Random matrices with external source and KP τ functions

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In this paper we prove that the partition function in the random matrix model with external source is a KP τ function. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Let $A \in \mathbf{H}^{n \times n}$ be an $n \times n$ Hermitian matrix and $d\mu(x) = w(x)dx$ be a measure on $\mathbb{R}$ with all moments finite. Then we define the partition function,

$$Z_n(A) = \int_{M \in \mathbf{H}^{n \times n}} e^{\mathrm{Tr}(AM)} d\mu(M), \tag{1}$$

where $d\mu(M)$ denote a unitary invariant measure on $\mathbf{H}^{n \times n}$ such that if eigenvalues of M are $\lambda_1, \ldots, \lambda_n$,

$$d\mu(M) = \prod_{i=1}^{n} w(\lambda_i) dM. \tag{2}$$

Due to the unitary invariance, we assume $A = \mathrm{diag}(a_1, \ldots, a_n)$, $a_i \in \mathbb{R}$ without loss of generality. We consider $Z_n(A)$ as a function of eigenvalues of A and find a KP (Kadomtsev-Petviashvili) τ function property of it.

$Z_n(A)$ arises in the random matrix model with external source.[7,8,30,31] Let $A \in \mathbf{H}^{n \times n}$ be an $n \times n$ Hermitian matrix, and $V(x)$ be a function defined on $\mathbb{R}$, such that $e^{-V(x)}$ decays sufficiently fast. We consider the ensemble of $n \times n$ Hermitian matrices with the probability density function,

$$P(M) = \frac{1}{Z_n^V(A;\mathbb{R})} e^{-\mathrm{Tr}(V(M)-AM)}, \tag{3}$$

where the normalization constant $Z_n^V(A;\mathbb{R})$, also called the partition function, is defined as

$$Z_n^V(A;\mathbb{R}) = \int_{M \in \mathbf{H}^{n \times n}} e^{-\mathrm{Tr}(V(M)-AM)} dM. \tag{4}$$

The ensemble is called the random matrix model with external source, and we can easily identify $Z_n^V(A;\mathbb{R})$ as the $Z_n(A)$ in (1) with $w(x) = e^{-V(x)}$. In the study of gap probability,[25,5,3,6,2] we need to consider also

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$$Z_n^V(A;E) = \int_{\substack{M \in \mathbf{H}^{n \times n} \\ \text{all eigenvalues of } M \in E}} e^{-\mathrm{Tr}(V(M)-AM)} dM, \tag{5}$$

such that the gap probability that all eigenvalues of M are in E, a subset of $\mathbb{R}$, is $Z_n^V(A;E)/Z_n^V(A;\mathbb{R})$. Similarly we identify $Z_n^V(A;E)$ as the $Z_n(A)$ with $w(x)=e^{-V(x)}\chi_E(x)$, where χ_E is the indicator function.

The random matrix model with external source was introduced in the 1990s (Refs. 7 and 8) as a generalization of the standard matrix model, i.e., the $A=0$ case, which is first proposed by Wigner in the 1950s.[19] However, a special form of the random matrix model with external source has been studied by statisticians since the 1920s, under the name of Wishart ensemble, one of the most important models in multivariate statistics.[24,13]

Consider N independent, identically distributed complex samples $\mathbf{x}_1,\dots,\mathbf{x}_N$, all of which are $n \times 1$ column vectors, and we further assume that the sample vectors $\mathbf{x}_i$'s are Gaussian with mean 0 and covariance matrix Σ, which is a fixed $n \times n$ positively defined Hermitian matrix. If we put $\mathbf{x}_i$'s into an $n \times N$ rectangular matrix $X=(\mathbf{x}_1:\dots:\mathbf{x}_N)$, then the sample covariance matrix $S =(1/N)XX^\dagger$ is an $n \times n$ positively defined Hermitian matrix matrix. If we assume $N \geq n$, then the probability density function of S is

$$P(S) = \frac{1}{Z_{n,N}^{\mathrm{Wis}}(\Sigma)} e^{-N \mathrm{Tr}(\Sigma^{-1}S)} (\det S)^{N-n}, \tag{6}$$

where the normalization constant $Z_{n,N}^{\mathrm{Wis}}(\Sigma)$, analogous to the partition function in (4), is

$$Z_{n,N}^{\mathrm{Wis}}(\Sigma) = \int_{\substack{S \in \mathbf{H}^{n \times n} \\ S \text{ is positively defined}}} e^{-N \mathrm{Tr}(\Sigma^{-1}S)} (\det S)^{N-n} dS, \tag{7}$$

which is the $Z_n(-N\Sigma^{-1})$ with $w(x)=x^{N-n}\chi_{[0,\infty)}(x)$. To study the distribution of the eigenvalues of S, we also need partition functions such as the $Z_n^V(A;E)$ in (5). See, e.g., Refs. 4, 12, 21, and 20.

