Biorthogonal polynomials related to quantum transport theory of disordered wires

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Abstract

We consider the Plancherel-Rotach type asymptotics of the biorthogonal polynomials associated to the biorthogonal ensemble with the joint probability density function

$$\frac{1}{C}\prod_{1\le i<j\le n}(\lambda_j-\lambda_i)(f(\lambda_j)-f(\lambda_i))\prod_{j=1}^{n}W_\alpha^{(n)}(\lambda_j)d\lambda_j,$$

where

$$f(x)=\sinh^2(\sqrt{x}), \qquad W_\alpha^{(n)}(x)=x^\alpha h(x)e^{-nV(x)}.$$

In the special case that the potential function V is linear, this biorthogonal ensemble arises in the quantum transport theory of disordered wires. We analyze the asymptotic problem via 2-component vector-valued Riemann-Hilbert problems, and solve it under the one-cut regular with a hard edge condition.

As a consequence of our result, we observe that the equilibrium measure of the biorthogonal ensemble with linear V is the limiting density of particles in the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation with the ballistic initial condition.

1 Introduction

1.1 Setup of the model

Let

$$f(x)=\frac{1}{4}\left(e^{2\sqrt{x}}-2+e^{-2\sqrt{x}}\right)=\sinh^2(\sqrt{x}) \tag{1.1}$$

On $[0,\infty)$. Let V be a real analytic function on $[0,\infty)$ and h be a positive valued real analytic function $h(x)$ on $[0,\infty)$, satisfying the limit condition

$$\lim_{x\to+\infty}\frac{V(x)}{\max(\log h(x),\sqrt{x}+1)}=+\infty. \tag{1.2}$$

We then denote the weight function $W_\alpha^{(n)}(x)$, depending on the parameter $\alpha>-1$ and , as

$$W_\alpha^{(n)}(x)=x^\alpha h(x)e^{-nV(x)}. \tag{1.3}$$

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We consider the monic polynomials $p_j^{(n)}(x)$ and $q_j^{(n)}(x)$, of degree $j \geq 0$, determined by the orthogonality conditions

$$\int_{\mathbb{R}_+} p_j^{(n)}(x) f(x)^k W_\alpha^{(n)}(x) dx = 0, \quad \int_{\mathbb{R}_+} x^k q_j^{(n)}(f(x)) W_\alpha^{(n)}(x) dx = 0, \quad k = 0, 1, \dots, j-1, \tag{1.4}$$

and define

$$h_j^{(n)} = \int_{\mathbb{R}_+} p_j^{(n)}(x) q_j^{(n)}(f(x)) W_\alpha^{(n)}(x) dx. \tag{1.5}$$

Because $p_j^{(n)}$ and $q_j^{(n)}$ satisfy the biorthogonal condition (1.4), they are called the biorthogonal polynomials with respect to $W_\alpha^{(n)}$.

In this paper, we are mainly concerned about the Plancherel-Rotach type asymptotics of $p_{n+k}^{(n)}(x)$ and $q_{n+k}^{(n)}(f(x))$, as $n \to \infty$ and k is a fixed integer.

These biorthogonal polynomials are related to the point process consisting of n particles at $\lambda_1, \dots, \lambda_n \in [0, \infty)$, with the joint probability density function (pdf)

$$\frac{1}{C} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)(f(\lambda_j) - f(\lambda_i)) \prod_{j=1}^{n} W_\alpha^{(n)}(\lambda_j) d\lambda_j. \tag{1.6}$$

Below we call the point process defined by (1.6) the biorthogonal ensemble.

The first relation between $p_j^{(n)}, q_j^{(n)}$ and the biorthogonal ensemble (1.6) is:

Proposition 1.1. *$p^{(n)}(x)$ and $q^{(n)}(f(x))$ have the representation*

$$p_n^{(n)}(z) = \mathbb{E}\left[\prod_{j=1}^{n}(z - \lambda_j)\right], \qquad q_n^{(n)}(f(z)) = \mathbb{E}\left[\prod_{j=1}^{n}(f(z) - f(\lambda_j))\right], \tag{1.7}$$

where $\mathbb{E}[\cdot]$ is with respect to the joint pdf (1.6).

he proof of the proposition is analogous to the proofs of [7, Proposition 2.1] and [13, Proposition 1], and we omit it. The most important consequence of this proposition in our paper is the existence and uniqueness of $p_n^{(n)}, q_n^{(n)}$, as well as $p_j^{(n)}, q_j^{(n)}$ with general j, if we allow the weight function $W_\alpha^{(n)}$ to be modified by a simple scaling transform.

The next and more important relation between $p_j^{(n)}, q_j^{(n)}$ and the biorthogonal ensemble (1.6) is:

Proposition 1.2. *The biorthogonal ensemble with joint pdf (1.6) is a determinantal point process, and the following $K_n(x, y)$ is its correlation kernel*

$$K_n(x, y) = \sqrt{W_\alpha^{(n)}(x) W_\alpha^{(n)}(y)} \sum_{j=0}^{n-1} \frac{1}{h_j^{(n)}} p_j^{(n)}(x) q_j^{(n)}(f(y)). \tag{1.8}$$

The proof is omitted. It is based on the general theory of determinantal point processes, see [37] for the framework and [8, Section 2] for a short discussion on the kernel formula of a similar model.

By Proposition 1.2, the Plancherel-Rotach type asymptotic result also implies the local limiting distribution of particles in this biorthogonal ensemble.

1.2 Motivation

Our study of the biorthogonal ensemble defined by the general weight function (1.3) is inspired by the concrete case with the weight function specialized by

$$V(x) = \frac{x}{\mathbb{M}}, \qquad \alpha = 0, \qquad \text{and} \qquad h(x) = f'(x)^{\frac{1}{2}} = \left(\frac{\sinh(\sqrt{x})\cosh(\sqrt{x})}{\sqrt{x}} \right)^{\frac{1}{2}}, \tag{1.9}$$

which is proposed in the study of the quantum transport theory of a disordered wire. See [4, Formula (19)] and [5, Section III A, Formula (3.4)] for the physical derivation of the biorthogonal ensemble. (In [5], $\mathbb{M}$ is denoted as s and n is denoted as N). We also refer the interested readers to the review article [3] for the physical theory that relates the biorthogonal ensemble with specialization (1.9).

For the purpose of our paper, it suffices to note that the biorthogonal ensemble with specialization (1.9) is an approximation of the distribution of the particles (representing the transmission eigenvalues) in the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation with the parameter $\beta = 2$ and the ballistic initial condition. The DMPK equation is the evolution equation for the density function $P(\tilde{\lambda}_1, \dots, \tilde{\lambda}_n; \mathbb{M})$ where $\mathbb{M}$ is the time parameter,

$$\frac{\partial P}{\partial \mathbb{M}} = \frac{1}{n} \sum_{j=1}^{n} \frac{\partial}{\partial \tilde{\lambda}_j} \left(\tilde{\lambda}_j (1 + \tilde{\lambda}_j) J \frac{\partial}{\partial \tilde{\lambda}_j} \frac{P}{J} \right), \qquad J = \prod_{i=1}^{n} \prod_{j=i+1}^{n} |\tilde{\lambda}_i - \tilde{\lambda}_j|^2. \tag{1.10}$$

(See [3, Equation (145)]. Here we take $\beta = 2$ in that formula, and use $\mathbb{M}$ to mean L/l there. In many occasions of [3], L/l is denoted as s.) $\tilde{\lambda}_j$ in (1.10) corresponds to $\sinh^2(\sqrt{\lambda_j})$ in (1.6). Here we remind the reders that the claim that the joint probability density function of $\{\sinh^2(\sqrt{\lambda_j})\}$ is approximately $P(\tilde{\lambda}_1, \dots, \tilde{\lambda}_n; \mathbb{M})$ was jusfified only in the regime that $\mathbb{M}, n$ are large and $1 \ll \mathbb{M} \ll n$. However, the approximation may be valid in the $\mathbb{M} = \mathcal{O}(1)$ and $n \to \infty$ (ballistic) regime. See Section 1.3.3.

The distribution of particles $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ in (1.10) (resp. $\lambda_1, \dots, \lambda_n$ in (1.6)) represents (resp. represents approximately) the distribution of the transmission eigenvalues of the disordered wire, and the sum

$$C_n(\mathbb{M}) := \sum_{j=1}^{n} \frac{1}{\cosh^2(\sqrt{\lambda_j})} \tag{1.11}$$

yields the conductance of the disordered wire at least in the regime $1 \ll \mathbb{M} \ll n$. Results from experimental physics imply the following mathematical results:

- Ohm's law
$$\lim_{\mathbb{M} \to \infty} M \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[C_n(\mathbb{M})] = 1. \tag{1.12}$$

- Universal conductance fluctuation
$$\lim_{\mathbb{M} \to \infty} \lim_{n \to \infty} \mathrm{Var}[C_n(\mathbb{M})] = \frac{1}{15}. \tag{1.13}$$

The framework of biorthogonal polynomials developed in our paper enables us to rigorously analyze the limiting distribution of $\lambda_1, \dots, \lambda_n$, and prove (1.12) and (1.13). We remark that the universal conductance fluctuation (1.13) has only been justified by physical argument [30], [28], [32], while the counterpart of (1.13) for a simpler model, the quantum dot, has been rigorously proved in [2], [22], [21], [36], and also see [23]. The Plancherel-Rotach type asymptotic result in this paper establishes the foundation to rigorously prove the universal conductance fluctuation (1.13).

1.3 Main results

1.3.1 Global results: Qualitative properties of equilibrium measure

Analogous to the determinantal point processes associated to orthogonal polynomials, (that is, replacing $f(\lambda_j) - f(\lambda_i)$ by $(\lambda_j - \lambda_i)$ in the joint pdf (1.6)), we define the equilibrium measure supported on $[0, \infty)$ as the minimizer of the functional

$$I_V(\mu) := \frac{1}{2}\iint \log|t-s|^{-1} d\mu(t)d\mu(s) + \frac{1}{2}\iint \log|f(t)-f(s)|^{-1} d\mu(t)d\mu(s) + \int V(s)d\mu(s). \tag{1.14}$$

Proposition 1.3. *Let V be a continuous function on $[0, \infty)$ that satisfies (1.2) with $h(x) = 1$. Then there exists a unique measure $\mu = \mu_V$ on $\mathbb{R}_+$ with compact support which minimizes the functional among all Borel probability measures on $\mathbb{R}_+$.*

The basic idea of the proof is contained in [17], which gives a detailed proof of the result if $f(x)$ is changed into x and the integer domain is changed into $\mathbb{R}$. If $f(x)$ is changed into e^x and the integral domain is changed into $\mathbb{R}$, an explanation is given in [13, Section 2]. Hence we omit the proof here.

Remark 1. Proposition 1.3 is only for the existence and uniqueness of $\mu = \mu_V$, and it does not tell us how to construct μ. Furthermore, it is only a potential theoretical result, and does not have direct relation to either the n-particle biorthogonal ensemble (1.6) or the biorthogonal polynomials $p^{(n)}_{n+k}, q^{(n)}_{n+k}$. Although we can follow the argument given in [17, Chapter 6] to establish the relations mentioned above, we do not pursue this approach.

We say that a potential function V satisfying (1.2) is "one-cut regular with a hard edge" if there exists an absolutely continuous measure $\mu = \mu_V$ such that

$$d\mu(x) = \psi(x)dx \quad \text{on } \mathbb{R}_+, \tag{1.15}$$

that satisfies:

Requirement 1.

1. $\operatorname{supp}\mu = [0, b]$ for $b > 0$ that depends on V, and $\int d\mu(x) = 1$.
2. $\psi(x)$ is continuous on $(0, b)$ and $\psi(x) > 0$ for $x \in (0, b)$.
3. For $x \in [0, b]$, there exists a constant ℓ depending on V such that

$$\int \log|t-x|^{-1} d\mu(t) + \int \log|f(t)-f(x)|^{-1} d\mu(t) + V(x) + \ell = 0. \tag{1.16}$$

4. For $x > b$,

$$\int \log|t-x|^{-1} d\mu(t) + \int \log|f(t)-f(x)|^{-1} d\mu(t) + V(x) + \ell > 0. \tag{1.17}$$

5. The two limits

$$\psi_0 := \lim_{x \to 0_+} x^{\frac{1}{2}}\psi(x) \qquad \text{and} \qquad \psi_b := \lim_{x \to b_-} (b-x)^{-\frac{1}{2}}\psi(x) \tag{1.18}$$

exist and both are positive.

It is clear that Item 1 implies that μ is a probability measure, and Item 2 further implies that $d\mu(x) = \psi(x)dx$ is a "one-cut" probability measure in the sense that its support is a compact interval and its probability density is positive everywhere in the interior of the support. Items 3 and 4 are slightly stronger than the Euler-Lagrange equation of the variational problem (1.14), so if $\psi(x)$ satisfies all Items 1, 2, 3 and 4, then $d\mu(x) = \psi(x)dx$ is the unique equilibrium measure defined by the minimization of I_V, as stated in Proposition 1.3. At last, Item 5 means that the equilibrium measure is regular at 0, the "hard edge", and b, the "soft edge". (The regularity of the equilibrium measure also includes Item 2 that means it is regular in the interior of the support, and includes Item 4 that means it is regular out of the support.) Throughout this paper, we only consider potential functions V that is one-cut regular with a hard edge.

Given a potential function V, generally it is hard to determine if V is one-cut regular with a hard edge. In this paper, we are satisfied with the following partial result:

Theorem 1.4. *If the potential function V is real analytic on $[0, \infty)$, satisfies (1.2), and*

$$U'(x) > 0 \quad \text{for all} \quad x \in (0, \infty), \quad \text{where} \quad U(x) = V'(x)\sqrt{x}. \tag{1.19}$$

then V is one-cut regular with a hard edge.

The proof of the theorem is given in Section 2.

1.3.2 Global results: quantitative properties of equilibrium measure

Given a one-cut regular with a hard edge potential V, it is still challenging to find the equilibrium measure μ_V, even its right end point b. If V satisfies the conditions in Theorem 1.4, then b and $\psi(x)$ can be computed in principal, but we need to define a few functions to state the result.

First we collect some properties of function f defined in (1.1).

