infty ic_{i,j}^{(1)}\frac{\partial}{\partial c_{i,j}^{(1)}} + \frac{n(n+1)}{2}\right]\tau_n,
\end{aligned} \tag{32}$$

and similarly

$$D^{m,2}\tau_n = \left[\sum_{i=1}^\infty it_i^{(m)}\frac{\partial}{\partial t_i^{(m)}} + \sum_{i=1}^\infty\sum_{j=1}^\infty jc_{i,j}^{(m)}\frac{\partial}{\partial c_{i,j}^{(m)}} + \frac{n(n+1)}{2}\right]\tau_n,$$

and for $l = 1, \ldots, m-1$

$$D^{l,2}\tau_n = \left[\sum_{i=1}^\infty it_i^{(l)}\frac{\partial}{\partial t_i^{(l)}} + \sum_{i=1}^\infty\sum_{j=1}^\infty jc_{i,j}^{(l)}\frac{\partial}{\partial c_{i,j}^{(l)}} + \sum_{i=1}^\infty\sum_{j=1}^\infty ic_{i,j}^{(l+1)}\frac{\partial}{\partial c_{i,j}^{(l+1)}} + n\right]\tau_n.$$

When variables are on the locus $\mathfrak{L}$ we get ($l = 1, \ldots, m-1$)

$$D^{0,2}\mathbb{P}_n = \left[-\frac{2}{1-c_1^2}\frac{\partial}{\partial t_2^{(0)}} + \frac{2c_1}{1-c_1^2}\frac{\partial}{\partial c_{1,1}^{(1)}} + \frac{n(n+1)}{2}\right]\mathbb{P}_n,$$

$$D^{l,2}\mathbb{P}_n = \left[-\left(\frac{2}{1-c_l^2} + \frac{2c_{l+1}^2}{1-c_{l+1}^2}\right)\frac{\partial}{\partial t_2^{(l)}} + \frac{2c_l}{1-c_l^2}\frac{\partial}{\partial c_{1,1}^{(l)}} + \frac{2c_{l+1}}{1-c_{l+1}^2}\frac{\partial}{\partial c_{1,1}^{(l+1)}} + n\right]\mathbb{P}_n,$$

$$D^{m,2}\mathbb{P}_n = \left[-\frac{2}{1-c_m^2}\frac{\partial}{\partial t_2^{(m)}} + \frac{2c_m}{1-c_m^2}\frac{\partial}{\partial c_{1,1}^{(m)}} + \frac{n(n+1)}{2}\right]\mathbb{P}_n.$$

If we define ($l = 1, 2, \ldots, m-1$)

$$E^{0,2} = D^{0,2} - c_{1,1}^{(1)}\frac{\partial}{\partial c_{1,1}^{(1)}}, \tag{33}$$

$$E^{l,2} = D^{l,2} - c_{1,1}^{(l)}\frac{\partial}{\partial c_{1,1}^{(l)}} - c_{1,1}^{(l+1)}\frac{\partial}{\partial c_{1,1}^{(l+1)}}, \tag{34}$$

$$E^{m,2} = D^{m,2} - c_{1,1}^{(m)}\frac{\partial}{\partial c_{1,1}^{(m)}}, \tag{35}$$

we have

Lemma 2. *For $k, l = 0, 1, \ldots, m$, when variables are on the locus $\mathfrak{L}$ in (16),*

$$E^{k,2}E^{l,1}\log\mathbb{P}_n = 2t_2^{(k)}|_{\mathfrak{L}}\frac{\partial^2\mathbb{P}_n}{\partial t_2^{(k)}\partial t_1^{(l)}} + \delta_k^l E^{l,1}\log\mathbb{P}_n. \tag{36}$$

Proof. With arguments similar to those for (26) and (28), we get

$$E^{k,2}E^{l,1}\log\mathbb{P}_n = \frac{E^{k,2}\mathbb{P}_n E^{l,1}\mathbb{P}_n}{\mathbb{P}_n^2} + \frac{E^{k,2}E^{l,1}\mathbb{P}_n}{\mathbb{P}_n} = -\frac{\left(2t_2^{(k)}|_{\mathfrak{L}}\frac{\partial\mathbb{P}_n}{\partial t_2^{(k)}} + C\mathbb{P}_n\right)\frac{\partial\mathbb{P}_n}{\partial t_1^{(l)}}}{\mathbb{P}_n^2} + \frac{E^{k,2}\frac{\partial}{\partial t_1^{(l)}}\mathbb{P}_n}{\mathbb{P}_n},$$

with

$$C = \begin{cases} \dfrac{n(n+1)}{2} & k = 1 \text{ or } m, \\ n & \text{otherwise.} \end{cases}$$