The main result in this paper is that the partition function $Z_n(A)$ is a KP τ function. To make the statement precise, we denote power sums of eigenvalues of A

$$t_k = \frac{1}{k}\sum_{i=1}^n a_i^k, \quad k = 1,2,\dots. \tag{8}$$

Since $Z_n(A)$ is a symmetric function in $a_1,\dots,a_n$, it can be regarded as a function of $\mathbf{t} =(t_1,t_2,\dots)$. We should be cautious that t_k's are not independent among one another, so that $Z_n(A)$ cannot be written as a function of t_k's in a unique way. However, we will eliminate the ambiguity in the following way. First, we define explicitly a function $\hat{Z}_n(\mathbf{t})$ in (26), and then identify

$$Z_n(A) = \hat{Z}_n(\mathbf{t})\big|_{t_k=1/k\sum_{i=1}^n a_i^k}. \tag{9}$$

Thus the rigorous statement is the following.

Theorem 1: *$\hat{Z}_n(\mathbf{t})$ defined by (26) is a KP τ function in variables $t_1,t_2,\dots$.*

Since $\hat{Z}_n(\mathbf{t})$ is a KP τ function, it satisfies Hirota bilinear equations,[9] which are equivalent to a series of partial differential equations (PDEs). For example,

$$\left(\frac{\partial^4}{\partial t_1^4} + 3\frac{\partial^2}{\partial t_2^2} - 4\frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \hat{Z}_n(\mathbf{t}) + 6\left(\frac{\partial^2}{\partial t_1^2} \log \hat{Z}_n(\mathbf{t}) \right)^2 = 0, \tag{10}$$

which corresponds to the first nontrivial Hirota bilinear equation.

In Refs. 16 and 32, it is proven that the HCIZ (Harish-Chandra-Itzykson-Zuber) integral is a Toda τ function. This result is closely related to Theorem 1, since the partition function $Z_n(A)$ is usually evaluated by the HCIZ formula,[7] although we do not use it in this paper, and KP τ functions are closely related to Toda τ functions.

$Z_n(A)$ is also very similar to the partition function of the generalized Kontsevich model (GKM) $Z_N^{\{V\}}[M]$ (following the notation in Ref. 17), which is also proven to be a KP τ function, with the negative power sums of eigenvalues of M as the KP flow parameters. See Refs. 17, 15, 23, and 22, for details. The definition of $Z_N^{\{V\}}[M]$ in the GKM is more complicated, and there is a crucial difference between $Z_N^{\{V\}}[M]$ and $Z_n(A)$: For different dimensions N, $Z_N^{\{V\}}[M]$ are specializations of the same τ function, however, $\hat{Z}_n(\mathbf{t})$ are different τ functions when the dimensions n are different. For example, for different n, the Virasoro constraints for $\hat{Z}_n(\mathbf{t})$ are not identical, and we can no longer find the Virasoro constraints by the trick "taking the large N limit," which works for the GKM.

$Z_n(A)$ was studied as a multicomponent KP τ function by Adler and van Moerbeke *et al.* In Ref. 1 it is proven that the determinant of a moment matrix for several weights, after adding deformation parameters, is a multicomponent KP τ function. By de Bruijn's formula (Lemma 3), we find that $Z_n(A)$ is a special case of these determinants of moment matrices. Especially, when eigenvalues of A have only two distinct values, $Z_n(A)$ is a three-component KP τ function, and is detailed in Ref. 2. For the nondegenerate case, i.e., eigenvalues $a_1, \ldots, a_n$ of A are distinct, $Z_n(A)$ is a $(1+n)$-component KP τ function.

The intriguing fact is that our KP τ function structure is independent of this multicomponent KP τ function structure. In particular, our set of KP flow parameters $\mathbf{t}$ is none of these sets of KP flow parameters in Ref. 1.

Remark 1: *With the help of Virasoro constraints, it is shown in Ref. 2 that for quadratic V, $Z_n^V(A;E)$ in (5) satisfies a PDE, which is derived from one Hirota bilinear equation of the multicomponent KP τ function. We can also find Virasoro constraints for $\hat{Z}_n(\mathbf{t})$ as the KP τ function and get new PDEs satisfied by $Z_n^V(A;E)$. It will be done in a forthcoming paper.*[29]

In Sec. II we summarize necessary preliminaries. Then in Sec. III we give the definition of $\hat{Z}_n(\mathbf{t})$ and prove Theorem 1.