Lemma 1.5. *$f(x)$ defined in (1.1) has a natural extension into an analytic function on $\mathbb{C}$, and it satisfies the follows:*

- *$f(x) \in \mathbb{R}_+$ for all $x \in \mathbb{R}_+$, and $f(x)$ increases from 0 to $+\infty$ as x runs from 0 to $+\infty$.*
- *$f(x) \in (-1, 0)$ for $x \in (-\pi^2/4, 0)$, and $f(x)$ increases from -1 to 0 as x runs from $-\pi^2/4$ to 0.*
- *As $x \in \mathbb{R}_+$ runs from 0 to $+\infty$, $f(\frac{1}{4}(x^2 - \pi^2) \pm \frac{1}{2}\pi x i) \in \mathbb{R}_-$ and it decreases from -1 to $-\infty$.*

Next, we define ρ the curve lying in $\mathbb{C}_-$ with formula

$$\rho = \{\frac{t^2}{\pi^2} - \frac{\pi^2}{4} - it \mid t \in (0, +\infty)\}. \tag{1.20}$$

Then $\rho \cup \{0\} \cup \bar{\rho}$ is a parabola. We define $\mathbb{P}$ as a region of $\mathbb{C}$ to the right of the parabola. From Lemma 1.5, we have that f : interior of $\mathbb{P} \to \mathbb{C} \setminus (-\infty, -1]$ is conformal, while f maps both ρ and $\bar{\rho}$ to $(-\infty, -1]$. Actually, if we glue ρ and $\bar{\rho}$ by identifying $z \in \rho$ with $\bar{z} \in \bar{\rho}$ and view $\mathbb{P}$ as a Riemann surface, then f is an conformal mapping between $\mathbb{P}$ and $\mathbb{C}$.

We define, for all $x \in \mathbb{R}_+$, the transformation

$$J_x(s) = x\sqrt{s} + \operatorname{arcosh}\left(\frac{s+1}{s-1}\right), \quad \text{and} \quad \mathbf{J}_x(s) = \frac{1}{4}(J_x(s))^2, \quad s \in \mathbb{C}_+ \cup (1, \infty), \tag{1.21}$$

such that $\sqrt{s}$ takes the principal branch on $\arg s \in (-\pi, \pi)$, and arcosh takes the branch that is the one-to-one mapping from $\mathbb{C} \setminus (-\infty, 1]$ to $\{s \in \mathbb{C} \mid \Re s > 0 \text{ and } -\pi < \Im s < \pi\}$. Below we

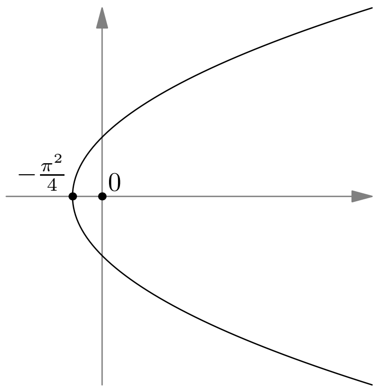

Figure 1: Shape of $\mathbb{P}$ (the region the the right of the parabola).

extend the domain of $J_x(s)$ and $\mathbf{J}_x(s)$. Naturally, They extend to $s \in \mathbb{C}_-$ by $J_x(\bar{s}) = \overline{J_x(s)}$ and $\mathbf{J}_x(\bar{s}) = \overline{\mathbf{J}_x(s))}$. $\mathbf{J}_x(s)$ also extends to $s \in (-\infty, 0)$ by continuation. For $s \in [0, 1]$, we leave $\mathbf{J}_x(s)$ undefined, since it has a branch cut there. We also define ad hoc that $J_x(s) = \lim_{\epsilon \to 0_+} J_x(s + \epsilon i)$ for $s \in (-\infty, 1)$.

Lemma 1.6. *$J_x(s)$ satisfies the following properties:*

1. *$J_x(s) \in \mathbb{R}_+$ for $s \in (1, +\infty)$; $J_x(s)$ decreases from $+\infty$ to $J_x(s_2(x)) = \sqrt{(x+1)^2 - 1} + \operatorname{arcosh}(x+1)$ as s runs from 1 to*
$$s_2(x) = 1 + 2/x, \tag{1.22}$$
and then it increases from $J_x(s_2(x))$ to $+\infty$ as s runs from $s_2(x)$ to ∞.

2. *$J_x(-s) \in i\mathbb{R}$ for $s \in (0, +\infty)$. $\Im J_x(-s)$ increases monotonically from $-\pi$ to $+\infty$ as s runs from 0 to ∞, and there is a unique $s_1(x) \in (-\infty, 0)$ such that $J_x(s_1(x)) = 0$. To be precise, $s_1(x)$ is the unique solution on $(-\infty, 0)$ of*
$$x\sqrt{-s} = \arccos\left(\frac{-s-1}{-s+1}\right). \tag{1.23}$$

3. *$J_x(s) + \pi i \in \mathbb{R}_+$ for $s \in (0, 1)$. $\Re J_x(s)$ increases from 0 to $+\infty$ as s runs from 0 to 1.*

4. *There is a unique curve $\gamma_1(x) \subseteq \mathbb{C}_+$ connecting $s_1(x)$ and $s_2(x)$, such that $J_x(z) \in \mathbb{R}$ as $z \in \gamma_1(x)$, and $J_x(z)$ increases from 0 to $J_x(s_2(x))$ as z moves from $s_1(x)$ to $s_2(x)$ along $\gamma_1(x)$.*

Part 4 of this lemma will be proved in Appendix A. Parts 1, 2 and 3 can be verified by direct computation, and we omit the detail.

Define
$$\gamma_2(x) = \{z \in \mathbb{C}_- \mid \bar{z} \in \gamma_1(x)\}, \tag{1.24}$$
Then $\gamma_1(x), \gamma_2(x)$ together enclose a region $D_x \subseteq \mathbb{C}$. In this paper, we orient $\gamma_1(x), \gamma_2(x)$ from $s_1(x)$ to $s_2(x)$, unless otherwise stated.

Lemma 1.7. *$\mathbf{J}_x$ maps $\mathbb{C} \setminus \overline{D_x}$ conformally to $\mathbb{C} \setminus [0, b(x)]$ and maps $D_x \setminus [0, 1]$ conformally to $\mathbb{P} \setminus [0, b(x)]$.*

This lemma will be proved in Appendix A.

For each $x \in (0, \infty)$, we let

$$b(x) = \mathbf{J}_x(s_2(x)) = \frac{1}{4} J_x(s_2(x))^2 = \frac{1}{4}\left(\sqrt{(x+1)^2 - 1} + \operatorname{arcosh}(x+1)\right)^2. \tag{1.25}$$

It is clear that $b(x)$ is a continuous function of x, and it increases monotonically from 0 to ∞ as x runs from 0 to ∞.

Let the functions $\mathbf{I}_{x,1}$ and $\mathbf{I}_{x,2}$ be inverse functions of $\mathbf{J}_x$, such that $\mathbf{I}_{x,1}$ is the inverse map of $\mathbf{J}_x$ from $\mathbb{C} \setminus [0, b(x)]$ to $\mathbb{C} \setminus \overline{D_x}$, and $\mathbf{I}_{x,2}$ is the inverse map of $\mathbf{J}_x$ from $\mathbb{P} \setminus [0, b(x)]$ to $D_x \setminus [0, 1]$:

$$\mathbf{I}_{x,1}(\mathbf{J}_x(s)) = s, \qquad s \in \mathbb{C} \setminus \overline{D_x}, \tag{1.26}$$

$$\mathbf{I}_{x,2}(\mathbf{J}_x(s)) = s, \qquad s \in D_x \setminus [0, 1]. \tag{1.27}$$

We then denote for $u \in (0, b(x))$

$$\mathbf{I}_{x,+}(u) := \lim_{\epsilon \to 0_+} \mathbf{I}_{x,1}(u + i\epsilon) = \lim_{\epsilon \to 0_+} \mathbf{I}_{x,2}(u - i\epsilon), \tag{1.28}$$

$$\mathbf{I}_{x,-}(u) := \lim_{\epsilon \to 0_+} \mathbf{I}_{x,1}(u - i\epsilon) = \lim_{\epsilon \to 0_+} \mathbf{I}_{x,2}(u + i\epsilon). \tag{1.29}$$

We have that $\mathbf{I}_{x,+}(x)$ lies in $\mathbb{C}_+$, $\mathbf{I}_{x,-}(x)$ lies in $\mathbb{C}_-$, and their loci are the upper and lower boundaries of D_x, that is, $\gamma_1(x)$ and $\gamma_2(x)$ respectively. For later use, we define for $\xi \in (0, b(x))$

$$F_x(u;\xi) = \log \left| \frac{(\sqrt{\mathbf{I}_{x,+}(u)} + \sqrt{\mathbf{I}_{x,+}(\xi)})(\sqrt{\mathbf{I}_{x,-}(u)} - \sqrt{\mathbf{I}_{x,+}(\xi)})}{(\sqrt{\mathbf{I}_{x,+}(u)} - \sqrt{\mathbf{I}_{x,+}(\xi)})(\sqrt{\mathbf{I}_{x,-}(u)} + \sqrt{\mathbf{I}_{x,+}(\xi)})} \right|. \tag{1.30}$$

The schematic illustration is given in Figures 2 and 3. (To simplify the notation, we assume $x = c$ in Figure 3; see (1.32).)

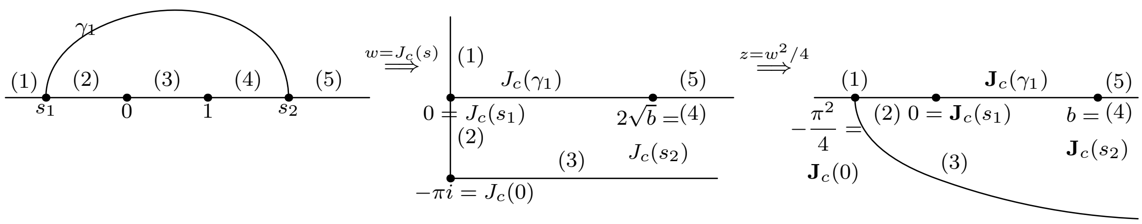

Figure 2: The schematic illustration of J_c and $\mathbf{J}_c$ on $\mathbb{C}_+$. (The definition of J_c and $\mathbf{J}_c$ is extended to $\mathbb{C}_-$ naturally by complex conjugation.) If c is changed to a general $x \in (0, \infty)$, then $J_x(s_2(x)) = 2\sqrt{b(x)}$ and $\mathbf{J}_x(s_2(x)) = b(x)$ will change, while $J_x(s_1(x)) = \mathbf{J}_x(s_1(x)) = 0$, $J_x(0) = \pi i$ and $\mathbf{J}_x(0) = -\pi^2/4$ are unchanged.

The definition of J_x and $\mathbf{J}_x$ is independent of V. Suppose V is one-cut regular with a hard edge, such that the equilibrium measure μ_V associated to V is supported on $[0, b]$ and the density function is $\psi(x)$. We let $c > 0$ be the unique solution to

$$b(x) = b, \qquad \text{or equivalently,} \qquad \sqrt{(x+1)^2 - 1} + \operatorname{arcosh}(x+1) = 2\sqrt{b}. \tag{1.31}$$

Throughout this paper, we denote

$$s_i = s_i(c), \quad \gamma_i = \gamma_i(c), \quad D = D_c, \quad \mathbf{I}_i(u) = \mathbf{I}_{c,i}(u), \quad \mathbf{I}_\pm(u) = \mathbf{I}_{c,\pm}(u), \quad F(u;\xi) = F_c(u;\xi). \tag{1.32}$$

Below we state the result on the constructive description of the equilibrium measure μ introduced in Proposition 1.3, under the assumption (1.19) in Theorem 1.4.

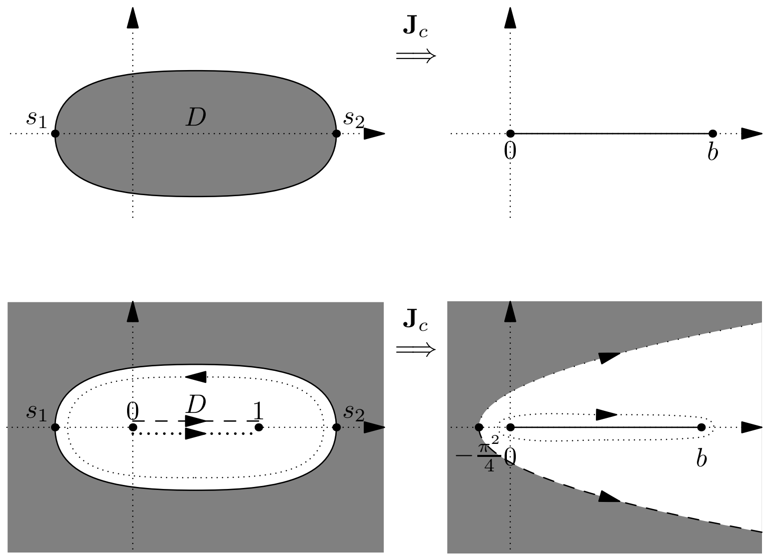

Figure 3: $\mathbf{J}_c$ maps $\mathbb{C} \setminus \overline{D}$ to $\mathbb{C} \setminus [0, b]$ and maps $D \setminus [0, 1]$ to $\mathbb{P} \setminus [0, b]$.

Theorem 1.8. *Suppose V satisfies* (1.19) *in Theorem 1.4.*

1. *The parameter c is the unique solution to the equation in $x \in (0, \infty)$*

$$\frac{1}{2\pi i} \oint_{\gamma(x)} \frac{\mathbf{J}'_x(\xi) V'(\mathbf{J}_x(\xi))}{\xi - s_2(x)} d\xi = \frac{1}{s_2(x) - 1}, \tag{1.33}$$

where $\gamma(x) = \gamma_1(x) \cup \gamma_2(x)$ is the boundary of D_x, with positive orientation. Hence b, the right-end point of the support of the equilibrium measure μ, is determined by $b = b(c)$.

2. *The density function $\psi(x)$ in* (1.15) *of the equilibrium measure μ is determined by*

$$\psi(x) = \frac{1}{4\pi^2 \sqrt{x}} \int_0^b U'(u) F(u; x) du, \tag{1.34}$$

where U is defined in (1.19) *and $F(u; x)$ is defined by* (1.32) *and* (1.30)*.*

This theorem will be proved in Section 2.