Similar to (29) and (30), we have

$$E^{k,2}\frac{\partial}{\partial t_1^{(l)}}\mathbb{P}_n = \frac{\partial}{\partial t_1^{(l)}}E^{k,2}\mathbb{P}_n$$

and

$$E^{k,2}\tau_n = \left(2t_2^{(k)}\frac{\partial}{\partial t_2^{(k)}} + t_1^{(k)}\frac{\partial}{\partial t_1^{(k)}} + C + \cdots\right)\tau_n, \tag{37}$$

such that all terms except for $2t_2^{(k)}\frac{\partial}{\partial t_2^{(k)}} + t_1^{(k)}\frac{\partial}{\partial t_1^{(k)}} + C$ in the operator on the right-hand side of (37) do not contain $t_l^{(l)}$ explicitly and vanish when variables are on the locus $\mathfrak{L}$. Therefore with an argument similar to that for (31), when variables are on $\mathfrak{L}$ we have

$$\begin{aligned}
E^{k,2}E^{l,1}\log\mathbb{P}_n &= -\frac{\left(2t_2^{(k)}|_{\mathfrak{L}}\frac{\partial\mathbb{P}_n}{\partial t_2^{(k)}} + C\mathbb{P}_n\right)\frac{\partial\mathbb{P}_n}{\partial t_1^{(l)}}}{\mathbb{P}_n^2} + \frac{\frac{\partial}{\partial t_1^{(l)}}\left(2t_2^{(k)}\frac{\partial}{\partial t_2^{(k)}} + t_1^{(k)}\frac{\partial}{\partial t_1^{(k)}} + C\right)\mathbb{P}_n}{\mathbb{P}_n} \\
&= 2t_2^{(k)}|_{\mathfrak{L}}\left(-\frac{\frac{\partial\mathbb{P}_n}{\partial t_2^{(k)}}\frac{\partial\mathbb{P}_n}{\partial t_1^{(l)}}}{\mathbb{P}_n^2} + \frac{\frac{\partial^2\mathbb{P}_n}{\partial t_2^{(k)}\partial t_1^{(l)}}}{\mathbb{P}_n}\right) + \frac{\delta_k^l\frac{\partial\mathbb{P}_n}{\partial t_1^{(l)}} + t_1^{(k)}|_{\mathfrak{L}}\frac{\partial^2\mathbb{P}_n}{\partial t_1^{(k)}\partial t_1^{(l)}}}{\mathbb{P}_n} \\
&= 2t_2^{(k)}|_{\mathfrak{L}}\frac{\partial^2\log\mathbb{P}_n}{\partial t_2^{(k)}\partial_1^{(l)}} + \delta_k^l E^{l,1}\log\mathbb{P}_n,
\end{aligned}$$

since when variables are on $\mathfrak{L}$

$$E^{l,1}\log\mathbb{P}_n = \frac{E^{l,1}\mathbb{P}_n}{\mathbb{P}_n} = \frac{\frac{\partial\mathbb{P}_n}{\partial t_1^{(l)}}}{\mathbb{P}_n}. \quad \square$$

When variables are on the locus $\mathfrak{L}$, as defined in (16), $c_{1,1}^{(k)}$ and $t_2^{(l)}$ are functions of $c_1 = e^{-s_1}, \ldots, c_m = e^{-s_m}$ and all other variables are 0. Now we regard $\mathbb{P}_n$ as a function in variables $s_1, \ldots, s_m$, and by the chain rule we get as operators on $\mathbb{P}_n$ ($l = 1, 2, \ldots, m$)

$$\frac{\partial}{\partial s_l} = \frac{2c_l^2}{(1-c_l^2)^2}\frac{\partial}{\partial t_2^{(l-1)}} + \frac{2c_l^2}{(1-c_l^2)^2}\frac{\partial}{\partial t_2^{(l)}} - \frac{2c_l(1+c_l^2)}{(1-c_l^2)^2}\frac{\partial}{\partial c_{1,1}^{(l)}}$$

and

$$c_{1,1}^{(l)}\frac{\partial}{\partial c_{1,1}^{(l)}} = \frac{2c_l^2}{1-c_l^4}\frac{\partial}{\partial t_2^{(l-1)}} + \frac{2c_l^2}{1-c_l^4}\frac{\partial}{\partial t_2^{(l)}} - \frac{1-c_l^2}{1+c_l^2}\frac{\partial}{\partial s_l}.$$