II. BOSON-FERMION CORRESPONDENCE AND KP τ FUNCTIONS

The definition of KP τ functions follows that in Ref. 14, and all materials on symmetric functions are from Ref. 27.

KP τ functions can be defined through representations of the Heisenberg algebra, an associative algebra. Over any field K with characteristic 0, such as $\mathbb{R}$ or $\mathbb{C}$, the Heisenberg algebra H is generated by $\{h_k\}_{k \in \mathbb{Z}\setminus\{0\}}$, satisfying

$$[h_k, h_l] = k\delta_{k,-l}. \tag{11}$$

We can construct a representation of H over the so-called boson Fock space, which is $K[t_1, t_2, \ldots]$, the space of polynomials with infinitely many variables. h_k's $(k \leq -1)$ act as multiplication operators, and h_k's $(k \geq 1)$ act as derivations,

$$h_k \to \begin{cases} -kt_{-k} & \text{for } k \leq -1 \\ \dfrac{\partial}{\partial t_k} & \text{for } k \geq 1. \end{cases} \tag{12}$$

This representation of H is called the Boson representation.

On the other hand, H has another representation over the so-called fermion Fock space. To define the fermion Fock space, we take an infinite dimensional vector space V with basis $\{v_i\}_{i \in \mathbb{Z}}$. The fermion Fock space Λ is composed of semi-infinite forms, spanned by the basis $v_{i_0, i_{-1}, i_{-2}, \ldots}$, which are defined as

$$v_{i_0, i_{-1}, i_{-2}, \dots} = v_{i_0} \wedge v_{i_{-1}} \wedge v_{i_{-2}} \wedge \dots, \tag{13}$$

with $i_0, i_{-1}, \dots$ strictly decreasing and $i_{-k} = -k$ for k sufficiently large.

We call the semi-infinite form $v_{0,-1,-2,\dots} = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots$ the *vacuum*, following the physical terminology. Later in this paper, we use *form* to mean a semi-infinite form, unless otherwise claimed.

Formally, we define the action of h_i on V by

$$h_k(v_l) = v_{l-k} \tag{14}$$

and get the induced action on Λ by

$$h_k(v_{i_0, i_{-1}, i_{-2}, \dots}) = \sum_{j \geq 0} v_{i_0} \wedge v_{i_{-1}} \wedge \cdots \wedge v_{i_{-j+1}} \wedge v_{i_{-j}-k} \wedge v_{i_{-j-1}} \wedge \cdots. \tag{15}$$

Although the action of h_k on V is not consistent with the commutation relation (11) of H, the action of h_k on Λ defines a representation of H. This is the fermion representation of H.

We can observe that the boson representation and the fermion representation of H are equivalent. The correspondence Φ between $K[t_1, t_2, \dots]$ and Λ is

$$\Phi(1) = \text{vacuum},$$

$$\Phi(kt_k) = h_{-k}(\text{vacuum}),$$

$$\Phi(f(t_1, 2t_2, \dots)) = f(h_{-1}, h_{-2}, \dots)(\text{vacuum}), \tag{16}$$

where $k \geq 0$ and f is a polynomial. Since $[h_k, h_l] = 0$ for $k, l \in \mathbb{Z}^-$, the polynomial of operators $f(h_{-1}, h_{-2}, \dots)$ is well defined. Although it is not difficult to check the validity of the correspondence Φ, the images of monomials on the boson Fock space become messy combinations of the basis of Λ. It is an interesting question what the preimage of $v_{i_0, i_{-1}, i_{-2}, \dots}$ is. The answer is nontrivial and can be best formulated in notations of symmetric functions.

Proposition 1: *Let* $i_0 = \kappa_0$, $i_{-1} = \kappa_1 - 1$, $i_{-2} = \kappa_2 - 2, \dots$ *with* $\kappa_0 \geq \kappa_1 \geq \kappa_2 \geq \dots$, *such that* $(\kappa_0, \kappa_1, \dots) = \kappa$ *is a partition, then* $\Phi^{-1}(v_{i_0, i_{-1}, i_{-2}, \dots}) = \tilde{s}_\kappa(\mathbf{t})$, *where* $\tilde{s}_\kappa$ *is a polynomial in the definition of Schur functions by power sums: If we regard* $\{t_k\}$ *as power sums defined in (8), then for any* n *the Schur functions* s_κ *satisfies*

$$s_\kappa(a_1, \dots, a_n) = \tilde{s}_\kappa(\mathbf{t}). \tag{17}$$

Since every symmetric function can be written uniquely as a polynomial of power sums, $\tilde{s}_\kappa$ is well defined.