1.3.3 Comparison with the limiting density of particles in DMPK equation

In this subsection we compare our result of equilibrium measure with the limiting density of particles in the DMPK equation (1.10). See [3, Section III B] for details of the limiting density, and the original derivations in [31] and [6].

On one hand, we have the following corollary of Theorem 1.8

Corollary 1.9. *When $V(x) = x/\mathbb{M}$ where $\mathbb{M}$ is a constant parameter, the parameter $c = 2\mathbb{M}$, and the end point*

$$b = \left(\frac{1}{2} \sqrt{(2\mathbb{M} + 1)^2 - 1} + \frac{1}{2} \operatorname{arcosh}(2\mathbb{M} + 1) \right)^2. \tag{1.35}$$

Furthermore, the density function $\psi(x)$ in (1.15) *is given by*

$$\psi(x) = \frac{1}{\pi} \Im \sqrt{\frac{\mathbf{I}_+(x)}{x}}. \tag{1.36}$$

The readers may compare (1.35) to [3, Equation (205)]. This corollary will be proved in Section 2.

On the other hand, if we make an physically convincing (but mathematically not proved) assumption that as $n \to \infty$, the limiting empirical density of the $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ with joint probability density function $P(\tilde{\lambda}_1, \dots, \tilde{\lambda}_n; \mathbb{M})$ is $\tilde{\rho}(\tilde{\lambda}; \mathbb{M})$, then after the change of variable $\tilde{\lambda} = \sinh^2 x$, the density function is changed into

$$\rho(x; \mathbb{M}) = \sinh(2x)\tilde{\rho}(\sinh^2 x; \mathbb{M}). \tag{1.37}$$

Then define (see [3, Equation (201)])

$$U(\zeta; \mathbb{M}) = \sinh\zeta \cosh\zeta \int \frac{\rho(x; \mathbb{M})}{\sinh^2\zeta - \sinh^2 x} dx, \tag{1.38}$$

for $\zeta \in \{\Re\zeta \in (0, +\infty)$, and $|\Im\zeta| \in (0, \pi/2)$. It is clear that from $U(\zeta; \mathbb{M})$ we can recover $\rho(x; \mathbb{M})$. With the help of mathematical tricks commonly used by physicists, it is shown that $U(\zeta; \mathbb{M})$ satisfies the Euler's equation (see [3, Equation (202)]), and so it satisfies the functional equation

$$U(\zeta; \mathbb{M}) = U_0(\zeta - \mathbb{M}U(\zeta; \mathbb{M})), \quad U_0(\zeta) := U(\zeta; 0). \tag{1.39}$$

The ballistic initial condition $\rho(x; 0) = \delta(x - 0_+)$ implies that $U_0(\zeta) = \coth\zeta$, and then (1.39) becomes

$$\sqrt{\zeta} = \mathbb{M}U(\zeta; \mathbb{M}) + \operatorname{arcoth} U(\zeta; \mathbb{M}). \tag{1.40}$$

Now we consider $\tilde{G}(z)$ defined in (2.15) that is an integral transform of $\psi(x)$ in (1.15). Conversely, $\psi(x)$ can also be computed from $\tilde{G}(z)$ as in (2.14). When $V(x)$ is linear, it is derived in the last part of Section 2.2 that $\tilde{G}$ is expressed by $\mathbf{I}_2(z)$ as in (2.44).

With some work, we can check that

$$U(\zeta; \mathbb{M}) := \zeta\tilde{G}(\zeta^2) = \sqrt{\mathbf{I}_2(\zeta^2)} \tag{1.41}$$

is the solution of (1.40). Hence the densify function is ($\psi(x)$ is given in (1.36))

$$\rho(x; \mathbb{M}) = 2x\psi(x^2), \quad \text{with} \quad V(x) = \frac{x}{\mathbb{M}}. \tag{1.42}$$

The calculation above supports the following claim:

Conjecture 1.10. *As $n \to \infty$ and $\mathbb{M}$ is fixed. Let the limiting empirical probability density function of the solution of the DMPK equation with the ballistic initial condition be $\tilde{\rho}(\tilde{\lambda}; \mathbb{M})$, and denote the density function after the change of variable $\tilde{\lambda} = \sinh^2 x$ as $\rho(x; \mathbb{M})$ given by* (1.37). *Then $\rho(x; \mathbb{M})$ is related to $\psi(x)$ by* (1.42), *where $\psi(x)$ is given in* (1.36) *with $V(x) = x/\mathbb{M}$.*

We call the claim above a conjecture, because part of the above arguments, including the existence of the limiting probability density function, is not rigorous. Nevertheless, this conjecture, which is very convincing, implies that the biorthogonal ensemble is a good approximation to the solution of the DMPK equation with ballistic initial condition, at least at the global density level, even in the regime $n \to \infty$ and $\mathbb{M}$ is fixed.

These result above, together with the explicit computation of b in (1.35), can be compared with the result about $\rho(x; \mathbb{M})$ given in [3, Section III B] (where the notation for $\rho(x; \mathbb{M})$ is $\rho(\zeta, s)$).

From our formula (1.36), we have the following limit results. In the limit that $\mathbb{M} \to 0$, we have that the support of the equilibrium measure is $[0, 4\mathbb{M} + \mathcal{O}(1)]$, and the the density function satisfies the limiting formula that for all $\epsilon > 0$,

$$\psi(x) = \frac{1}{2\pi\mathbb{M}}\sqrt{\frac{4\mathbb{M}-x}{x}}(1+\mathcal{O}(\mathbb{M})), \quad x \in (\epsilon\mathbb{M}, (4-\epsilon)\mathbb{M}), \tag{1.43}$$

$$\psi_0 = \frac{\mathbb{M}^{-\frac{1}{2}}}{\pi}(1+\mathcal{O}(\mathbb{M})), \quad \psi_b = \frac{\mathbb{M}^{-\frac{3}{2}}}{4\pi}(1+\mathcal{O}(\mathbb{M})). \tag{1.44}$$

In the limit that $\mathbb{M} \to \infty$, we have that the support of the equilibrium measure is $[0, \mathbb{M}^2 + \mathcal{O}(\mathbb{M})]$, and the density function satisfies the limiting formula that for all $\epsilon > 0$,

$$\psi(x) = \frac{1}{2\mathbb{M}\sqrt{x}}(1+\mathcal{O}(\mathbb{M}^{-1})), \quad x \in (\epsilon\mathbb{M}^2, (1-\epsilon)\mathbb{M}^2), \tag{1.45}$$

$$\psi_0 = \frac{1}{2\mathbb{M}}(1+\mathcal{O}(\mathbb{M})), \quad \psi_b = \frac{1}{\sqrt{2}\pi\mathbb{M}^{\frac{5}{2}}}(1+\mathcal{O}(\mathbb{M})). \tag{1.46}$$

Here we note that (1.43) is comparable to the Marčenko-Pastur law [1, Chapter 3], and (1.45) is comparable to [3, Equation (191)].

1.3.4 Local results: Plancherel-Rotach asymptotics

Since both $p_j^{(n)}(z)$ and $q_j^{(n)}(f(z))$ are real analytic functions and $p_j^{(n)}(\bar{z}) = \overline{p_j^{(n)}(z)}$, $q_j^{(n)}(f(\bar{z})) = \overline{q_j^{(n)}(f(z))}$, we only need to give their asymptotics in the upper half plane and the real line. To be precise, we let $\delta > 0$ be a small enough constant, and let $C_\delta = \{z \in \mathbb{C}_+ \cup \mathbb{R} \mid |z| \leq \delta\}$ and $D_\delta = \{z \in \mathbb{C}_+ \cup \mathbb{R} \mid |z - b| \leq \delta\}$ be the two semicircles centred at 0 and b respectively, $B_\delta = \{z \in \mathbb{C}_+ \cup \mathbb{R} \mid \Im z \leq \delta/2 \text{ and } |z| > \delta \text{ and } |z - b| > \delta\}$, and $A_\delta = (\mathbb{C}_+ \cup \mathbb{R}) \setminus (B_\delta \cup C_\delta \cup D_\delta\}$. See Figure 4 to see the shapes of the regions. We assume that $V(z)$ and $h(z)$ are analytic in $B_\delta \cup C_\delta \cup D_\delta$.

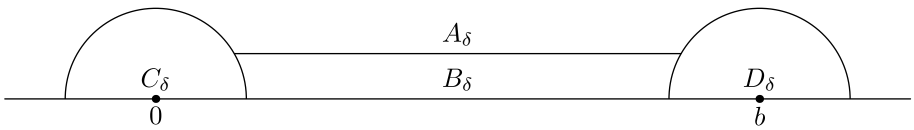

Figure 4: The four regions in the upper complex plane where the asymptotics of $p_{n+k}^{(n)}(z)$ and $q_{n+k}^{(n)}(f(z))$ are given.

Let $\psi(x)$ be the density function of μ_V on $(0, b)$. We then define the functions

$$\mathbf{g}(z) := \int_0^b \log(z-x)\psi(x)dx, \qquad \tilde{\mathbf{g}}(z) := \int_0^b \log(f(z)-f(x))\psi(x)dx, \tag{1.47}$$

with the branch cut of the logarithms for $z \in (-\infty, x)$ and $f(z) \in (-\infty, f(x))$ respectively. Let

$$\phi(z) = \mathbf{g}(z) + \tilde{\mathbf{g}}(z) - V(z) - \ell \tag{1.48}$$

for $z \in \mathbb{P} \setminus (-\infty, b)$, where ℓ is a constant to make $\phi(0) = \phi(b) = 0$. (See (1.16) and (1.17).) Then as explained in Section 3.6, we have that

$$f_b(z) := \left(-\frac{3}{4}\phi(z)\right)^{\frac{2}{3}}, \quad \text{with } f_b(b) = 0 \text{ and } f_b'(b) = (\pi\psi_b)^{\frac{2}{3}} > 0 \tag{1.49}$$

is a well defined analytic function in a certain neighbourhood of b. Similarly, as explained in Section 3.7,

$$f_0(z) = \frac{1}{16}(\phi(z) \mp \pi i)^2, \quad \text{with } f_0(0) = 0 \text{ and } f_0'(0) = -(\pi\psi_0)^2 < 0 \tag{1.50}$$

(where the sign is $-$ in $\mathbb{C}_+$ and $-$ in $\mathbb{C}_-$), is a well defined analytic function in a certain neighbourhood of 0.

Recall the contours γ_1, γ_2 defined in (1.32), $\gamma = \gamma_1 \cup \gamma_2$ with positive orientation, and $h(z)$ is a real analytic function on $[0, \infty)$. Let γ' and γ'' be positively oriented contours such that γ' encloses γ, and γ'' is enclosed by γ, such that $h(\mathbf{J}_c(s))$ is well defined and analytic in the annular region between γ' and γ''. We also assume that $\{\mathbf{I}_1(z), \mathbf{I}_2(z) : z \in B_\delta \cup C_\delta \cup D_\delta\}$ lies inside the annular region. Then define

$$D(s) = \exp\left(\frac{1}{2\pi i}\oint_{\gamma''} \log h(\mathbf{J}_c(\zeta))\frac{d\zeta}{\zeta - s}\right), \qquad z \text{ is outside } \gamma'', \tag{1.51}$$

$$\tilde{D}(s) = \exp\left(\frac{-1}{2\pi i}\oint_{\gamma'} \log h(\mathbf{J}_c(\zeta))\frac{d\zeta}{\zeta - s}\right), \qquad z \text{ is inside } \gamma'. \tag{1.52}$$

We have that between γ' and γ'', both $D_k(s)$ and $\tilde{D}_k(s)$ are defined, and

$$D(s)\tilde{D}(s) = h(\mathbf{J}_c(s))^{-1}, \quad z \text{ is between } \gamma' \text{ and } \gamma''. \tag{1.53}$$

If $h(z)$ is defined in (1.9), then it can be verified that

$$D(s) = \frac{1}{\sqrt{c}}\sqrt{\frac{s-1}{s-s_1}}\frac{J_c(s)^{1/2}}{s^{1/4}}, \quad \tilde{D}(s) = \sqrt{c}\sqrt{\frac{s-s_1}{s-1}}\frac{s^{1/4}}{\sinh(J_c(s))^{1/2}}, \quad \tilde{D}(1) = \sqrt{\frac{c(1-s_1)}{2e^c}}, \tag{1.54}$$

such that all power funcitons take the principal branch.

We define

$$G_k(s) = \frac{(\frac{c^2}{4})^{\alpha+\frac{1}{2}+k}(s-s_1)^{\alpha+1}s^{\frac{1}{2}}(s-1)^k}{\mathbf{J}_c(s)^{\alpha+\frac{1}{2}}\sqrt{(s-s_1)(s-s_2)}}D(s), \quad s \text{ is outside } \gamma'' \text{ and not on } \gamma_1 \text{ or } [s_1, 1], \tag{1.55}$$

where $\sqrt{(s-s_1)(s-s_2)}$ is analytic in $\mathbb{C} \setminus \gamma_1$ and is $\sim s$ as $s \to \infty$, and $(s-s_1)^{\alpha+1}s^{\frac{1}{2}}/\mathbf{J}_c(s)^{\alpha+\frac{1}{2}}$ is analytic in $\mathbb{C} \setminus [s_1, 1]$, and is $\sim (4/c^2)^{\alpha+1/2}$ as $s \to \infty$. We also define

$$\tilde{G}_k(s) = \frac{(1-s_1)^{\alpha+\frac{1}{2}}\sqrt{s_2-1}i\tilde{D}(1)^{-1}e^{kc}}{(s-s_1)^\alpha(s-1)^k\sqrt{(s-s_1)(s-s_2)}}\tilde{D}(s), \quad s \text{ is inside } \gamma' \text{ and not on } \gamma_2 \text{ or } (-\infty, s_1], \tag{1.56}$$

where $\sqrt{(s-s_1)(s-s_2)}$ is analytic in $\mathbb{C} \setminus \gamma_2$ and is $\sim s$ as $s \to \infty$, and $(s-s_1)^\alpha$ is analytic in $\mathbb{C} \setminus (-\infty, s_1]$, and takes the principal branch.

where all power functions take the principal branch, and $G_k(s)$ and $\tilde{G}_k(s)$ are then extended to their domains in (1.55) and (1.56) respectively.