Therefore by (33)–(35) we get that as operators acting on $\mathbb{P}_n$, ($l = 1, 2, \ldots, m-1$)

$$E^{0,2} = D^{0,2} - \frac{2c_1^2}{1-c_1^4}\frac{\partial}{\partial t_2^{(0)}} - \frac{2c_1^2}{1-c_1^4}\frac{\partial}{\partial t_2^{(1)}} + \frac{1-c_1^2}{1+c_1^2}\frac{\partial}{\partial s_1}, \tag{38}$$

$$E^{l,2} = D^{l,2} - \frac{2c_l^2}{1-c_l^4}\frac{\partial}{\partial t_2^{(l-1)}} - \left(\frac{2c_l^2}{1-c_l^4} + \frac{2c_{l+1}^2}{1-c_{l+1}^4}\right)\frac{\partial}{\partial t_2^{(l)}} - \frac{2c_{l+1}^2}{1-c_{l+1}^4}\frac{\partial}{\partial t_2^{(l+1)}} + \frac{1-c_l^2}{1+c_l^2}\frac{\partial}{\partial s_l} + \frac{1-c_{l+1}^2}{1+c_{l+1}^2}\frac{\partial}{\partial s_{l+1}}, \tag{39}$$

$$E^{m,2} = D^{m,2} - \frac{2c_m^2}{1-c_m^4}\frac{\partial}{\partial t_2^{(m-1)}} - \frac{2c_m^2}{1-c_m^4}\frac{\partial}{\partial t_2^{(m)}} + \frac{1-c_m^2}{1+c_m^2}\frac{\partial}{\partial s_m}. \tag{40}$$

Now we denote ($l = 1, 2, \ldots, m-1$)

$$F^{0,2} = D^{0,2} + \frac{1-c_1^2}{1+c_1^2}\frac{\partial}{\partial s_1},$$

$$F^{l,2} = D^{l,2} + \frac{1-c_l^2}{1+c_l^2}\frac{\partial}{\partial s_l} + \frac{1-c_{l+1}^2}{1+c_{l+1}^2}\frac{\partial}{\partial s_{l+1}},$$

$$F^{m,2} = D^{m,2} + \frac{1-c_m^2}{1+c_m^2}\frac{\partial}{\partial s_m},$$

and have

Lemma 3. *For $l = 1, \ldots, m-1$,*

$$F^{0,2}E^{m,1}\log \mathbb{P}_n = \left[-\frac{2}{1-c_1^4}\frac{\partial^2}{\partial t_2^{(0)}\partial t_1^{(m)}} + \frac{2c_1^2}{1-c_1^4}\frac{\partial^2}{\partial t_2^{(1)}\partial t_1^{(m)}}\right]\log \mathbb{P}_n, \tag{41}$$

$$F^{l,2}E^{m,1}\log \mathbb{P}_n = \left[\frac{2c_l^2}{1-c_l^4}\frac{\partial^2}{\partial t_2^{(l-1)}\partial t_1^{(m)}} - \left(\frac{2}{1-c_l^4} + \frac{2c_{l+1}^4}{1-c_{l+1}^4}\right)\frac{\partial^2}{\partial t_2^{(l)}\partial t_1^{(m)}} + \frac{2c_{l+1}^2}{1-c_{l+1}^4}\frac{\partial^2}{\partial t_2^{(l+1)}\partial t_1^{(m)}}\right]\log \mathbb{P}_n, \tag{42}$$

$$F^{m,2}E^{m,1}\log \mathbb{P}_n = \left[\frac{2c_m^2}{1-c_m^4}\frac{\partial^2}{\partial t_2^{(m-1)}\partial t_1^{(m)}} - \frac{2}{1-c_m^4}\frac{\partial^2}{\partial t_2^{(m)}\partial t_1^{(m)}}\right]\log \mathbb{P}_n + E^{m,1}\log \mathbb{P}_n. \tag{43}$$

Proof. We only prove (41) and (43), and (42) can be proved similarly. By (38) and (40), as operators acting on $\mathbb{P}_n$, we have

$$F^{0,2} = E^{0,2} + \frac{2c_1^2}{1-c_1^4}\frac{\partial}{\partial t_2^{(0)}} + \frac{2c_1^2}{1-c_1^4}\frac{\partial}{\partial t_2^{(1)}}, \tag{44}$$

$$F^{m,2} = E^{m,2} + \frac{2c_m^2}{1-c_m^4}\frac{\partial}{\partial t_2^{(m-1)}} + \frac{2c_m^2}{1-c_m^4}\frac{\partial}{\partial t_2^{(m)}}. \tag{45}$$

Similar to (28) and (31), we have

$$\frac{\partial}{\partial t_2^{(k)}}E^{l,1}\log \mathbb{P}_n = -\frac{\frac{\partial \mathbb{P}_n}{\partial t_2^{(k)}}E^{l,1}\mathbb{P}_n}{\mathbb{P}_n^2} + \frac{\frac{\partial}{\partial t_2^{(k)}}E^{l,1}\mathbb{P}_n}{\mathbb{P}_n} = \frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(k)}\partial t_1^{(l)}}.$$