To define KP τ functions, we need the concept of decomposability of forms. We call $v \in \Lambda$ decomposable, if and only if

$$v = u_0 \wedge u_{-1} \wedge u_{-2} \wedge \dots, \tag{18}$$

where u_i's are linear combinations of v_i's. Now we are ready to give the definition.

Definition 1: $f(\mathbf{t})$ *is a KP τ function if and only if* $f = \Phi^{-1}(v)$, *where* v *is a decomposable form.*

Here we note that the $\hat{Z}_n(\mathbf{t})$ that we are going to study is not a polynomial in $t_1, t_2, \dots$, but rather a series. We can generalize the correspondence Φ such that it maps "convergent" series $f(\mathbf{t})$ into a form that is an infinite sum of $v_{i_0, i_{-1}, i_{-2}, \dots}$. Then we define that $f(\mathbf{t})$ is a KP τ function if and only if $\Phi(f)$ is a decomposable form. Thus Definition 1 is also applicable to $\hat{Z}_n(\mathbf{t})$. For the rigorous statement of convergence condition, see Ref. 26.

III. PROOF OF THEOREM 1

In order to define $\hat{Z}_n(\mathbf{t})$, we first expand $Z_n(A)$ in Schur polynomials. By the Weyl integral formula, we have

$$Z_n(A) = \frac{1}{C_n} \int \cdots \int \Delta(\lambda)^2 \int_{U(n)} e^{\mathrm{Tr}(AU \, \mathrm{diag}(\lambda)U^{-1})} dU d\mu(\lambda_1) \ldots d\mu(\lambda_n), \tag{19}$$

where C_n is a constant and dU is the Haar measure over $U(n)$. Then we use the identity[18]

$$\int_{U(n)} e^{\mathrm{Tr}(AU \, \mathrm{diag}(\lambda)\mathrm{U}^{-1})} dU = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \le n}} \frac{C_\kappa(l_1, \ldots, l_n) C_\kappa(\lambda_1, \ldots, \lambda_n)}{C_\kappa(1, \ldots, 1)}, \tag{20}$$

where κ's are partitions of k, $l(\kappa)$ is the length of κ, and C_κ's are constant multiples of Schur polynomials s_κ, with the normalization[11]

$$\sum_{\kappa \vdash k} C_\kappa(x_1, \ldots, x_n) = (x_1 + \ldots + x_n)^k. \tag{21}$$

Furthermore, we know[27,11]

$$C_\kappa = \frac{k!}{H(\kappa)} s_\kappa \quad \text{and} \quad s_\kappa(1, \ldots, 1) = \frac{(n)_\kappa}{H(\kappa)}, \tag{22}$$

where $H(\kappa)$ is the hook length product of κ and $(n)_\kappa$ is the extended Pochhammer symbol: if $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_l)$, then

$$(n)_\kappa = \prod_{i=1}^{l} \prod_{j=1}^{\kappa_i} (n - i + j). \tag{23}$$

Thus if we denote

$$G_\kappa = \int \cdots \int \Delta^2(\lambda) s_\kappa(\lambda_1, \ldots, \lambda_n) d\mu(\lambda_1) \cdots d\mu(\lambda_n), \tag{24}$$

we have

$$Z_n(A) = \frac{1}{C_n} \int \cdots \int \Delta(\lambda)^2 \sum_{k=0}^{\infty} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \le n}} \frac{s_\kappa(a_1, \ldots, a_n) s_\kappa(\lambda_1, \ldots, \lambda_n)}{(n)_\kappa} d\mu(\lambda_1) \ldots d\mu(\lambda_n)$$

$$= \frac{1}{C_n} \sum_{k=0}^{\infty} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \le n}} \frac{1}{(n)_\kappa} G_\kappa s_\kappa(a_1, \ldots, a_n), \tag{25}$$

and we define

$$\hat{Z}_n(\mathbf{t}) = \frac{1}{C_n} \sum_{k=0}^{\infty} \sum_{\substack{\kappa \vdash k \\ l(\kappa) \le n}} \frac{1}{(n)_\kappa} G_\kappa \tilde{s}_\kappa(\mathbf{t}), \tag{26}$$

where $\tilde{s}_\kappa$ is the polynomial defined in (17). Then it is clear that (9) holds.

To determine whether $\hat{Z}_n(\mathbf{t})$ is a KP function, we turn to the fermion representation and map it as

$$\Phi(\hat{Z}_n(\mathbf{t})) = \frac{1}{C_n}\mathcal{V}, \quad \text{with } \mathcal{V} = \sum_{k=0}^{\infty}\sum_{\substack{\kappa \vdash k \\ l(\kappa) \leq n}} \frac{1}{(n)_\kappa}G_\kappa v_\kappa, \tag{27}$$

where for $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_l)$, we denote $v_\kappa = v_{0+\kappa_0, -1+\kappa_1, \ldots, -l+\kappa_l, -l-1, \ldots}$.