Based on G_k and $\tilde{G}_k$, we then define

$$r_k(x) = 2|G_{k,+}(\mathbf{I}_+(x))|, \qquad \theta_k(x) = \arg(G_{k,+}(\mathbf{I}_+(x))), \tag{1.57}$$

$$\tilde{r}_k(x) = 2|\tilde{G}_{k,+}(\mathbf{I}_-(x))|, \qquad \tilde{\theta}_k(x) = \arg(\tilde{G}_{k,+}(\mathbf{I}_-(x))), \tag{1.58}$$

where $G_{k,+}(\mathbf{I}_+(x))$ is the limit of $G_k(z)$ as z approaches $\mathbf{I}_+(x) \in \gamma_1$ in $\mathbb{C} \setminus \overline{D}$, and $\tilde{G}_{k,+}(\mathbf{I}_-(x))$ is the limit of $\tilde{G}_k(z)$ as z approaches $\mathbf{I}_-(x) \in \gamma_2$ in D.

Theorem 1.11. *Let V be one-cut regular with a hard edge and assume $V(z)$ is analytic in an open set containing $[0,+\infty)$. As $n \to \infty$, we have the following asymptotics of $p_{n+k}^{(n)}(z)$ and $q_{n+k}^{(n)}(f(z))$, $k \in \mathbb{Z}$, uniformly for z in regions $A_\delta, B_\delta, C_\delta$ and D_δ, if $\delta > 0$ is small enough.*

1. *In region A_δ we have*

$$p_{n+k}^{(n)}(z) = (1+\mathcal{O}(n^{-1}))G_k(\mathbf{I}_1(z))e^{n\mathbf{g}(z)}, \qquad z \in A_\delta, \tag{1.59}$$

$$q_{n+k}^{(n)}(f(z)) = (1+\mathcal{O}(n^{-1}))\tilde{G}_k(\mathbf{I}_2(z))e^{n\tilde{\mathbf{g}}(z)}, \qquad z \in A_\delta \cap \mathbb{P}. \tag{1.60}$$

2. *In region B_δ we have*

$$p_{n+k}^{(n)}(z) = (1+\mathcal{O}(n^{-1}))G_k(\mathbf{I}_1(z))e^{n\mathbf{g}(z)} + (1+\mathcal{O}(n^{-1}))G_k(\mathbf{I}_2(z))e^{n(V(z)-\tilde{\mathbf{g}}(z)+\ell)}, \tag{1.61}$$

$$q_{n+k}^{(n)}(f(z)) = (1+\mathcal{O}(n^{-1}))\tilde{G}_k(\mathbf{I}_2(z))e^{n\tilde{\mathbf{g}}(z)} + (1+\mathcal{O}(n^{-1}))\tilde{G}_k(\mathbf{I}_1(z))e^{n(V(z)-\mathbf{g}(z)+\ell)}. \tag{1.62}$$

Especially, if $x \in (\delta, b-\delta)$, we have

$$p_{n+k}^{(n)}(x) = r_k(x)e^{n\int \log|x-y|d\mu(y)}\left[\cos\left(n\pi\mu([x,b]) + \theta_k(x)\right) + \mathcal{O}(n^{-1})\right], \tag{1.63}$$

$$q_{n+k}^{(n)}(f(x)) = \tilde{r}_k(x)e^{n\int \log|f(x)-f(y)|d\mu(y)}\left[\cos\left(n\pi\mu([x,b]) + \tilde{\theta}_k(x)\right) + \mathcal{O}(n^{-1})\right]. \tag{1.64}$$

3. *In region D_δ we have*

$$\begin{aligned} &e^{-\frac{n}{2}(\mathbf{g}(z)-\tilde{\mathbf{g}}(z)+V(z)+\ell)}p_{n+k}^{(n)}(z) = \\ &\quad\sqrt{\pi}\Big[n^{\frac{1}{6}}f_b^{\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))G_k(\mathbf{I}_1(z)) - (1+\mathcal{O}(1))iG_k(\mathbf{I}_2(z))\right)\mathrm{Ai}(n^{\frac{2}{3}}f_b(z)) \\ &\qquad - n^{-\frac{1}{6}}f_b^{-\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))G_k(\mathbf{I}_1(z)) + (1+\mathcal{O}(1))iG_k(\mathbf{I}_2(z))\right)\mathrm{Ai}'(n^{\frac{2}{3}}f_b(z))\Big], \end{aligned} \tag{1.65}$$

$$\begin{aligned} &e^{-\frac{n}{2}(\tilde{\mathbf{g}}(z)-\mathbf{g}(z)+V(z)+\ell)}q_{n+k}^{(n)}(f(z)) = \\ &\quad\sqrt{\pi}\Big[n^{\frac{1}{6}}f_b^{\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))\tilde{G}_k(\mathbf{I}_2(z)) - (1+\mathcal{O}(1))i\tilde{G}_k(\mathbf{I}_1(z))\right)\mathrm{Ai}(n^{\frac{2}{3}}f_b(z)) \\ &\qquad - n^{-\frac{1}{6}}f_b^{-\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))\tilde{G}_k(\mathbf{I}_2(z)) + (1+\mathcal{O}(1))i\tilde{G}_k(\mathbf{I}_1(z))\right)\mathrm{Ai}'(n^{\frac{2}{3}}f_b(z))\Big]. \end{aligned} \tag{1.66}$$

In particular, if $z = b + f_b'(b)^{-1}n^{-2/3}t$ with t bounded, then

$$\begin{aligned} &n^{-\frac{1}{6}}e^{-\frac{n}{2}(\mathbf{g}(z)-\tilde{\mathbf{g}}(z)+V(z)+\ell)}p_{n+k}^{(n)}(z) = \\ &\quad 2^{\frac{1}{4}}\sqrt{\pi}b^{\frac{1}{8}}s_2^{\frac{3}{8}}c^{\frac{1}{2}}\left(\frac{c^2(s_2-s_1)}{4b}\right)^{\alpha+\frac{1}{2}}\left(\frac{c}{2}\right)^k f_b'(b)^{\frac{1}{4}}D(s_2)\left(\mathrm{Ai}(t)+\mathcal{O}(n^{-\frac{1}{3}})\right), \end{aligned} \tag{1.67}$$

$$\begin{aligned} &n^{-\frac{1}{6}}e^{-\frac{n}{2}(\tilde{\mathbf{g}}(z)-\mathbf{g}(z)+V(z)+\ell)}q_{n+k}^{(n)}(f(z)) = \\ &\quad 2^{\frac{3}{4}}\sqrt{\pi}\left(\frac{1-s_1}{s_2-s_1}\right)^{\alpha+\frac{1}{2}}\left(\frac{c}{2}\right)^k e^{kc}\left(\frac{b}{s_2}\right)^{\frac{1}{8}} f_b'(b)^{\frac{1}{4}}\frac{\tilde{D}(s_2)}{\tilde{D}(1)}\left(\mathrm{Ai}(t)+\mathcal{O}(n^{-\frac{1}{3}})\right). \end{aligned} \tag{1.68}$$

Here Ai *is the Airy function.*

4. In region C_δ we have

$$e^{-\frac{n}{2}(\mathbf{g}(z)-\tilde{\mathbf{g}}(z)+V(z)+\ell)}p^{(n)}_{n+k}(z) =$$
$$\sqrt{\pi}\Big[n^{\frac{1}{2}}f_0^{\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))G_k(\mathbf{I}_1(z)) - (1+\mathcal{O}(1))ie^{\alpha\pi i}G_k(\mathbf{I}_2(z))\right)I_\alpha(2n\sqrt{f_0(z)})$$
$$+ n^{-\frac{1}{2}}f_0^{-\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))G_k(\mathbf{I}_1(z)) + (1+\mathcal{O}(1))ie^{\alpha\pi i}G_k(\mathbf{I}_2(z))\right)I_\alpha(2n\sqrt{f_0(z)})\Big], \tag{1.69}$$

$$e^{-\frac{n}{2}(\tilde{\mathbf{g}}(z)-\mathbf{g}(z)+V(z)+\ell)}q^{(n)}_{n+k}(f(z)) =$$
$$\sqrt{\pi}\Big[n^{\frac{1}{2}}f_0^{\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))\tilde{G}_k(\mathbf{I}_2(z)) - (1+\mathcal{O}(1))ie^{-\alpha\pi i}\tilde{G}_k(\mathbf{I}_1(z))\right)I_\alpha(2n\sqrt{f_0(z)})$$
$$+ n^{-\frac{1}{2}}f_0^{-\frac{1}{4}}(z)\left((1+\mathcal{O}(n^{-1}))\tilde{G}_k(\mathbf{I}_2(z)) + (1+\mathcal{O}(1))ie^{-\alpha\pi i}\tilde{G}_k(\mathbf{I}_1(z))\right)I_\alpha(2n\sqrt{f_0(z)})\Big]. \tag{1.70}$$

In particular, if $z = -f_0'(0)^{-1}n^{-2}t$, with t bounded, then

$$n^{-\frac{1}{2}}e^{-\frac{n}{2}(\mathbf{g}(z)-\tilde{\mathbf{g}}(z)+V(z)+\ell)}z^{\frac{\alpha}{2}}p^{(n)}_{n+k}(z) = 2\sqrt{\pi}\sqrt{\frac{-s_1}{s_2-s_1}}$$
$$\times\left(\frac{c(1-s_1)\sqrt{-s_1}}{s_2-s_1}\right)^{\alpha+\frac{1}{2}}\left(\frac{c^2}{4}(s_1-1)\right)^k D(s_1)(-f_0(0))^{\frac{1}{4}}\left(J_\alpha(2\sqrt{t})+\mathcal{O}(n^{-1})\right), \tag{1.71}$$

$$n^{-\frac{1}{2}}e^{-\frac{n}{2}(\tilde{\mathbf{g}}(z)-\mathbf{g}(z)+V(z)+\ell)}z^{\frac{\alpha}{2}}q^{(n)}_{n+k}(f(z)) = 2\sqrt{\pi}\sqrt{\frac{s_2-1}{s_2-s_1}}$$
$$\times\left(\frac{c(s_2-s_1)}{4\sqrt{-s_1}}\right)^{\alpha+\frac{1}{2}}(s_1-1)^{-k}e^{kc}\frac{\tilde{D}(s_1)}{\tilde{D}(1)}(-f_0(z))^{\frac{1}{4}}\left(J_\alpha(2\sqrt{t})+\mathcal{O}(n^{-1})\right). \tag{1.72}$$

Here J_α is the Bessel function and I_α is the modified Bessel function.

5.
$$e^{-n\ell}h^{(n)}_{n+k} = \frac{2\pi}{\tilde{D}(1)}\left(\frac{c^2(1-s_1)}{4}\right)^{\alpha+\frac{1}{2}}\left(\frac{c^2}{4}\right)^{k+\frac{1}{4}}e^{kc}+\mathcal{O}(n^{-1}). \tag{1.73}$$

The proof of the theorem will be given in Section 5 based on the Riemann-Hilbert analysis in Sections 3 and 4.

Based on Theorem 1.11, we can use the method in [14] to show that the correlation kernel $K_n(x,y)$ in (1.8) has the limit as the Airy kernel around b, and the sine kernel in $(0,b)$, upon proper scaling transform and conjugation. Also we can show that $K_n(x,y)$ has the limit as the Bessel kernel, using the method in [38]. (In [38] the limit of the Muttalib-Borodin correlation kernel is shown to converge to a Meijer-G kernel with a parameter θ. If $\theta = 1$, the Meijer G kernel specializes into the Bessel kernel.) Hence, the biorthogonal ensemble defined in (1.6) has the desired limiting universal behaviour at the bulk $(0,b)$, the soft edge b, and the hard edge 0. We put off the proof of the above claims to a subsequent paper.

1.4 Related models and previous results

Orthogonal polynomials were related to Riemann-Hilbert problems by [18], [19], and then the powerful Deift-Zhou nonlinear steepest-descent method was successfully applied to such Riemann-Hilbert problems [16], [15], and opened the door to manifold of limiting results of orthogonal polynomials and their generalizations, see [26] for a review. These various Riemann-Hilbert problems are all matrix valued, some 2×2 and some of larger sizes.

In the study of a special kind of biorthogonal polynomials related to the random matrix model with equispaced external source [13], Claeys and the author related the biorthogonal polynomials to 2-component vector-valued Riemann-Hilbert problems, and applied the Deift-Zhou method on them to find the limiting results. The biorthogonal polynomials in [13] has, in our notation, $f(x) = e^x$ in (1.4) where the integral domain is replaced by $\mathbb{R}$.

After [13] and before the current paper, the method of vector-valued Riemann-Hilbert problems was, to the author's limited knowledge, only applied to the Muttalib-Borodin biorthogonal polynomials [12], [38], [10]. The Muttalib-Borodin biorthogonal polynomials is characterized by $f(x) = x^\theta$ in (1.4), and the particle model given in (1.6) with $f(x) = x^\theta$ is called the Muttalib-Borodin ensemble.

We remark that Muttalib-Borodin ensemble was proposed by physicist Muttalib [34] as a simplification of the biorthogonal ensemble considered in our paper, and the study of the Muttalib-Borodin ensemble [8], [11], [20], [39] is partially motivated by its indirect relation to the quantum transport theory of disordered wires.

We also remark that technically the Muttalib-Borodin ensemble is more challenging, for its limit behaviour at the hard edge is the more complicated Meijer G kernel, rather than the Bessel kernel.

At last, we note that biorthogonal ensembles are also investigated from other aspects, for instance, [9], [29], [27], [25], [33].

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2 Construction of the equilibrium measure

In this section we assume that V is a potential function that satisfies the conditions in Theorem 1.4, and show that V is one-cut regular with a hard edge at 0, by an explicit construction of its equilibrium measure. Like in [13], we first give the support of the equilibrium measure as an ansatz, then compute the density within the support of the equilibrium measure, and at last verify that the measure constructed satisfies the criteria of one-cut regularity, and conclude that it is the unique equilibrium measure.

At every step, we analyze the $V = x/\mathbb{M}$ special case and get explicit formulas for it.

2.1 A technical lemma

Recall the mapping $\mathbf{J}_x$ defined in (1.21) and $\gamma(x)$ defined in Part 1 of Theorem 1.8.