Thus with the results of (27), (36), (44) and (45) we have

$$\begin{aligned} F^{0,2}E^{m,1}\log \mathbb{P}_n &= E^{0,2}E^{m,1}\log \mathbb{P}_n + \frac{2c_1^2}{1-c_1^4}\frac{\partial}{\partial t_2^{(0)}}E^{m,1}\log \mathbb{P}_n + \frac{2c_1^2}{1-c_1^4}\frac{\partial}{\partial t_2^{(1)}}E^{m,1}\log \mathbb{P}_n \\ &= 2t_2^{(0)}|_{\mathfrak{L}}\frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(0)}\partial t_1^{(m)}} + \frac{2c_1^2}{1-c_1^4}\frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(0)}\partial t_1^{(m)}} + \frac{2c_1^2}{1-c_1^4}\frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(1)}\partial t_1^{(m)}} \\ &= \left[-\frac{2}{1-c_1^4}\frac{\partial^2}{\partial t_2^{(0)}\partial t_1^{(m)}} + \frac{2c_1^2}{1-c_1^4}\frac{\partial^2}{\partial t_2^{(1)}\partial t_1^{(m)}}\right]\log \mathbb{P}_n, \end{aligned}$$

and

$$\begin{aligned} F^{m,2}E^{m,1}\log \mathbb{P}_n &= E^{m,2}E^{m,1}\log \mathbb{P}_n + \frac{2c_m^2}{1-c_m^4}\frac{\partial}{\partial t_2^{(m-1)}}E^{m,1}\log \mathbb{P}_n + \frac{2c_m^2}{1-c_m^4}\frac{\partial}{\partial t_2^{(m)}}E^{m,1}\log \mathbb{P}_n \\ &= \left(2t_2^{(m)}|_{\mathfrak{L}}\frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(m)}\partial t_1^{(m)}} + E^{m,1}\log \mathbb{P}_n\right) + \frac{2c_m^2}{1-c_m^4}\frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(m-1)}\partial t_1^{(m)}} + \frac{2c_m^2}{1-c_m^4}\frac{\partial^2 \log \mathbb{P}_n}{\partial t_2^{(m)}\partial t_1^{(m)}} \\ &= \left[-\frac{2}{1-c_m^4}\frac{\partial^2}{\partial t_2^{(m)}\partial t_1^{(m)}} + \frac{2c_m^2}{1-c_m^4}\frac{\partial^2}{\partial t_2^{(m-1)}\partial t_1^{(m)}}\right]\log \mathbb{P}_n + E^{m,1}\log \mathbb{P}_n. \quad \square \end{aligned}$$

Finally we define

$$\begin{pmatrix} G^{0,2} \\ G^{1,2} \\ \vdots \\ G^{m,2} \end{pmatrix} = K \begin{pmatrix} F^{0,2} \\ F^{1,2} \\ \vdots \\ F^{m,2} \end{pmatrix},$$

where

$$K=\begin{pmatrix} -\frac{2}{1-c_1^4} & \frac{2c_1^2}{1-c_1^4} & & & \\ \frac{2c_1^2}{1-c_1^4} & -\frac{2}{1-c_1^4}-\frac{2c_2^4}{1-c_2^4} & \frac{2c_2^2}{1-c_2^4} & & \\ & \frac{2c_2^2}{1-c_2^4} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{2c_m^2}{1-c_m^4} \\ & & & \frac{2c_m^2}{1-c_m^4} & -\frac{2}{1-c_m^4} \end{pmatrix}^{-1}.$$

We can get K^{-1} by substituting each c_i in $J^{-1}|_{\mathfrak{L}}$ by c_i^2, so we have

$$K_{0,l}=-\frac{1}{2}\prod_{i=1}^{l}c_i^2,\qquad K_{m,l}=-\frac{1}{2}\prod_{i=1}^{m-l}c_{m-i+1}^2$$

and get by (41)–(43),

Lemma 4.

$$G^{0,2}E^{m,1}\log\mathbb{P}_n=\frac{\partial^2}{\partial t_2^{(0)}\partial t_1^{(m)}}\log\mathbb{P}_n+K_{0,m}E^{m,1}\log\mathbb{P}_n. \tag{46}$$

Symmetrically, we can get by the same method

Lemma 5.

$$G^{m,2}E^{0,1}\log\mathbb{P}_n=\frac{\partial^2}{\partial t_2^{(m)}\partial t_1^{(0)}}\log\mathbb{P}_n+K_{m,0}E^{0,1}\log\mathbb{P}_n. \tag{47}$$