To prove that $\hat{Z}_n(\mathbf{t})$ is a KP τ function is equivalent to prove that $\mathcal{V}$ is a decomposable form. We have a simple criterion (Plücker relations) for decomposability of forms, and first introduce two kinds of linear operators $v_i \wedge (v)$ and $\iota_{v_i^*}(v)$ on any $v \in \Lambda$,

$$v_i \wedge (v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) = v_i \wedge v_{i_0} \wedge v_{i_{-1}} \wedge \cdots, \tag{28}$$

$$\iota_{v_i^*}(v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) = \begin{cases} (-1)^j v_{i_0} \wedge \cdots \wedge v_{i_{-j+1}} \wedge v_{i_{-j-1}} \wedge \cdots & \text{if } v_{i_{-j}} = v_i \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Here we note that $v_i \wedge (v)$ and $\iota_{v_i^*}(v)$ are not in Λ: they are in Λ^+ and Λ^-, respectively. Λ^+ is spanned by forms $v_{i_0, i_{-1}, \ldots}^+ = v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$, such that $i_0, i_{-1}, \ldots$ are strictly decreasing and $i_{-k} = -k+1$ for k sufficiently large; Λ^- is spanned by forms $v_{i_0, i_{-1}, \ldots}^- = v_{i_0} \wedge v_{i_{-1}} \wedge \cdots$, such that $i_0, i_{-1}, \ldots$ are strictly decreasing and $i_{-k} = -k-1$ for k sufficiently large.

Now we can state the criterion.[14]

Proposition 2: *$v \in \Lambda$ is decomposable, if and only if the tensor of forms in $\Lambda^+ \otimes \Lambda^-$,*

$$\sum_{i=-\infty}^{\infty} v_i \wedge (v) \otimes \iota_{v_i^*}(v) = 0. \tag{30}$$

For notational simplicity, for a partition $\kappa = (\kappa_0, \ldots, \kappa_l)$, we denote v_κ^+ and v_κ^-, analogous to v_κ,

$$v_\kappa^+ = v_{1+\kappa_0, 0+\kappa_1, \ldots, -l+1+\kappa_l, -l, \ldots}, \quad v_\kappa^- = v_{-1+\kappa_0, -2+\kappa_1, \ldots, -l-1+\kappa_l, -l-2, \ldots}. \tag{31}$$

For κ a partition and i an integer, we define partitions $\kappa + i$ and $\kappa - i$. First, $\kappa = (\kappa_0, \ldots, \kappa_l)$ corresponds to a $v_\kappa^- \in \Lambda^-$, whose subscript $(-1+\kappa_0, -2+\kappa_1, \ldots, -l-1+\kappa_l, -l-2, \ldots)$ is a decreasing sequence of integers. If $i \in \{-1+\kappa_0, -2+\kappa_1, \ldots, -l-1+\kappa_l, -l-2, \ldots\}$, then we say $\kappa + i$ is not well defined; otherwise we can arrange elements in $\{-1+\kappa_0, -2+\kappa_1, \ldots, -l-1+\kappa_l, -l-2, \ldots\} \cup \{i\}$ into a decreasing sequence which is the subscript of a form $v_{\kappa'} \in \Lambda$, and we define $\kappa + i = \kappa'$. Symmetrically, we can define $\kappa - i$: If $i \notin \{1+\kappa_0, 0+\kappa_1, \ldots, -l+1+\kappa_l, -l, \ldots\}$, then $\kappa - i$ is not well defined; otherwise we can arrange $\{1+\kappa_0, 0+\kappa_1, \ldots, -l+1+\kappa_l, -l, \ldots\} \setminus \{i\}$ into a decreasing sequence, which is the subscript of a $v_{\kappa''} \in \Lambda$, and we define $\kappa - i = \kappa''$.

Now we consider the tensor of forms $\sum_{i=-\infty}^{\infty} v_i \wedge (\mathcal{V}) \otimes \iota_{v_i^*}(\mathcal{V}) \in \Lambda^+ \otimes \Lambda^-$ and compute its coefficients of every $v_\alpha^+ \otimes v_\beta^-$ term. By book keeping, we get the result.