Lemma 2.1. *Suppose V satisfies the condition required in Theorem 1.4. There is a unique $x \in (0, \infty)$ such that the equation with unknown x (1.33) holds.*

Proof. Using the formulas (1.21) and (1.22), we can rewrite (1.33) as

$$F(x) = 2, \quad \text{where} \quad F(x) = \frac{1}{2\pi i} \oint_{\gamma'} \frac{J_x(\xi) V'(\mathbf{J}_x(\xi))}{\sqrt{\xi}(\xi - 1)} d\xi. \tag{2.1}$$

Here γ' can be $\gamma(x)$, but can also be a slightly bigger contour circling $\gamma(x)$, as long as $V'(\mathbf{J}_c(\xi))$ is well defined there. Let $U(z) = V'(z)\sqrt{z}$ where $\sqrt{z}$ takes the principal branch. By direct computation, we have

$$F'(x) = \frac{1}{4\pi i} \oint_{\gamma'} U'(\mathbf{J}_x(\xi)) J_x(\xi) \frac{d\xi}{\xi - 1}. \tag{2.2}$$

Then taking $\gamma' = \gamma(x)$, and change the variable $y = \mathbf{J}_x(\xi)$, we have

$$\begin{aligned} F'(x) &= \frac{1}{x\pi i} \int_0^{b(x)} U'(y) \left(\frac{\sqrt{\mathbf{I}_{x,-}(y)}}{\mathbf{I}_{x,-}(y) - s_2(x))} - \frac{\sqrt{\mathbf{I}_{x,+}(y)}}{\mathbf{I}_{x,+}(y) - s_2(x))} \right) dy \\ &= \frac{2}{x\pi} \int_0^{b(x)} U'(y) \Im \frac{\sqrt{\mathbf{I}_{x,+}(y)}}{\mathbf{I}_{x,+}(y) - s_2(x))} dy. \end{aligned} \tag{2.3}$$

Suppose $y \in (0, b(x))$. We note that $U'(y) = V''(y)x^{1/2} + \frac{1}{2}V'(y)x^{-1/2} > 1$. By the definition (1.28) of $\mathbf{I}_{x,+}$, we know that $\mathbf{I}_{x,+}(y) \in \gamma_1(x)$ for all $y \in (0, b(x))$. In the proof of part 4 of Lemma 1.6 in Appendix A, we have the parametrization of $\gamma_1(x)$ in (A.3). (It is proved there that $\gamma_1'(x)$ in (A.3) is $\gamma_1(x)$.) It is clear that $\arg \mathbf{I}_{x,+}(y) \in (0, \pi)$ and $\arg \sqrt{\mathbf{I}_{x,+}(y)} \in (0, \pi/2)$. Also we have $\arg(\mathbf{I}_{x,+}(y) - s_2(x)) \in (\pi/2, \pi)$. Hence $\arg(\sqrt{\mathbf{I}_{x,+}(y)}/(\mathbf{I}_{x,+}(y) - s_2(x))) \in (-\pi, 0)$, and similarly $\arg(\sqrt{\mathbf{I}_{x,+}(y)}/(\mathbf{I}_{x,+}(y) - s_2(x))) \in (-\pi, 0)$. We conclude that the integrand on the right-hand side of (2.3) is positive, and so $F(x)$ is a monotonically increasing function.

In the special case $V'(\mathbf{J}_c(s)) = C > 0$ is a constant, which is equivalent to $U'(y) = (C/2)y^{-1/2}$, then we have

$$F(x)|_{V'(\mathbf{J}_c(s))=C} = \frac{C}{2\pi i} \oint_{\gamma'} \frac{J_x(\xi)}{\sqrt{\xi}(\xi - 1)} d\xi = \frac{C}{2\pi i} \oint_{\gamma'} \frac{x}{\xi - 1} d\xi + \frac{C}{2\pi i} \oint_{\gamma'} \frac{\operatorname{arcosh} \frac{\xi+1}{\xi-1}}{\sqrt{\xi}(\xi - 1)} d\xi = Cx, \tag{2.4}$$

since the integral of $x/(\xi - 1)$ is Cx, and the other term vanishes, which can be verified easily by deforming γ' into a very large circular contour. Then for general V that satisfies assumption (1.19), by a comparison argument of $U'(y)$ and $(C/2)y^{-1/2}$ and the integral formula (2.3), we also have that $F(x) \to 0$ as $x \to 0$ and $F(x) \to \infty$ as $x \to \infty$. So $F(x) = 2$ has a unique solution on $(0, \infty)$. □

Now we name this solution c'. Later in Section 2.2 we will see that the unique solution to (1.33) is equal to c, the parameter in (1.31).

In the special case $V(x) = x/\mathbb{M}$, we have that $V'(\mathbf{J}_c(s)) = \mathbb{M}^{-1}$, and then by (2.4), we have that the solution to (1.33) is

$$c' = 2\mathbb{M}. \tag{2.5}$$

2.2 The $\tilde{\mathbf{g}}$-functions and the density of the equilibrium measure

In Section 1.3.4, we define the functions $\mathbf{g}(z)$ and $\tilde{\mathbf{g}}(z)$ in (1.47) under the assumption that the end point b and the density function $\psi(x)$ are known. In this subsection, we first assume the existence of b and ψ, and derive a Riemann-Hilbert problem satisfied by $\mathbf{g}'(z)$ and $\tilde{\mathbf{g}}'(z)$. Then we solve the Riemann-Hilbert problem by direct calculation, and then confirm the value of b and express ψ in a computable way. Thus we prove Theorems 1.4 and 1.8.

We recall that $\mathbf{g}(z)$ and $\tilde{\mathbf{g}}(z)$ defined in (1.47) are analytic on $\mathbb{C} \setminus (-\infty, b]$ and on $\mathbb{P} \setminus (-\pi^2/4, b]$ respectively.

Lemma 2.2. *Suppose V is one-cut regular with a hard edge. Then*

1. *For $x \in (-\infty, 0)$, $\mathbf{g}_\pm(x)$ are continuous, and*

$$\mathbf{g}_+(x) = \mathbf{g}_-(x) + 2\pi i; \tag{2.6}$$

for $x \in (-\pi^2/4, 0)$, $\tilde{\mathbf{g}}_\pm(x)$ are continuous, and

$$\tilde{\mathbf{g}}_+(x) = \tilde{\mathbf{g}}_-(x) + 2\pi i; \tag{2.7}$$

and for $z \in \rho$, the lower parabola defined in (1.20), $\tilde{\mathbf{g}}(z)$ and $\tilde{\mathbf{g}}(z)$ are continuous, and

$$\tilde{\mathbf{g}}_-(\bar{z}) = \tilde{\mathbf{g}}_+(z) + 2\pi i, \tag{2.8}$$

where $\bar{z}$ is the complex conjugate of z lying on the upper parabola, with both ρ and $\bar{\rho}$ oriented from left to right.

2. *For $x \in (0, b)$, we have*

$$-\frac{1}{2\pi i}(\mathbf{g}'(x)_+ - \mathbf{g}'_-(x)) = -\frac{1}{2\pi i}(\tilde{\mathbf{g}}'(x)_+ - \tilde{\mathbf{g}}'_-(x)) > 0, \tag{2.9}$$

and the left-hand side is equal to $\psi(x)$.

3. *As $z \to b$, the limits of $\mathbf{g}(z), \tilde{\mathbf{g}}(z)$ and $\mathbf{g}'(z), \tilde{\mathbf{g}}'(z)$ exist, and as $x \to b$, $\mathbf{g}'(z) - \mathbf{g}'(b) = \mathcal{O}(|z-b|^{1/2})$, $\tilde{\mathbf{g}}'(z) - \tilde{\mathbf{g}}'(b) = \mathcal{O}(|z-b|^{1/2})$, and*

$$\lim_{x \to b_-} \frac{i(\mathbf{g}'_+(x) - \mathbf{g}'_-(x))}{\sqrt{b-x}} = \lim_{x \to b_-} \frac{i(\tilde{\mathbf{g}}'_+(x) - \tilde{\mathbf{g}}'_-(x))}{\sqrt{b-x}} \in (0, +\infty), \tag{2.10}$$

and as $z \to 0$ in $\mathbb{C}_+$ or $\mathbb{C}_-$, the limits of $\mathbf{g}(z), \tilde{\mathbf{g}}(z)$ exist, and as $x \to 0$, $\mathbf{g}'(z) = \mathcal{O}(|z|^{-1/2})$, $\tilde{\mathbf{g}}'(z) = \mathcal{O}(|z|^{-1/2})$, and

$$\lim_{x \to 0_+} i(\mathbf{g}'_+(x) - \mathbf{g}'_-(x))\sqrt{x} = \lim_{x \to 0_+} i(\tilde{\mathbf{g}}'_+(x) - \tilde{\mathbf{g}}'_-(x))\sqrt{x} \in (0, +\infty). \tag{2.11}$$

The two limits in (2.10) and (2.11) are $2\pi\psi_b$ and $2\pi\psi_0$ respectively.

4. *As $z \to \infty$ in $\mathbb{C}$, $\mathbf{g}'(z) = z^{-1} + \mathcal{O}(z^{-2})$, and as $f(z) \to \infty$ (i.e., $\Re z \to \infty$), $\tilde{\mathbf{g}}'(z) = z^{-1/2}(1 + \mathcal{O}(f(z)^{-1})$.*

5. *For $x \in [0, b]$, there exists a constant ℓ such that*

$$\mathbf{g}_\pm(x) + \tilde{\mathbf{g}}_\mp(x) - V(x) - \ell = 0. \tag{2.12}$$

For $x \in (b, \infty)$, we have

$$\mathbf{g}_\pm(x) + \tilde{\mathbf{g}}_\mp(x) - V(x) - \ell < 0. \tag{2.13}$$

Conversely, if functions $\mathbf{g}(x)$ and $\tilde{\mathbf{g}}(x)$ which are analytic on $\mathbb{C} \setminus (-\infty, b]$ and on $\mathbb{P} \setminus (-\pi^2/4, b]$ respectively satisfies all the properties listed above, then V is one-cut regular with a hard edge, and its equilibrium measure is $d\mu(x) = \psi(x)dx$ supported on $[0, b]$ with

$$\psi(x) = -\frac{1}{2\pi i}(G(x)_+ - G_-(x)) = -\frac{1}{2\pi i}(\tilde{G}(x)_+ - \tilde{G}_-(x)), \tag{2.14}$$

where

$$G(z) = \mathbf{g}'(z) = \int_0^b \frac{\psi(x)dx}{z-x}, \qquad \tilde{G}(z) = \tilde{\mathbf{g}}'(z) = \int_0^b \frac{f'(z)\psi(x)dx}{f(z)-f(x)}. \tag{2.15}$$

The proof of Lemma 2.2 is straightforward and we omit it. Below we construct $\mathbf{g}(z)$ and $\tilde{\mathbf{g}}(z)$ that satisfies the properties, under the condition of V given in Theorem 1.4. We note that in the construction procedure, the value of b is unknown and needs to be determined.

To construct $\mathbf{g}(z)$ and $\tilde{\mathbf{g}}(z)$, it is equivalent to construct their the derivatives $G(z)$ and $\tilde{G}(z)$. We recall the function $b(x)$ defined in (1.25) and $s_1(x), s_2(x)$ defined in Lemma 1.6. We define the following two Riemann-Hilbert (RH) problem for $(H^{(x)}(z), \tilde{H}^{(x)}(z))$ and $N^{(x)}(s)$ with a parameter $x > 0$:

RH Problem 2.3.

1. *$H^{(x)}(z)$ is analytic in $\mathbb{C} \setminus [0, b(x)]$ and $\tilde{H}^{(x)}(z)$ is analytic in $\mathbb{P} \setminus [0, b(x)]$.*

2. *We have the boundary conditions that $H^{(x)}_{\pm}(z)$ and $\tilde{H}^{(x)}_{\pm}(z)$ are continuous functions on $(0, b(x))$ [1] and*

$$H^{(x)}(z) = \frac{1}{z} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty, \tag{2.16}$$

$$\tilde{H}^{(x)}(z) = z^{-\frac{1}{2}}(1 + \mathcal{O}(f(z)^{-1}), \qquad \text{as } f(z) \to \infty \text{ (i.e., } \Re z \to +\infty), \tag{2.17}$$

$$H^{(x)}(z) = \mathcal{O}(1), \quad \tilde{H}^{(x)}(z) = \mathcal{O}(1), \qquad \text{as } z \to b(x), \tag{2.18}$$

$$H^{(x)}(z) = \mathcal{O}(z^{-\frac{1}{2}}), \quad \tilde{H}^{(x)}(z) = \mathcal{O}(z^{-\frac{1}{2}}), \qquad \text{as } z \to 0. \tag{2.19}$$

3. *For $z \in (0, b(x))$, we have*

$$H^{(x)}_{\pm}(z) + \tilde{H}^{(x)}_{\mp}(z) - V'(z) = 0. \tag{2.20}$$

4. *$\tilde{H}^{(x)}(z)$ is continuous up to the boundary $\rho \cup \{0\} \cup \bar{\rho}$ of $\mathbb{P}$. For $z \in \rho \in \mathbb{C}_-$ such that $z = \lim_{\epsilon \to 0_+} \mathbf{J}'_x(y + \epsilon i)$ with $y \in (0, 1)$, we have*

$$\lim_{\epsilon \to 0_+} \tilde{H}^{(x)}(z + \epsilon i)\mathbf{J}'_x(y + \epsilon i) = \lim_{\epsilon \to 0_+} \tilde{H}^{(x)}(\bar{z} - \epsilon i)\mathbf{J}'_x(y - \epsilon i). \tag{2.21}$$

Our RH problem 2.3 is motivated by the properties satisfied by $(G(z), \tilde{G}(z))$ defined by (2.15). Analogous to (2.14), we define for $y \in (0, b(x))$

$$\psi^{(x)}(y) = -\frac{1}{2\pi i}(H^{(x)}(y)_+ - H^{(x)}_-(y)) = -\frac{1}{2\pi i}(\tilde{H}^{(x)}(y)_+ - \tilde{H}^{(x)}_-(y)), \tag{2.22}$$

and then have, analogous to (2.15),

$$H^{(x)}(z) = \int_0^{b(x)} \frac{\psi^{(x)}(y)dy}{z - y}, \qquad \tilde{H}^{(x)}(z) = \int_0^{b(x)} \frac{f'(z)\psi^{(x)}(y)dy}{f(z) - f(y)}. \tag{2.23}$$

RH Problem 2.4.