By the result of [6]

$$\frac{\partial}{\partial t_1^{(0)}}\log\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{\frac{\partial^2}{\partial t_2^{(0)}\partial t_1^{(m)}}\log\tau_n}{\frac{\partial^2}{\partial t_1^{(0)}\partial t_1^{(m)}}\log\tau_n},$$

$$\frac{\partial}{\partial t_1^{(m)}}\log\frac{\tau_{n+1}}{\tau_{n-1}}=\frac{\frac{\partial^2}{\partial t_2^{(m)}\partial t_1^{(0)}}\log\tau_n}{\frac{\partial^2}{\partial t_1^{(m)}\partial t_1^{(0)}}\log\tau_n},$$

and by (27), which implies the differential equation when variables of τ_{n-1} and τ_{n+1} are on the locus $\mathfrak{L}$

$$\frac{\partial}{\partial t_1^{(0)}}\log\frac{\tau_{n+1}}{\tau_{n-1}}\bigg|_{\mathfrak{L}}=E^{0,1}\log\frac{\tau_{n+1}}{\tau_{n-1}}\bigg|_{\mathfrak{L}}, \tag{48}$$

we get the differential equation with respect to parameters $a_i^{(l)}$, when variables of τ_{n-1}, τ_n and τ_{n+1} are on the locus $\mathfrak{L}$, by (25) and (46)–(48)

$$E^{0,1}\log\frac{\tau_{n+1}}{\tau_{n-1}}\bigg|_{\mathfrak{L}}=\frac{G^{0,2}E^{m,1}\log\tau_n|_{\mathfrak{L}}-K_{0,m}E^{m,1}\log\tau_n|_{\mathfrak{L}}}{E^{0,1}E^{m,1}\log\tau_n|_{\mathfrak{L}}-nJ_{m,0}},$$

$$E^{m,1}\log\frac{\tau_{n+1}}{\tau_{n-1}}\bigg|_{\mathfrak{L}}=\frac{G^{m,2}E^{0,1}\log\tau_n|_{\mathfrak{L}}-K_{m,0}E^{0,1}\log\tau_n|_{\mathfrak{L}}}{E^{m,1}E^{0,1}\log\tau_n|_{\mathfrak{L}}-nJ_{0,m}}.$$

By the identity

$$E^{0,1}E^{m,1}\log\frac{\tau_{n+1}}{\tau_{n-1}}\bigg|_{\mathfrak{L}}=E^{m,1}E^{0,1}\log\frac{\tau_{n+1}}{\tau_{n-1}}\bigg|_{\mathfrak{L}},$$

we get the final result

$$E^{0,1}\frac{G^{m,2}E^{0,1}\log\tau_n-K_{m,0}E^{0,1}\log\tau_n}{E^{m,1}E^{0,1}\log\tau_n-nJ_{0,m}}\bigg|_{\mathfrak{L}}=E^{m,1}\frac{G^{0,2}E^{m,1}\log\tau_n-K_{0,m}E^{m,1}\log\tau_n}{E^{0,1}E^{m,1}\log\tau_n-nJ_{m,0}}\bigg|_{\mathfrak{L}}. \tag{49}$$

Now we denote specializations of differential operators when variables are on the locus $\mathfrak{L}$

$$\mathcal{A}_1 = -2E^{0,1},$$
$$\mathcal{B}_1 = -2E^{m,1},$$
$$\mathcal{A}_2 = -2(G^{0,2} - K_{0,m}),$$
$$\mathcal{B}_2 = -2(G^{m,2} - K_{m,0}),$$

which agree with (7)–(10), and we get the Eq. (11), after the notation change $\mathbb{P}_n^{\text{Dyson}} \leftrightarrow \tau_n|_{\mathfrak{L}}$.

Remark 5. In the 2-time case, i.e. $m = 1$,

$$\mathcal{A}_1 = D^{0,1} + c_1 D^{1,1},$$
$$\mathcal{B}_1 = c_1 D^{0,1} + D^{1,1},$$
$$\mathcal{A}_2 = F^{0,2} + c_1^2 F^{1,2} - c_1^2 = D^{0,1} + c_1^2 D^{1,1} + (1 - c_1^2)\frac{\partial}{\partial t_1} - c_1^2,$$
$$\mathcal{B}_2 = c_1^2 F^{0,2} + F^{1,2} - c_1^2 = c_1^2 D^{0,1} + D^{1,1} + (1 - c_1^2)\frac{\partial}{\partial t_1} - c_1^2.$$

Our PDE (11) agrees with that in [5].

3. The joint probability in the Airy process

In this section we adapt notations defined in (3)–(5), and by Remark 1, $a_1^{(l)} = \bar{a}_1^{(l)} = -\infty$. We denote ($l = 0, 1, \ldots, m$)

$$\bar{D}^{l,1} = \sum_{k=1}^{2r_l} \frac{\partial}{\partial \bar{a}_k^{(l)}}, \qquad \bar{D}^{l,2} = \sum_{k=1}^{2r_l} \bar{a}_k^{(l)} \frac{\partial}{\partial \bar{a}_k^{(l)}},$$

if all $a_{r_l}^{(l)} < +\infty$, otherwise drop the $a_{r_l}^{(l)}$ part. We can write the differential operators defined for the Dyson process as

$$\mathcal{A}_1 = \sqrt{2\bar{n}} \sum_{l=0}^{m} \mathrm{e}^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}, \tag{50}$$

$$\mathcal{B}_1 = \sqrt{2\bar{n}} \sum_{l=0}^{m} \mathrm{e}^{(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1}, \tag{51}$$

$$\mathcal{A}_2 = \sum_{l=0}^{m} \mathrm{e}^{-2\bar{t}_l/\bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} \mathrm{e}^{-2\bar{t}_l/\bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} (1 - \mathrm{e}^{-2\bar{t}_l/\bar{n}}) \frac{\partial}{\partial \bar{t}_l} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}, \tag{52}$$

$$\mathcal{B}_2 = \sum_{l=0}^{m} \mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} \mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} (\mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}) \frac{\partial}{\partial \bar{t}_l} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}. \tag{53}$$