Lemma 1: *For any two partitions α and β, the coefficient of the $v_\alpha^+ \otimes v_\beta^-$ term of $\mathcal{V}$ is*

$$c(\alpha, \beta) = \sum_{\substack{i \in \mathbb{Z}; \\ \alpha-i \text{ and } \beta+i \text{ are both well defined;} \\ l(\alpha-i) \leq n, \ l(\beta+i) \leq n}} \frac{\mathrm{sgn}^+(\alpha, i)\mathrm{sgn}^-(\beta, i)}{(n)_{\alpha-i}(n)_{\beta+i}}G_{\alpha-i}G_{\beta+i}, \tag{32}$$

where $\mathrm{sgn}^+(\alpha, i) = (-1)^j$ if $v_\alpha^+ = v_{a_0, a_{-1}, \ldots}$ and $i = a_{-j}$; $\mathrm{sgn}^-(\beta, i) = (-1)^{j'}$ if $v_\beta^- = v_{b_0, b_{-1}, \ldots}$ and $b_{-j'} < i < b_{-j'+1}$.

To simplify $c(\alpha, \beta)$, we first prove the following.

Lemma 2: *For any partitions α and β, and integer i such that $\alpha - i$ and $\beta + i$ are both well defined, we have for any $n \in \mathbb{Z}^+$*

$$(n)_{\alpha-i}(n)_{\beta+i} = n(n+1)_\alpha(n-1)_\beta. \tag{33}$$

Proof: Let $\alpha=(\alpha_0,\alpha_1,\ldots,\alpha_q)$, $\beta=(\beta_0,\beta_1,\ldots,\beta_r)$, and $(j=0,1,\ldots)$

$$a_{-j}=\begin{cases}\alpha_j-j+1 & \text{for } j\leq q\\ -j+1 & \text{for } j>q,\end{cases}\tag{34}$$

$$b_{-j}=\begin{cases}\beta_j-j-1 & \text{for } j\leq r\\ -j-1 & \text{for } j>r.\end{cases}\tag{35}$$

We assume that $i=a_{-k}$ and $b_{-l}>i>b_{-l-1}$, then we have

$$(n)_{\alpha-i}=\left(\prod_{j=0}^{k-1}\prod_{p=n-j}^{n+a_{-j}-1}p\right)\left(\prod_{j=k+1}^{\infty}\prod_{p=n-j+1}^{n+a_{-j}-1}p\right),\tag{36}$$

$$(n)_{\beta+i}=\left(\prod_{j=0}^{l}\prod_{p=n-j}^{n+b_{-j}-1}p\right)\left(\prod_{p=n-l-1}^{n+i-1}p\right)\left(\prod_{j=l+1}^{\infty}\prod_{p=n-j-1}^{n+b_{-j}-1}p\right).\tag{37}$$

Here we take the convention that $\Pi_{p'}^{p''}=1$ if $p''<p'$.

On the other hand, we have

$$(n+1)_{\alpha}=\left(\prod_{j=0}^{k-1}\prod_{p=n-j+1}^{n+a_{-j}-1}p\right)\left(\prod_{p=n-k-1}^{n+a_{-k}-1}p\right)\left(\prod_{j=k+1}^{\infty}\prod_{p=n-j+1}^{n+a_{-j}-1}p\right),\tag{38}$$

$$(n-1)_{\beta}=\left(\prod_{j=0}^{l}\prod_{p=n-j-1}^{n+b_{-j}-1}p\right)\left(\prod_{j=l+1}^{\infty}\prod_{p=n-j-1}^{n+b_{-j}-1}p\right).\tag{39}$$

Thus, we get

$$\frac{(n)_{\alpha-i}(n)_{\beta+i}}{(n+1)_{\alpha}(n-1)_{\beta}}=\frac{\prod_{j=0}^{k-1}(n-j)\prod_{n-l-1}^{n+i-1}p}{\prod_{j=0}^{l}(n-j-1)\prod_{n-k+1}^{n+a_{-k}-1}p}.\tag{40}$$

Notice that $a_{-k}=i$, we can verify that

$$\frac{(n)_{\alpha-i}(n)_{\beta+i}}{(n+1)_{\alpha}(n-1)_{\beta}}=n,\tag{41}$$

and prove the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

From this result we observe that if both the $\alpha-i$ and $\beta+i$ are well defined, the condition $\max(l(\alpha-i),l(\beta+i))\leq n$ is equivalent to $\max(l(\alpha)-1,l(\beta)+1)\leq n$. Then we have for $l(\alpha)\leq n+1$ and $l(\beta)\leq n-1$,

$$c(\alpha,\beta)=\frac{1}{n(n+1)_{\alpha}(n-1)_{\beta}}\sum_{\substack{i\in\mathbb{Z};\\ \alpha-i \text{ and } \beta+i \text{ are}\\ \text{both well defined;}\\ \max(l(\alpha)-1,l(\beta)+1)\leq n}}\text{sgn}^+(\alpha,i)\text{sgn}^-(\beta,i)G_{\alpha-i}G_{\beta+i},\tag{42}$$