1. *$N^{(x)}(s)$ is analytic in $\mathbb{C} \setminus \gamma(x)$, where the contour $\gamma(x) = \gamma_1(x) \cup \gamma_2(x)$ defined in Theorem 1.8.*

2. *$N^{(x)}_{\pm}(s)$ is bounded on $\gamma_1(x)$ and $\gamma_2(x)$ and $N^{(x)}(s)$ is bounded if $s \to s_1(x)$ or $s \to s_2(x)$ or $s \to 0$. $N^{(x)}(s)$ has the behaviour*

$$N^{(x)}(s) = \mathcal{O}(s^{-1}), \quad \text{as} \quad s \to \infty. \tag{2.24}$$

[1] In all RH problems in this paper, the boundary values of functions are continuous on the two sides of the jump curves, unless otherwise stated. In later RH problems we omit the statement of continuity.

3. $N^{(x)}(s)$ satisfies the jump condition

$$N_+^{(x)}(s) + N_-^{(x)}(s) = \frac{s-1}{s-s_2(x)}\mathbf{J}_x'(s)V'(\mathbf{J}_x(s)), \quad s \in \gamma(x) \setminus \{s_1(x), s_2(x)\}, \tag{2.25}$$

where the mapping $\mathbf{J}_x(s)$ is defined in (1.21)*, constant $s_2(x)$ is defined in* (1.22)*.*

If $(H^{(x)}(z), \tilde{H}^{(x)}(z))$ is a solution to RH problem 2.3, then the function $\tilde{N}^{(x)}(s)$ defined by

$$\tilde{N}^{(x)}(s) = \frac{s-1}{s-s_2(x)} \times \begin{cases} \mathbf{J}_x'(s)H^{(x)}(\mathbf{J}_x(s)), & s \in \mathbb{C} \setminus \overline{D_x}, \\ \mathbf{J}_x'(s)\tilde{H}^{(x)}(\mathbf{J}_x(s)), & s \in D_x \setminus [0,1], \end{cases} \tag{2.26}$$

is a solution to RH problem 2.4. (Although by (2.26) $\tilde{N}^{(x)}(s)$ is undefined on $(0,1)$, it can be naturally extended to $(0,1)$ by continuation, due to Item 4 of RH problem 2.3.) Moreover, $N^{(x)}(s)$ given by (2.26) satisfies a condition stronger than (2.24)

$$\tilde{N}^{(x)}(1) = (s_2 - 1)^{-1}, \qquad \tilde{N}^{(x)}(s) = s^{-1} + \mathcal{O}(s^{-2}), \quad s \to \infty. \tag{2.27}$$

We also have that for each $x > 0$, RH problem 2.4 may have at most one solution. Suppose both $N^{(x),1}(s)$ and $N^{(x),2}(s)$ are solutions to RH problem 2.4, then the function

$$M^{(x)}(s) = \begin{cases} N^{(x),1}(s) - N^{(x),2}(s), & s \in \mathbb{C} \setminus \bar{D}, \\ -N^{(x),1}(s) + N^{(x),2}(s), & s \in D. \end{cases} \tag{2.28}$$

We have that $M^{(x)}(s)$ is analytic in $\mathbb{C} \setminus \{s_1(x), s_2(x)\}$ after analytic continuation on $\gamma(x) \setminus \{s_1(x), s_2(x)\}$, and it is bounded as s approaches $s_1(x), s_2(x)$ and $M^{(x)}(s) \to 0$ as $s \to \infty$. By Liouville's theorem, $M^{(x)}(s) = 0$, and then $\tilde{N}^{(x),1}(s) = N^{(x),2}(s)$.

The unique solution to RH problem 2.4 has an explicit formula

$$N^{(x)}(s) = \begin{cases} -\frac{1}{2\pi i}\int_{\gamma(x)} \frac{(\xi-1)\mathbf{J}_x'(\xi)V'(\mathbf{J}_x(\xi))}{(\xi-s_2(x))(\xi-s)}d\xi, & s \in \mathbb{C} \setminus \overline{D_x}, \\ \frac{1}{2\pi i}\int_{\gamma(x)} \frac{(\xi-1)\mathbf{J}_x'(\xi)V'(\mathbf{J}_x(\xi))}{(\xi-s_2(x))(\xi-s)}d\xi, & s \in D_x \setminus [0,1]. \end{cases} \tag{2.29}$$

By direct computation, we have that

$$N^{(x)}(1) = (s_2(x) - 1)^{-1}C_x, \qquad \lim_{s\to\infty} N^{(x)}(s) = C_x s^{-1} + \mathcal{O}(s^{-2}), \tag{2.30}$$

where

$$C_x = \frac{1}{2\pi i}\int_{\gamma(x)} \frac{(\xi-1)\mathbf{J}_x'(\xi)V'(\mathbf{J}_x(\xi))}{\xi - s_2(x)}d\xi = \frac{s_2(x)-1}{2\pi i}\int_{\gamma(x)} \frac{\mathbf{J}_x'(\xi)V'(\mathbf{J}_x(\xi))}{\xi - s_2(x)}d\xi \tag{2.31}$$

We conclude that RH problem 2.3 has a solution, which implies that RH problem 2.4 has a solution that in addition satisfies (2.27), only if $C_x = 1$, which is equivalent to equation (1.33) in Theorem 1.8. Hence RH problem 2.4 has a solution only if $x = c'$, the unique solution of equation (1.33) as in Lemma 2.1. We then find that under the condition required in Theorem 1.4, and assuming the one-cut regular with a hard edge property of V, the only possible value of the right-end point of the support of the equilibrium measure is $b' = b(c')$, the only candidates of functions $G, \tilde{G}$ defined by (1.47) and (2.15) are

$$H^{(c')}(z) = \frac{2}{c'}\sqrt{\frac{\mathbf{I}_{c',1}(z)}{z}}N^{(c')}(\mathbf{I}_{c',1}(z)), \qquad \tilde{H}^{(c')}(z) = \frac{2}{c'}\sqrt{\frac{\mathbf{I}_{c',2}(z)}{z}}N^{(c')}(\mathbf{I}_{c',2}(z)), \tag{2.32}$$

and the only candidate of the density function is $\psi^{(c')}(x)$ defined in (2.22).

The remaining part of the proof is to show that the $\psi^{(c')}(x)dx$ on $[0,b']$ does satisfy Requirement 1. Identities (2.16) and (2.17) in Item 2 of RH problem 2.3 implies that the total mass of the (possibly signed) measure $\psi^{(c')}(x)dx$ is 1, and identity (1.16) in Part 3 of Requirement 1 is implied by (2.20) in Item 3 of RH problem 2.3. Hence, we only need to verify Parts 2, 4 and 5 of Requirement 1.

To this end, for $s \in \mathbb{C} \setminus \overline{D}$, we express ($U(u)$ is defined in (1.19))

$$\begin{aligned}
N^{(c')}(s) &= \frac{-1}{2\pi i}\frac{c'}{2}\oint_{\gamma(c')} \frac{V'(\mathbf{J}_{c'}(\xi))\sqrt{\mathbf{J}_{c'}(\xi)}}{\sqrt{\xi}(\xi - s)}d\xi \\
&= \frac{-1}{2\pi i}\frac{c'}{2}\int_{\gamma_1(c')} \frac{V'(\mathbf{J}_{c'}(\xi))\sqrt{\mathbf{J}_{c'}(\xi)}}{\sqrt{\xi}(\xi - s)}d\xi + \frac{1}{2\pi i}\frac{c'}{2}\int_{\gamma_2(c')} \frac{V'(\mathbf{J}_{c'}(\xi))\sqrt{\mathbf{J}_{c'}(\xi)}}{\sqrt{\xi}(\xi - s)}d\xi \\
&= \frac{-1}{2\pi i}\frac{c'}{2}\left[\int_0^{b'} \frac{U(u)\mathbf{I}'_{c',+}(u)}{\sqrt{\mathbf{I}_{c',+}(u)}(\mathbf{I}_{c',+}(u) - s)}du - \int_0^{b'} \frac{U(u)\mathbf{I}'_{c',-}(u)}{\sqrt{\mathbf{I}_{c',-}(u)}(\mathbf{I}_{c',-}(u) - s)}du\right] \\
&= \frac{-1}{2\pi i}c'\left[\int_0^{b'} \frac{U(u)\sqrt{\mathbf{I}_{c',+}(u)}'}{(\sqrt{\mathbf{I}_{c',+}(u)} + \sqrt{s})(\sqrt{\mathbf{I}_{c',+}(u)} - \sqrt{s})}du \right. \\
&\qquad \left. - \int_0^{b'} \frac{U(u)\sqrt{\mathbf{I}_{c',-}(u)}'}{(\sqrt{\mathbf{I}_{c',-}(u)} + \sqrt{s})(\sqrt{\mathbf{I}_{c',-}(u)} - \sqrt{s})}du\right] \\
&= \frac{-1}{2\pi i}\frac{c'}{2\sqrt{s}}\int_0^{b'} U(u)\frac{d}{du}\left(\log\frac{\sqrt{\mathbf{I}_{c',+}(u)} - \sqrt{s}}{\sqrt{\mathbf{I}_{c',+}(u)} + \sqrt{s}} - \log\frac{\sqrt{\mathbf{I}_{c',-}(u)} - \sqrt{s}}{\sqrt{\mathbf{I}_{c',-}(u)} + \sqrt{s}}\right)du \\
&= \frac{1}{2\pi i}\frac{c'}{2\sqrt{s}}\int_0^{b'} U'(u)\left(\log\frac{\sqrt{\mathbf{I}_{c',+}(u)} - \sqrt{s}}{\sqrt{\mathbf{I}_{c',+}(u)} + \sqrt{s}} - \log\frac{\sqrt{\mathbf{I}_{c',-}(u)} - \sqrt{s}}{\sqrt{\mathbf{I}_{c',-}(u)} + \sqrt{s}}\right)du,
\end{aligned} \tag{2.33}$$

where $\sqrt{s}$ takes the principal branch. Hence, for $x \in (0,b)$,

$$\begin{aligned}
&\psi^{(c')}(x) \\
= &-\frac{x^{-\frac{1}{2}}}{4\pi^2}\Re\lim_{\epsilon\to 0_+}\int_0^{b'} U'(u)\log\frac{(\sqrt{\mathbf{I}_{c',+}(u)} - \sqrt{\mathbf{I}_{c',1}(x+\epsilon i)})(\sqrt{\mathbf{I}_{c',-}(u)} + \sqrt{\mathbf{I}_{c',1}(x+\epsilon i)})}{(\sqrt{\mathbf{I}_{c',+}(u)} + \sqrt{\mathbf{I}_{c',1}(x+\epsilon i)})(\sqrt{\mathbf{I}_{c',-}(u)} - \sqrt{\mathbf{I}_{c',1}(x+\epsilon i)})}du \\
= &\frac{x^{-\frac{1}{2}}}{4\pi^2}\int_0^{b'} U'(u)F_{c'}(u;x)du,
\end{aligned} \tag{2.34}$$

where $F_{c'}(u;x)$ is defined in (1.30). Since $F_{c'}(u;x)$ is a continuous function on $u \in [0,x) \cup (x,b')$ and blows up at $u = x$ as $\mathcal{O}(\log(|u-x|)$, the integral (2.34) is well defined.

Positivity and regularity in the bulk Using the fact that for all $y \in (0,b')$, $\Re\mathbf{I}_{c',+}(y) = \Re\mathbf{I}_{c',-}(y) > 0$ and $\Im\mathbf{I}_{c',+}(y) = -\Im\mathbf{I}_{c',-}(y) > 0$, it is clear that for all $x,u \in (0,b')$,

$$\left|\frac{\sqrt{\mathbf{I}_{c',+}(u)} + \sqrt{\mathbf{I}_{c',+}(x)}}{\sqrt{\mathbf{I}_{c',-}(u)} + \sqrt{\mathbf{I}_{c',+}(x)}}\right| > 1, \qquad \left|\frac{\sqrt{\mathbf{I}_{c',-}(u)} - \sqrt{\mathbf{I}_{c',+}(x)}}{\sqrt{\mathbf{I}_{c',+}(u)} - \sqrt{\mathbf{I}_{c',+}(x)}}\right| > 1. \tag{2.35}$$

Hence, as a function in u,

$$F_{c'}(u;x) > 0, \quad \text{for all} \quad u \in (0,x) \cup (x,b'). \tag{2.36}$$

Together with the assumption $U'(x) > 0$ in (1.19), we conclude that $\psi^{(c')}(x) > 0$ for all $x \in (0,b')$.

Regularity at the edges We need to show that $\psi^{(c')}(x)$ satisfies that $\lim_{x\to b'_-} \psi^{(c')}(x)/\sqrt{b'-x} = c_1(1+\mathcal{O}(x-b'))$ and $\lim_{x\to 0^+} \psi^{(c')}(x)\sqrt{x} = c_2(1+\mathcal{O}(x))$ for some $c_1 > 0$ and $c_2 > 0$. By (2.34), this is equivalent to that the integral

$$\int_0^{b'} U'(u)F_{c'}(u;x)du \tag{2.37}$$

decreases like a square root as x moves to b' and approaches a nonzero limit as x moves to 0. The desired property of (2.37) relies on the properties of $U'(u)$ and $F_{c'}(u;x)$. They are both continuous functions as $u \in (0,b')$, and we have the positivity results (1.19) for $U'(u)$ on $(0,b')$ and (2.36) for $F_{c'}(u;x)$ as $u,x \in (0,b')$.

Moreover, since $V(z)$ is analytic at 0, $U(x)$ either converges to 0 (when $V'(0)=0$) or blows up like $x^{-1/2}$ as $u \to 0_+$ (when $V'(0) > 0$).