It is not difficult to see that (11) implies

$$[\mathcal{A}_1\mathcal{B}_2\mathcal{A}_1 - \mathcal{B}_1\mathcal{A}_2\mathcal{B}_1] \log \mathbb{P}_n \cdot (\mathcal{A}_1\mathcal{B}_1 \log \mathbb{P}_n + 2n\mathrm{e}^{-t_m}) = \mathcal{B}_2\mathcal{A}_1 \log \mathbb{P}_n \cdot \mathcal{A}_1\mathcal{B}_1\mathcal{A}_1 \log \mathbb{P}_n - \mathcal{A}_2\mathcal{B}_1 \log \mathbb{P}_n \cdot \mathcal{B}_1\mathcal{A}_1\mathcal{B}_1 \log \mathbb{P}_n. \tag{54}$$

Substituting (50)–(53) into (54), we get

$$\begin{aligned}
&\left(\left[\sum_{l=0}^{m} \mathrm{e}^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}\right] \left[\sum_{l=0}^{m} \mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} \mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1}\right.\right. \\
&\quad \left. + \bar{n} \sum_{l=1}^{m} (\mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}) \frac{\partial}{\partial \bar{t}_l} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}\right] \left[\sum_{l=0}^{m} \mathrm{e}^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}\right] \log \mathbb{P}_n \\
&\quad \left. - \left[\sum_{l=0}^{m} \mathrm{e}^{(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1}\right] \left[\sum_{l=0}^{m} \mathrm{e}^{-2\bar{t}_l/\bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} \mathrm{e}^{-2\bar{t}_l/\bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} (1 - \mathrm{e}^{-2\bar{t}_l/\bar{n}}) \frac{\partial}{\partial \bar{t}_l} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}\right] \left[\sum_{l=0}^{m} \mathrm{e}^{(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1}\right] \log \mathbb{P}_n\right) \\
&\quad \times \left(\left[\sum_{l=0}^{m} \mathrm{e}^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}\right] \left[\sum_{l=0}^{m} \mathrm{e}^{(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1}\right] \log \mathbb{P}_n + \bar{n}^2 \mathrm{e}^{-\bar{t}_m/\bar{n}}\right) \\
&= \left[\sum_{l=0}^{m} \mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,2} + 2\bar{n}^2 \sum_{l=0}^{m} \mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1} + \bar{n} \sum_{l=1}^{m} (\mathrm{e}^{2(\bar{t}_l - \bar{t}_m)/\bar{n}} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}) \frac{\partial}{\partial \bar{t}_l} - \mathrm{e}^{-2\bar{t}_m/\bar{n}}\right] \\
&\quad \times \left[\sum_{l=0}^{m} \mathrm{e}^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}\right] \log \mathbb{P}_n \times \left[\sum_{l=0}^{m} \mathrm{e}^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}\right] \left[\sum_{l=0}^{m} \mathrm{e}^{(\bar{t}_l - \bar{t}_m)/\bar{n}} \bar{D}^{l,1}\right] \left[\sum_{l=0}^{m} \mathrm{e}^{-\bar{t}_l/\bar{n}} \bar{D}^{l,1}\right] \log \mathbb{P}_n
\end{aligned}$$

$$-\left[\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,2}+2\bar{n}^2\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,1}+\bar{n}\sum_{l=1}^{m}(1-\mathrm{e}^{-2\bar{t}_l/\bar{n}})\frac{\partial}{\partial\bar{t}_l}-\mathrm{e}^{-2\bar{t}_m/\bar{n}}\right]$$
$$\times\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\log\mathbb{P}_n\times\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\log\mathbb{P}_n. \tag{55}$$