and later in this section we assume $\max(l(\alpha)-1,l(\beta)+1)\leq n$. We find that for all but finitely many $i\in\mathbb{Z}$, either $G_{\alpha-i}$ or $G_{\beta+i}$ is not well defined, and we can write (42) as a finite summation,

$$c(\alpha,\beta) = \frac{1}{n(n+1)_\alpha(n-1)_\beta} \sum_{j=0}^{n} \operatorname{sgn}^+(\alpha,a_{-j})\operatorname{sgn}^-(\beta,a_{-j}) G_{\alpha-a_{-j}} G_{\beta+a_{-j}}, \tag{43}$$

where a_{-j} is given by (34), and we assume $G_{\beta+a_{-j}}=0$ if $\beta+a_{-j}$ is not well defined.

Now we recall the determinantal formula for Schur polynomials[28] that if $\kappa=(\kappa_0,\ldots,\kappa_l)$ with $l\le n$, then

$$s_\kappa(x_1,\ldots,x_n) = \frac{\begin{vmatrix} x_1^{n-1+\kappa_0} & x_2^{n-1+\kappa_0} & \ldots & x_n^{n-1+\kappa_0} \\ x_1^{n-2+\kappa_1} & x_2^{n-2+\kappa_1} & \ldots & x_n^{n-2+\kappa_1} \\ \vdots & \vdots & \ldots & \vdots \\ x_1^{n-l-1+\kappa_l} & x_2^{n-l-1+\kappa_l} & \ldots & x_n^{n-l-1+\kappa_l} \\ x_1^{n-l-2} & x_2^{n-l-2} & \ldots & x_n^{n-l-2} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 1 & \ldots & 1 \end{vmatrix}}{\Delta(x_1,\ldots,x_n)}. \tag{44}$$

Therefore

$$\Delta(\lambda)^2 s_\kappa(\lambda_1,\ldots,\lambda_n) = \Delta(\lambda) \begin{vmatrix} \lambda_1^{n-1+\kappa_0} & \ldots & \lambda_n^{n-1+\kappa_0} \\ \lambda_1^{n-2+\kappa_1} & \ldots & \lambda_n^{n-2+\kappa_1} \\ \vdots & \ldots & \vdots \\ 1 & \ldots & 1 \end{vmatrix}. \tag{45}$$

To simplify the integrals in (42), we need another formula.[10]

Lemma 3: (de Bruijn's) *For any* $f_0,\ldots,f_{n-1},g_0,\ldots,g_{n-1}\in L^2(\mathbb{R})$,

$$\int\cdots\int d\lambda_1\ldots d\lambda_n \begin{vmatrix} f_{n-1}(\lambda_1) & f_{n-1}(\lambda_2) & \ldots & f_{n-1}(\lambda_n) \\ f_{n-2}(\lambda_1) & f_{n-2}(\lambda_2) & \ldots & f_{n-2}(\lambda_n) \\ \vdots & \vdots & \ldots & \vdots \\ f_0(\lambda_1) & f_0(\lambda_2) & \ldots & f_0(\lambda_n) \end{vmatrix} \begin{vmatrix} g_{n-1}(\lambda_1) & g_{n-1}(\lambda_2) & \ldots & g_{n-1}(\lambda_n) \\ g_{n-2}(\lambda_1) & g_{n-2}(\lambda_2) & \ldots & g_{n-2}(\lambda_n) \\ \vdots & \vdots & \ldots & \vdots \\ g_0(\lambda_1) & g_0(\lambda_2) & \ldots & g_0(\lambda_n) \end{vmatrix}$$

$$= n!\, \det\left(\int f_i(x)g_j(x)dx\right)_{0\le i,\ j\le n-1}. \tag{46}$$

Now we denote

$$G_i = \int x^i d\mu(x), \tag{47}$$

and by (45) and Lemma 3, for $0\le j\le n$ we have [a_{-j} is given by (34)]

$$G_{\alpha-a_{-j}} = \int \cdots \int \Delta(\lambda)^2 s_{\alpha-a_{-j}}(\lambda_1, \ldots, \lambda_n) d\mu(\lambda_1) \ldots d\mu(\lambda_n)$$

$$= n! \begin{vmatrix} G_{n-1+a_0} & G_{n-1+a_0+1} & \ldots & G_{n-1+a_0+n-1} \\ G_{n-1+a_{-1}} & G_{n-1+a_{-1}+1} & \ldots & G_{n-1+a_{-1}+n-1} \\ \vdots & \vdots & \ldots & \vdots \\ \hat{G}_{n-1+a_{-j}} & \hat{G}_{n-1+a_{-j}+1} & \cdots & \hat{G}_{n-1+a_{-j}+n-1} \\ \vdots & \vdots & \ldots & \vdots \\ G_{n-1+a_{-n}} & G_{n-1+a_{-n}+1} & \ldots & G_{n-1+a_{-n}+n-1} \end{vmatrix}, \tag{48}$$