For $x \in (0,b')$ in the vicinity of 0, from the properties of $\mathbf{I}_{c',\pm}(x)$ given in Appendix A, we have that the function

$$\frac{F_{c'}(u;x)}{\sqrt{\log^2(|u-x|)+1}} \tag{2.38}$$

is uniformly bounded, and it converges uniformly to a non-vanishing limit as $x \to 0$. Hence (2.37) converges to a non-zero limit as $x \to 0$. For $x \in (0,b')$ in the vicinity of b', we have ($d_1(x)$ is defined in (A.16))

$$F_{c'}(u;x) = \log\left|\frac{2b' + d_1(c')(\sqrt{b'-u}+\sqrt{b'-x})i}{2b' + d_1(c')(\sqrt{b'-u}-\sqrt{b'-x})i}\frac{\sqrt{b'-u}+\sqrt{b'-x}}{\sqrt{b'-u}-\sqrt{b'-x}}\right|(1+\sqrt{b'-u}f(u;x)), \tag{2.39}$$

such that $f(u;x)$ is continuous on $[0,b']$ and converge uniformly to a limit function as $x \to b'$. Hence the product of the integral in (2.37) and $(b-x)^{-1/2}$ converges to a non-zero limit as $x \to b_-$.

Regularity away from the support We want to show that if $\psi(x)$ is defined as $\psi^{(c')}(x)$ in (2.22) and $b = b'$, then Item 4 of Requirement 1 is satisfied. To this end, we denote

$$g(x) = H^{(c')}(x) + \tilde{H}^{(c')}(x) - V'(x), \quad x \in (b', +\infty), \tag{2.40}$$

and it suffices to show that $g(x) < 0$ for all $x > b'$. Since by (2.20), the continuity of $H^{(c')}(z)$, and that $\tilde{H}^{(c')}(z)$ at $z = b'$, we have $\lim_{x\to b_+} g(x) = 0$. Hence it suffices to show that $\frac{d}{dx}(g(x)\sqrt{x}) < 0$ on (b',∞). By (2.23), we can express $\frac{d}{dx}(g(x)\sqrt{x})$ as

$$-\int_0^{b'}\left(\frac{x+t}{2\sqrt{x}(x-t)^2} + \frac{\sinh^2(\sqrt{x}) + \cosh(2\sqrt{x})\sinh^2(\sqrt{t})}{2\sqrt{x}(\sinh^2(\sqrt{x}) - \sinh^2(\sqrt{t})}\right)\psi^{(c')}(t)dt - U'(x), \tag{2.41}$$

and it is negative for all $x \in (b', +\infty)$, due to the positivity of $\psi^{(c')}(t)$ and $U'(x)$.

Proof of Theorems 1.4 and 1.8. We only need to prove Theorem 1.8 that is a quantitative version of Theorem 1.4.

By the computation above in this subsection, we find that with c' being the unique solution of (1.33), the explicitly constructed density function $\psi^{(c')}(x)$ and the measure it defines satisfy all the items in Requirement 1, so it is the desired equilibrium measure. (The uniqueness is guaranteed by Proposition 1.3.) Hence we also verify that $c = c'$ and $b = b(c') = b'$. □

In the special case $V(x) = x/\mathbb{M}$, we have that for $s \in D$,

$$N(s) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathbb{M}^{-1}\frac{c}{4}(c\sqrt{\xi} + \operatorname{arcosh}\frac{\xi+1}{\xi-1})}{\sqrt{\xi}(\xi - s)} d\xi \tag{2.42a}$$

$$= \frac{c^2}{4\mathbb{M}} \frac{1}{2\pi i} \oint_\gamma \frac{d\xi}{\xi - s} + \frac{c}{4\mathbb{M}} \frac{1}{2\pi i} \oint_\gamma \frac{\operatorname{arcosh}\frac{\xi+1}{\xi-1}}{\sqrt{\xi}(\xi - s)} d\xi, \tag{2.42b}$$

and we find that the first term in (2.42b) is $c^2(4\mathbb{M})^{-1}s$ and the second term vanishes, by deforming γ into a large circle. Similarly we can evaluate $N(s)$ with $s \in \mathbb{C} \setminus \overline{D}$, and have (recalling $c = 2\mathbb{M}$ in this case)

$$N(s) = \begin{cases} \mathbb{M}, & s \in D, \\ \frac{1}{2\sqrt{s}} \operatorname{arcosh}\frac{s+1}{s-1}, & s \in \mathbb{C} \setminus \overline{D}. \end{cases} \tag{2.43}$$

Hence, we have that

$$G(z) = \frac{1}{\mathbb{M}} - \sqrt{\frac{\mathbf{I}_1(z)}{z}}, \qquad \tilde{G}(z) = \sqrt{\frac{\mathbf{I}_2(z)}{z}}, \tag{2.44}$$

and derive (1.36) by (2.14).

3 Asymptotic analysis for $p_{n+k}^{(n)}(x)$

3.1 RH problem of the polynomials

Consider the following modified Cauchy transform of p_j:

$$Cp_j(z) := \frac{1}{2\pi i} \int_{\mathbb{R}_+} \frac{p_j(x)}{f(x) - f(z)} W_\alpha^{(n)}(x) dx, \tag{3.1}$$

which is well defined for $z \in \mathbb{P} \setminus \mathbb{R}_+$. Since $W_\alpha^{(n)}(x)$ is real analytic and vanishes rapidly as $x \to +\infty$, we have the following asymptotic expansion for $Cp_j(z)$ as $z \in \mathbb{P} \setminus \mathbb{R}_+$ and $\Re z \to +\infty$:

$$\begin{aligned} Cp_j(z) &= \frac{-1}{2\pi i f(z)} \int_{\mathbb{R}_+} \frac{p_j(x)}{1 - f(x)/f(z)} W_\alpha^{(n)}(x) dx \\ &= \frac{-1}{2\pi i f(z)} \sum_{k=0}^{M} \left(\int_{\mathbb{R}_+} p_j(x) f^k(x) W_\alpha^{(n)}(x) dx \right) f^{-(k+1)}(z) + \mathcal{O}(f^{-(M+2)}(z)), \end{aligned} \tag{3.2}$$

for any $M \in \mathbb{N}$ and uniformly in $\Im z$. Thus due to the orthogonality,

$$Cp_j(z) = \frac{-h^{(n)}}{2\pi i} f^{-(j+1)}(z) + \mathcal{O}(f^{-(j+2)}(z)), \tag{3.3}$$

where $h_j^{(n)}$ is given in (1.5).

Hence we conclude that if we define the array

$$Y(z) = Y^{(j,n)}(z) := (p_j(z), Cp_j(z)), \tag{3.4}$$

then they satisfy the following conditions:

RH Problem 3.1.

1. $Y = (Y_1, Y_2)$, *where* Y_1 *is analytic on* $\mathbb{C}$, *and* Y_2 *is analytic on* $\mathbb{P} \setminus \mathbb{R}_+$.

2. *With the standard orientation of* $\mathbb{R}_+$,

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & W_\alpha^{(n)}(x)/f'(x) \\ 0 & 1 \end{pmatrix}, \quad \textit{for } x \in \mathbb{R}_+. \tag{3.5}$$

3. *As* $z \to \infty$ *in* $\mathbb{C}$, $Y_1(z) = z^j + \mathcal{O}(z^{j-1})$.

4. *As* $f(z) \to \infty$ *in* $\mathbb{P}$ *(i.e.,* $\Re z \to +\infty$*)*, $Y_2(z) = \mathcal{O}(f^{-(j+1)}(z))$.

5. *As* $z \to 0$ *in* $\mathbb{C}$ *or* $\mathbb{P}$,

$$Y_1(z) = \mathcal{O}(1), \qquad\qquad Y_2(z) = \begin{cases} \mathcal{O}(1), & \alpha > 0, \\ \mathcal{O}(\log z), & \alpha = 0, \\ \mathcal{O}(z^\alpha), & \alpha \in (-1, 0). \end{cases} \tag{3.6}$$

6. *At* $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, *the limit* $Y_2(z) := \lim_{w \to z \text{ in } \mathbb{P}} Y_2(w)$ *exists and is continuous, and*

$$Y_2(z) = Y_2(\bar{z}). \tag{3.7}$$

Below we take $j = n + k$ where k is a constant integer, and our goal is to obtain the asymptotics for $Y = Y^{(n+k,n)}$ as $n \to \infty$.

3.1.1 Uniqueness of RH problem 3.1

For later use in the proof of Lemma 3.15, we consider the uniqueness of a weaker form of RH problem 3.1 such that Item 5 of RH problem 3.1 is replace by

5'. *As* $z \to 0$ *in* $\mathbb{C}$ *or* $\mathbb{P}$,

$$Y_1(z) = \begin{cases} \mathcal{O}(z^{-\alpha}), & \alpha > 0 \textit{ and } \arg z \in [-\frac{\pi}{3}, \frac{\pi}{3}], \\ \mathcal{O}(\log z), & \alpha = 0, \\ \mathcal{O}(1), & \alpha \in (-1, 0) \textit{ or } \alpha > 0 \textit{ and } \arg(-z) \in (-\frac{2\pi}{3}, \frac{2\pi}{3}), \end{cases} \tag{3.8}$$

$$Y_2(z) = \begin{cases} \mathcal{O}(1), & \alpha > 0, \\ \mathcal{O}(\log z), & \alpha = 0, \\ \mathcal{O}(z^\alpha), & \alpha \in (-1, 0). \end{cases} \tag{3.9}$$

and $Y_1(z)$ *and* $Y_2(z)$ *are allowed to have a mild blowup at* b, *such that*

$$Y_1(z) = \mathcal{O}((z-b)^{-\frac{1}{2}}), \qquad Y_2(z) = \mathcal{O}((z-b)^{-\frac{1}{2}}), \qquad \textit{as } z \to b, \tag{3.10}$$

Proof of the uniqueness of RH problem 3.1 with Item 5 weakened to 5'. Despite nominally Y_1 may have blowups at 0 and b, since Y_1 has pole singularities at $1, b$, and it may only blow up at b like an inverse square root and may only blow up at 0 like an inverse logarithm at 0 in the sector $\arg(-z) \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$, we find that $Y_1(z)$ actually has no singularities at $1, b$. Hence, by Item 3 of RH problem 3.1, we find that Y_1 is a polynomial of degree j. Next, define

$$Z_2(z) = Y_2(z) - \frac{1}{2\pi i} \int_{\mathbb{R}_+} \frac{p_j(x)}{f(x) - f(z)} W_\alpha^{(n)}(x) dx. \tag{3.11}$$

Then $Z_2(z)$ has only a trivial jump on $\mathbb{R}_+$, so its can be defined analytically on $\mathbb{P} \setminus \{0, b\}$, and by an argument similar to that applied to Y_1 above, b is not a singular point of $Z_2(z)$, and $Z_2(z)$ can be defined analytically on $\mathbb{P} \setminus \{0\}$. Now consider the function $Z_2(f^{-1}(z))$ where $z \in \mathbb{C} \setminus (-\infty, -1]$. By Item 6 of RH problem 3.1, $Z_2(f^{-1}(z))$ can be extended analytically to $\mathbb{C} \setminus \{0\}$. Then by (3.6), $Z_2(f^{-1}(z)) = o(z^{-1})$ as $z \to 0$. Hence $Z_2(f^{-1}(z))$ is analytic on $\mathbb{C}$, and we conclude that $Z_2(z) = 0$ by Item 4 of RH problem 3.1. At last, Item 4 of RH problem 3.1 entails the orthogonality of $Y_1(z)$, so the uniqueness of biorthogonal polynomials given in Proposition 1.1 concludes the proof. □

3.2 First transformation $Y \mapsto T$

Recall $\mathbf{g}(z)$ and $\tilde{\mathbf{g}}(z)$ defined in (1.47) on $\mathbb{C} \setminus [0, b]$ and $\mathbb{P} \subseteq [0, b]$. Denote $Y = Y^{(n+k,n)}$ and define T as

$$T(z) = e^{-\frac{n\ell}{2}} Y(z) \begin{pmatrix} e^{-n\mathbf{g}(z)} & 0 \\ 0 & e^{n\tilde{\mathbf{g}}(z)} \end{pmatrix} e^{\frac{n\ell}{2}\sigma_3}, \tag{3.12}$$

where ℓ is the constant appearing in (1.16), and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then T satisfies a RH problem with the same domain of analyticity as Y, but with a different asymptotic behaviour and a different jump relation.

RH Problem 3.2.

1. *$T = (T_1, T_2)$, where T_1 is analytic in $\mathbb{C} \setminus \mathbb{R}_+$, and T_2 is analytic in $\mathbb{P} \setminus \mathbb{R}_+$.*
2. *T satisfies the jump relation*

$$T_+(x) = T_-(x) J_T(x), \quad \text{for } x \in \mathbb{R}_+, \tag{3.13}$$

where

$$J_T(x) = \begin{pmatrix} e^{n(\mathbf{g}_-(x) - \mathbf{g}_+(x))} & \frac{x^\alpha h(x)}{f'(x)} e^{n(\mathbf{g}_-(x) + \tilde{\mathbf{g}}_+(x) - V(z) - \ell)} \\ 0 & e^{n(\tilde{\mathbf{g}}_+(x) - \tilde{\mathbf{g}}_-(x))} \end{pmatrix}. \tag{3.14}$$

3.

$$T_1(z) = z^k + \mathcal{O}(z^{k-1}) \quad \text{as } z \to \infty \text{ in } \mathbb{C}, \quad T_2(z) = \mathcal{O}(f^{-(k+1)}(z)) \quad \text{as } f(z) \to \infty \text{ in } \mathbb{P}. \tag{3.15}$$

4. *As $z \to 0$ in $\mathbb{C}$ or in $\mathbb{P}$, $T(z)$ has the same limit behaviour as $Y(z)$ in (3.6).*
5.

$$T_1(z) = \mathcal{O}(1), \qquad T_2(z) = \mathcal{O}(1), \qquad \text{as } z \to b. \tag{3.16}$$

6. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $T_2(z)$ satisfies the same boundary condition as $Y_2(z)$ in (3.7).*

3.3 Second transformation $T \mapsto S$

For $x \in (b, +\infty)$, it follows from the analyticity of $\mathbf{g}(z)$ and $\tilde{\mathbf{g}}(z)$ there and (2.13) that the jump matrix $J_T(x)$ tends to the identity matrix exponentially fast in the limit $n \to \infty$. For $x \in (0, b)$, we decompose the jump matrix $J_T(x)$ as

$$\begin{pmatrix} 1 & 0 \\ \frac{f'(x)}{x^\alpha h(x)} e^{-n\phi_-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{e^{n(\mathbf{g}_-(x) + \tilde{\mathbf{g}}_+(x) - V(x) - \ell)}}{f'(x) x^{-\alpha} h(x)^{-1}} \\ -\frac{f'(x) x^{-\alpha} h(x)^{-1}}{e^{n(\mathbf{g}_-(x) + \tilde{\mathbf{g}}_+(x) - V(x) - \ell)}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{f'(x)}{x^\alpha h(x)} e^{-n\phi_+(x)} & 1 \end{pmatrix}. \tag{3.17}$$

Here $\phi(z) = \mathbf{g}(z) + \tilde{\mathbf{g}}(z) - V(z) - \ell$ is defined as in (1.48). The function $\phi(z)$ has discontinuity on $\mathbb{R}_-$ and $(0, b)$, such that

$$\phi_+(x) = \phi_-(x) + 4\pi i, \qquad x < 0, \tag{3.18}$$

$$\phi_+(x) = -\phi_-(x), \qquad x \in (0, b). \tag{3.19}$$

Then we "open the lens", where the lens Σ_S is a contour consisting of $\mathbb{R}_+$ and two arcs from 0 to b. We assume that one of the two arcs lies in the upper half plane and denote it by Σ_1, the other lies in the lower half plane and denote it by Σ_2, see Figure 5. (We may take Σ_1 and Σ_2 symmetric about the real axis.) We do not fix the shape of Σ_S at this state, but only require that Σ_S is in $\mathbb{P}$ and V is analytic in a simply-connected region containing Σ_S. The exact shape of Σ_1 and Σ_2 will be given in Sections 3.6, 3.7 and 3.8.