Since we have commutator formulas

$$\left[\bar{n}(\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}-\mathrm{e}^{-2\bar{t}_m/\bar{n}})\frac{\partial}{\partial\bar{t}_l},\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}\right]=\sum_{l=0}^{m}(\mathrm{e}^{(\bar{t}_l-2\bar{t}_m)/\bar{n}}-\mathrm{e}^{-(\bar{t}_l+2\bar{t}_m)/\bar{n}})\bar{D}^{l,1},$$
$$\left[\bar{n}\sum_{l=1}^{m}(1-\mathrm{e}^{-2\bar{t}_l/\bar{n}})\frac{\partial}{\partial\bar{t}_l},\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]=\sum_{l=0}^{m}(\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}-\mathrm{e}^{-(\bar{t}_l+\bar{t}_m)/\bar{n}})\bar{D}^{l,1},$$
$$\left[\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1},\sum_{l=0}^{m}\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,2}\right]=\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-2\bar{t}_m)/\bar{n}}\bar{D}^{l,1},$$
$$\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1},\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,2}\right]=\sum_{l=0}^{m}\mathrm{e}^{-(\bar{t}_l+\bar{t}_m)/\bar{n}}\bar{D}^{l,1},$$

we can change some orders of operator multiplications in (55) and make some cancelations, so that it becomes

$$\begin{aligned}
&\Bigg(\left[\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}\right]^2\left[\sum_{l=0}^{m}\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,2}+2\bar{n}^2\sum_{l=0}^{m}\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}+\bar{n}\sum_{l=1}^{m}(\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}-\mathrm{e}^{-2\bar{t}_m/\bar{n}})\frac{\partial}{\partial\bar{t}_l}\right]\log\mathbb{P}_n\\
&\quad-\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]^2\left[\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,2}+2\bar{n}^2\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,1}+\bar{n}\sum_{l=1}^{m}(1-\mathrm{e}^{-2\bar{t}_l/\bar{n}})\frac{\partial}{\partial\bar{t}_l}\right]\log\mathbb{P}_n\Bigg)\\
&\quad\times\left(\left[\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\log\mathbb{P}_n+\bar{n}^2\mathrm{e}^{-\bar{t}_m/\bar{n}}\right)\\
&=\Bigg(\left[\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,2}+2\bar{n}^2\sum_{l=0}^{m}\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}+\bar{n}\sum_{l=1}^{m}(\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}-\mathrm{e}^{-2\bar{t}_m/\bar{n}})\frac{\partial}{\partial\bar{t}_l}\right]\log\mathbb{P}_n\\
&\quad-\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-2\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\log\mathbb{P}_n\Bigg)\times\left[\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}\right]^2\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\log\mathbb{P}_n\\
&\quad\times\Bigg(-\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,2}+2\bar{n}^2\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,1}+\bar{n}\sum_{l=1}^{m}(1-\mathrm{e}^{-2\bar{t}_l/\bar{n}})\frac{\partial}{\partial\bar{t}_l}\right]\log\mathbb{P}_n\\
&\quad-\left[\sum_{l=0}^{m}\mathrm{e}^{-(\bar{t}_l+\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]\log\mathbb{P}_n\Bigg)\times\left[\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}\right]^2\log\mathbb{P}_n.
\end{aligned}$$

Since all terms of the PDE involves $\bar{n}$, we can expand the PDE with respect to $\bar{n}$, with formulas ($*$ can be 1 or 2)

$$\sum_{l=0}^{m}\mathrm{e}^{-\bar{t}_l/\bar{n}}\bar{D}^{l,1}=\sum_{l=0}^{m}\bar{D}^{l,1}-\frac{1}{\bar{n}}\sum_{l=0}^{m}\bar{t}_l\bar{D}^{l,1}+\frac{1}{2\bar{n}^2}\sum_{l=0}^{m}\bar{t}_l^2\bar{D}^{l,1}-\frac{1}{6\bar{n}^3}\sum_{l=0}^{m}\bar{t}_l^3\bar{D}^{l,1}+O\left(\frac{1}{\bar{n}^4}\right),$$
$$\sum_{l=0}^{m}\mathrm{e}^{-2\bar{t}_l/\bar{n}}\bar{D}^{l,*}=\sum_{l=0}^{m}\bar{D}^{l,*}-\frac{2}{\bar{n}}\sum_{l=0}^{m}\bar{t}_l\bar{D}^{l,*}+\frac{2}{\bar{n}^2}\sum_{l=0}^{m}\bar{t}_l^2\bar{D}^{l,*}-\frac{4}{3\bar{n}^3}\sum_{l=0}^{m}\bar{t}_l^3\bar{D}^{l,*}+O\left(\frac{1}{\bar{n}^4}\right),$$
$$\sum_{l=0}^{m}\mathrm{e}^{(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,1}=\sum_{l=0}^{m}\bar{D}^{l,1}+\frac{1}{\bar{n}}\sum_{l=0}^{m}(\bar{t}_l-\bar{t}_m)\bar{D}^{l,1}+\frac{1}{2\bar{n}^2}\sum_{l=0}^{m}(\bar{t}_l-\bar{t}_m)^2\bar{D}^{l,1}+\frac{1}{6\bar{n}^3}\sum_{l=0}^{m}(\bar{t}_l-\bar{t}_m)^3\bar{D}^{l,1}+O\left(\frac{1}{\bar{n}^4}\right),$$
$$\sum_{l=0}^{m}\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}\bar{D}^{l,*}=\sum_{l=0}^{m}\bar{D}^{l,*}+\frac{2}{\bar{n}}\sum_{l=0}^{m}(\bar{t}_l-\bar{t}_m)\bar{D}^{l,*}+\frac{2}{\bar{n}^2}\sum_{l=0}^{m}(\bar{t}_l-\bar{t}_m)^2\bar{D}^{l,*}+\frac{4}{3\bar{n}^3}\sum_{l=0}^{m}(\bar{t}_l-\bar{t}_m)^3\bar{D}^{l,*}+O\left(\frac{1}{\bar{n}^4}\right),$$
$$\bar{n}\sum_{l=1}^{m}(\mathrm{e}^{2(\bar{t}_l-\bar{t}_m)/\bar{n}}-\mathrm{e}^{2\bar{t}_m/\bar{n}})\frac{\partial}{\partial\bar{t}_l}=2\sum_{l=1}^{m}\bar{t}_l\frac{\partial}{\partial\bar{t}_l}+\frac{2}{\bar{n}}\sum_{l=1}^{m}\bar{t}_l(\bar{t}_l-2\bar{t}_m)\frac{\partial}{\partial\bar{t}_l}+O\left(\frac{1}{\bar{n}^2}\right),$$
$$\bar{n}\sum_{l=1}^{m}(1-\mathrm{e}^{-2\bar{t}_l/\bar{n}})\frac{\partial}{\partial\bar{t}_l}=2\sum_{l=1}^{m}\bar{t}_l\frac{\partial}{\partial\bar{t}_l}-\frac{2}{\bar{n}}\sum_{l=1}^{m}\bar{t}_l^2\frac{\partial}{\partial\bar{t}_l}+O\left(\frac{1}{\bar{n}^2}\right).$$