where $\hat{\ }$ means the entry is deleted. For $\beta+a_{-j}$, we denote $v_\beta^+ = b_{b_0,b_{-1},\ldots}$, and similarly have that if $\beta+a_{-j}$ is well defined and $l(\beta+a_{-j}) \leq n$, then

$$\mathrm{sgn}^-(\beta, a_{-j}) G_{\beta+a_{-j}} = \mathrm{sgn}^-(\beta, a_{-j}) \int \cdots \int \Delta(\lambda)^2 s_{\beta+a_{-j}}(\lambda_1, \ldots, \lambda_n) d\mu(\lambda_1) \ldots d\mu(\lambda_n)$$

$$= n! \begin{vmatrix} G_{n-1+a_{-j}} & G_{n-1+a_{-j}+1} & \ldots & G_{n-1+a_{-j}+n-1} \\ G_{n-1+b_0} & G_{n-1+b_0+1} & \ldots & G_{n-1+b_0+n-1} \\ G_{n-1+b_{-1}} & G_{n-1+b_{-1}+1} & \ldots & G_{n-1+b_{-1}+n-1} \\ \vdots & \vdots & \ldots & \vdots \\ G_{n-1+b_{-n+2}} & G_{n-1+b_{-n+2}+1} & \ldots & G_{n-1+b_{-n+2}+n-1} \end{vmatrix}. \tag{49}$$

Here we notice that the $n \times n$ matrix in (48) is constructed from an $(n+1) \times n$ matrix with the $(j+1)$th row eliminated, and the first row in the matrix in (49) is the same as the deleted row in the construction of the matrix in (48). Thus we have a determinantal formula,

$$\sum_{j=0}^{n} \mathrm{sgn}^+(\alpha, a_{-j}) \mathrm{sgn}^-(\beta, a_{-j}) G_{\alpha-a_{-j}} G_{\beta+a_{-j}} = (n!)^2 \begin{vmatrix} B_0 & B_1 & \ldots & B_{n-1} \\ G_{n-1+b_0} & G_{n-1+b_0+1} & \ldots & G_{n-1+b_0+n-1} \\ G_{n-1+b_{-1}} & G_{n-1+b_{-1}+1} & \ldots & G_{n-1+b_{-1}+n-1} \\ \vdots & \vdots & \ldots & \vdots \\ G_{n-1+b_{-n+2}} & G_{n-1+b_{-n+2}+1} & \ldots & G_{n-1+b_{-n+2}+n-1} \end{vmatrix}, \tag{50}$$

where all rows except for the first one are the same as those in the matrix in (49), and the first row $(B_0, B_1, \ldots, B_{n-1})$ is

$$(B_0, B_1, \ldots, B_{n-1}) = \sum_{j=0}^{n} \mathrm{sgn}^+(\alpha, a_{-j}) G_{\alpha-a_{-j}} (G_{n-1+a_{-j}}, G_{n-1+a_{-j}+1}, \ldots, G_{n-1+a_{-j}+n-1}). \tag{51}$$

Notice that $\mathrm{sgn}^+(\alpha, a_{-j}) = (-1)^j$, we find that (51) is equivalent to that B_k is the determinant of a $(n+1) \times (n+1)$ matrix $(k=0,1,\ldots,n-1)$,

$$B_k = \begin{vmatrix} G_{n-1+a_0+k} & G_{n-1+a_0} & G_{n-1+a_0+1} & \ldots & G_{n-1+a_0+n-1} \\ G_{n-1+a_{-1}+k} & G_{n-1+a_{-1}} & G_{n-1+a_{-1}+1} & \ldots & G_{n-1+a_{-1}+n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ G_{n-1+a_{-n}+k} & G_{n-1+a_{-n}} & G_{n-1+a_{-n}+1} & \ldots & G_{n-1+a_{-n}+n-1} \end{vmatrix}. \tag{52}$$

Since in the matrix, the first column is identical to the $(k+2)$th column, we get $B_k = 0$. Therefore $\sum_{j=0}^{n} \mathrm{sgn}^+(\alpha, a_{-j}) \mathrm{sgn}^-(\beta, a_{-j}) G_{\alpha-a_{-j}} G_{\beta+a_{-j}} = 0$, which means that $c(\alpha, \beta) = 0$ by (43).

Summing up these results, we get the conclusion that $\mathcal{V}$ is a decomposable form, and by (27) prove Theorem 1.

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