Although the left-hand side of (55) contains $O(\bar{n}^4)$ terms, after careful calculation we find all $O(\bar{n}^4)$, $O(\bar{n}^3)$ and $O(\bar{n}^2)$ terms disappear, and the equation becomes

$$\begin{aligned}
&\left[\sum_{l=0}^{m}\bar{D}^{l,1}\right]^2\left[\sum_{l=0}^{m}(\bar{t}_m-2\bar{t}_l)\bar{D}^{l,2}+\sum_{l=0}^{m}((\bar{t}_m-\bar{t}_l)^3-\bar{t}_l^3)\bar{D}^{l,1}+2\sum_{l=1}^{m}\bar{t}_l(\bar{t}_m-\bar{t}_l)\frac{\partial}{\partial\bar{t}_l}\right]\log\mathbb{P}_n\\
&\quad+\left[\sum_{l=0}^{m}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}(2\bar{t}_l-\bar{t}_m)\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\bar{D}^{l,2}+\sum_{l=0}^{m}(\bar{t}_l^2+(\bar{t}_m-\bar{t}_l)^2)\bar{D}^{l,1}+2\sum_{l=1}^{m}\bar{t}_l\frac{\partial}{\partial\bar{t}_l}\right]\log\mathbb{P}_n\\
&\quad+\left[\sum_{l=0}^{m}\bar{D}^{l,1}\right]\left[\left[\sum_{l=0}^{m}\bar{t}_l\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}(\bar{t}_m-\bar{t}_l)^2\bar{D}^{l,1}\right]-\left[\sum_{l=0}^{m}(\bar{t}_m-\bar{t}_l)\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\bar{t}_l^2\bar{D}^{l,1}\right]\right]\log\mathbb{P}_n\\
&\quad+2\left[\sum_{l=0}^{m}\bar{t}_l\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}(\bar{t}_m-\bar{t}_l)\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}(2\bar{t}_l-\bar{t}_m)\bar{D}^{l,1}\right]\log\mathbb{P}_n\\
&=\left\{\left[\sum_{l=0}^{m}(2\bar{t}_l-\bar{t}_m)\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\bar{D}^{l,1}\right]\log\mathbb{P}_n,\left[\sum_{l=0}^{m}\bar{D}^{l,1}\right]\left[\sum_{l=0}^{m}\bar{D}^{l,1}\right]\log\mathbb{P}_n\right\}_{\sum_{l=0}^{m}\bar{D}^{l,1}}+O\left(\frac{1}{\bar{n}}\right).
\end{aligned}\tag{56}$$

The term $O\left(\frac{1}{\bar{n}}\right)$ in (56) is a quadratic function in term of $\log\mathbb{P}_n$ and its derivatives with coefficients $O\left(\frac{1}{\bar{n}}\right)$. By the definition of $\mathbb{P}^{\text{Airy}}$ in (6) and the convergence result in [5], we take the limit $n\to\infty$, and get the PDE (12) after the changing of notations, i.e. cleaning all "bars" for variables and operators.

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