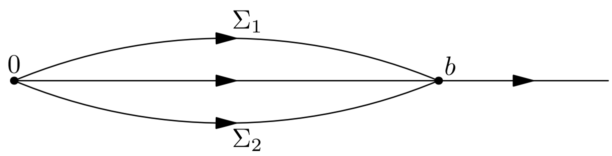

Figure 5: The lens Σ_S.

Define

$$S(z) := \begin{cases} T(z), & \text{outside of the lens,} \\ T(z) \begin{pmatrix} 1 & 0 \\ z^{-\alpha} h(z)^{-1} f'(z) e^{-n\phi(z)} & 1 \end{pmatrix}, & \text{in the lower part of the lens,} \\ T(z) \begin{pmatrix} 1 & 0 \\ -z^{-\alpha} h(z)^{-1} f'(z) e^{-n\phi(z)} & 1 \end{pmatrix}, & \text{in the upper part of the lens.} \end{cases} \tag{3.20}$$

From the definition of S, identity (2.12) and decomposition of $J_T(x)$ in (3.17), we have that S satisfies the following:

RH Problem 3.3.

1. *$S = (S_1, S_2)$, where S_1 is analytic in $\mathbb{C} \setminus \Sigma_S$, and S_2 is analytic in $\mathbb{P} \setminus \Sigma_S$.*

2. *We have*

$$S_+(z) = S_-(z) J_S(z), \quad \text{for } z \in \Sigma_S, \tag{3.21}$$

where

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-\alpha} h(z)^{-1} f'(z) e^{-n\phi(z)} & 1 \end{pmatrix}, & \text{for } z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & z^{\alpha} h(z) f'(z)^{-1} \\ -z^{-\alpha} h(z)^{-1} f'(z) & 0 \end{pmatrix}, & \text{for } z \in (0, b), \\ \begin{pmatrix} 1 & z^{\alpha} h(z) f'(z)^{-1} e^{n\phi(z)} \\ & 1 \end{pmatrix}, & \text{for } z \in (b, \infty). \end{cases} \tag{3.22}$$

3. *As $z \to \infty$ in $\mathbb{C}$ or $\mathbb{P}$, $S(z)$ has the same limit behaviour as $T(z)$ in (3.15).*

4. *As $z \to 0$ in $\mathbb{C} \setminus \Sigma$, we have*

$$S_1(z) = \begin{cases} \mathcal{O}(z^{-\alpha}), & \alpha > 0 \text{ and } z \text{ inside the lens}, \\ \mathcal{O}(\log z), & \alpha = 0 \text{ and } z \text{ inside the lens}, \\ \mathcal{O}(1), & z \text{ outside the lens or } -1 < \alpha < 0. \end{cases} \tag{3.23}$$

5. *As $z \to 0$ in $\mathbb{P}$, S_2 has the same limit behaviour as $Y_2(z)$ in (3.6).*

6. *As $z \to b$, $S(z)$ has the same limit behaviour as $T(z)$ as in (3.16).*

7. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $S_2(z)$ satisfies the same boundary condition as $Y_2(z)$ in (3.7).*

By (2.12), for $x \in (0, b)$, we have

$$\phi'_\pm(x) = \mathbf{g}'_\pm(x) + \tilde{\mathbf{g}}'_\pm(x) - V'(x) = \mathbf{g}'_\pm(x) - \tilde{\mathbf{g}}'_\mp(x) = \mp 2\pi i \psi(x). \tag{3.24}$$

Since $\psi(x) > 0$ for all $x \in (0, b)$, we have, by the Cauchy-Riemann condition, $\Re\phi(z) > 0$ on both the upper arc Σ_1 and lower arc Σ_2, if these arcs are sufficiently close to $(0, b)$. As a consequence, the jump matrix for S on the lenses tend to the identity matrix as $n \to \infty$. Uniform convergence breaks down when x approaches the end points 0 and b, so we need to use special local parametrices near these points.

Remark 2. Here and in subsequent RH problems, we may deform the jump contour $(b, +\infty)$ locally, as long as $\Re\phi(z) < 0$ there, so that the entry $z^\alpha h(z) f'(z)^{-1} e^{n\phi(z)}$ remains exponentially small.

3.4 Construction of the global parametrix

Since $J_S(z)$ converges to I on $\Sigma_1 \cup \Sigma_2 \cup (b, \infty)$, we construct the following

RH Problem 3.4.

1. *$P^{(\infty)} = (P_1^{(\infty)}, P_2^{(\infty)})$, where $P_1^{(\infty)}$ is analytic in $\mathbb{C} \setminus [0, b]$, and $P_2^{(\infty)}$ is analytic in $\mathbb{P} \setminus [0, b]$.*

2. *For $x \in (0, b)$, we have*

$$P_+^{(\infty)}(x) = P_-^{(\infty)}(x) \begin{pmatrix} 0 & x^\alpha h(x) f'(x)^{-1} \\ -x^{-\alpha} h(x)^{-1} f'(x) & 0 \end{pmatrix}.$$

3. *As $z \to \infty$ in $\mathbb{C}$ or $\mathbb{P}$, $P^{(\infty)}(z)$ has the same limit behaviour as $T(z)$ in (3.15).*

4. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $P_2^{(\infty)}(z)$ satisfies the same boundary condition as $Y_2(z)$ in (3.7).*

To construct a solution to the above RH problem, we follow the idea in [13] to map the RH problem for $P^{(\infty)}$ to a scalar RH problem which can be solved explicitly. More precisely, using the function $\mathbf{J}_c(s)$ defined in (1.21), we set

$$\mathcal{P}(s) := \begin{cases} P_1^{(\infty)}(\mathbf{J}_c(s)), & s \in \mathbb{C} \setminus \overline{D}, \\ P_2^{(\infty)}(\mathbf{J}_c(s)), & s \in D \setminus [0, 1], \end{cases} \tag{3.25}$$

where D is the region bounded by the curves γ_1 and γ_2, as shown in Figure 2. Due to Item 4 of RH Problem 3.4, the function $\mathcal{P}$ is then well defined onto $[0, 1)$ by continuation. It is straightforward to check that $\mathcal{P}$ satisfies the following:

RH Problem 3.5.

1. *$\mathcal{P}$ is analytic in $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2 \cup \{1\})$.*

2. *For $s \in \gamma_1 \cup \gamma_2$, $\mathcal{P}_+(s) = \mathcal{P}_-(s) J_{\mathcal{P}}(s)$, where, with γ_1 and γ_2 oriented from s_1 to s_2,*

$$J_{\mathcal{P}}(s) = \begin{cases} -\frac{\sinh(J_c(s))}{\mathbf{J}_c^{\alpha}(s) h(\mathbf{J}_c(s)) J_c(s)}, & s \in \gamma_1, \\ \frac{\mathbf{J}_c^{\alpha}(s) h(\mathbf{J}_c(s)) J_c(s)}{\sinh(J_c(s))}, & s \in \gamma_2. \end{cases} \tag{3.26}$$

3. *As $s \to \infty$, $\mathcal{P}(s) = (\frac{c^2}{4})^k s^k + \mathcal{O}(s^{k-1})$.*

4. *As $s \to 1$, $\mathcal{P}(s) = \mathcal{O}((s-1)^{k+1})$.*

The solution to RH problem 3.5 may not be unique. One solution is

$$\mathcal{P}(s) = \begin{cases} G_k(s), & s \in \mathbb{C} \setminus \bar{D}, \\ 2\frac{(\frac{c^2}{4})^{\alpha+\frac{1}{2}+k}(s-s_1)^{\alpha+1} s^{\frac{1}{2}}(s-1)^k}{\sinh(J_c(s))\sqrt{(s-s_1)(s-s_2)}} \tilde{D}(s)^{-1}, & s \in D. \end{cases} \tag{3.27}$$

where the power function of $(s-s_1)^{\alpha+1}$ takes the principal branch and $\sqrt{(s-s_1)(s-s_2)} \sim s$ in $\mathbb{C} \setminus \gamma_1$. Later in this paper we take (3.27) as the definition of $\mathcal{P}(s)$. We note that

$$\mathcal{P}(s) = \begin{cases} \mathcal{O}((s-s_1)^{-\alpha-\frac{1}{2}}), & s \to s_1 \text{ in } \mathbb{C} \setminus \bar{D}, \\ \mathcal{O}((s-s_1)^{\alpha-\frac{1}{2}}), & s \to s_1 \text{ in } D, \end{cases} \qquad \mathcal{P}(s) = \mathcal{O}((s-s_2)^{-\frac{1}{2}}), \quad s \to s_2. \tag{3.28}$$

Based on this solution, we construct the solution to RH problem 3.4 as

$$P_1^{(\infty)}(z) = \mathcal{P}(\mathbf{I}_1(z)) = G_k(\mathbf{I}_1(z)), \qquad z \in \mathbb{C} \setminus [0,b], \tag{3.29}$$

$$P_2^{(\infty)}(z) = \mathcal{P}(\mathbf{I}_2(z)), \qquad z \in \mathbb{P} \setminus [0,b]. \tag{3.30}$$

By direct calculation in Section A, we have

$$P_1^{(\infty)}(z) = \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}), \qquad P_2^{(\infty)}(z) = \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}), \qquad \text{as } z \to 0, \tag{3.31}$$

$$P_1^{(\infty)}(z) = \mathcal{O}(z^{-\frac{1}{4}}), \qquad P_2^{(\infty)}(z) = \mathcal{O}(z^{-\frac{1}{4}}), \qquad \text{as } z \to b. \tag{3.32}$$

3.5 Third transformation $S \mapsto Q$

Noting that $P_1^{(\infty)}(z) \neq 0$ for all $z \in \mathbb{C} \setminus [0,b]$ and $P_2^{(\infty)}(z) \neq 0$ for all $z \in \mathbb{P} \setminus [0,b]$, we define the third transformation by

$$Q(z) = (Q_1(z), Q_2(z)) = \left(\frac{S_1(z)}{P_1^{\infty}(z)}, \frac{S_2(z)}{P_2^{\infty}(z)} \right). \tag{3.33}$$

In view of the RH problems 3.3 and 3.4, and the properties of $P^{(\infty)}(z)$ in (3.31) and (3.32), it is then easily seen that Q satisfies the following RH problem:

RH Problem 3.6.

1. *$Q = (Q_1, Q_2)$, where Q_1 is analytic in $\mathbb{C} \setminus \Sigma$, and Q_2 is analytic in $\mathbb{P} \setminus \Sigma$.*

2. *For $z \in \Sigma$, we have*

$$Q_+(z) = Q_-(z) J_Q(z), \tag{3.34}$$

where

$$J_Q(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-\alpha} h(z)^{-1} f'(z) \frac{P_2^{(\infty)}(z)}{P_1^{(\infty)}(z)} e^{-n\phi(z)} & 1 \end{pmatrix}, & z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & z \in (0, b), \\ \begin{pmatrix} 1 & z^{\alpha} h(z) f'(z)^{-1} \frac{P_1^{(\infty)}(z)}{P_2^{(\infty)}(z)} e^{-n\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (b, \infty). \end{cases} \tag{3.35}$$

3.

$$Q_1(z) = 1 + \mathcal{O}(z^{-1}), \quad \text{as } z \to \infty \text{ in } \mathbb{C}, \quad Q_2(z) = \mathcal{O}(f^{-1}(z)), \quad \text{as } f(z) \to \infty \text{ in } \mathbb{P}. \tag{3.36}$$

4. *As $z \to 0$ in $\mathbb{C} \setminus \Sigma$, we have*

$$Q_1(z) = \begin{cases} \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}), & \alpha > 0 \text{ and } z \text{ inside the lens}, \\ \mathcal{O}(z^{\frac{1}{4}} \log z), & \alpha = 0 \text{ and } z \text{ inside the lens}, \\ \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}), & z \text{ outside the lens or } -1 < \alpha < 0. \end{cases} \tag{3.37}$$

5. *As $z \to 0$ in $\mathbb{P}$, we have*

$$Q_2(z) = \begin{cases} \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}), & \alpha > 0, \\ \mathcal{O}(z^{\frac{1}{4}} \log z), & \alpha = 0, \\ \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}), & \alpha \in (-1, 0). \end{cases} \tag{3.38}$$

6.

$$Q_1(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad Q_2(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad \text{as } z \to b. \tag{3.39}$$

7. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $Q_2(z)$ satisfies the same boundary condition as $Y_2(z)$ in (3.7).*

3.6 Construction of local parametrix near b

First we consider the local parametrix near b. Using Part 5 of Lemma 2.2, we have that $\lim_{z\to b} \phi(z) = 0$ where ϕ is defined in (1.48). Then by Part 3 of Lemma 2.2 and Part 5 of Requirement 1, we obtain the local behaviour for ϕ in the vicinity of b that (ψ_b is defined in (1.18))

$$\phi(z) = -\frac{4\pi}{3} \psi_b (z-b)^{\frac{3}{2}} + \mathcal{O}(|z-b|^{\frac{5}{2}}), \tag{3.40}$$

$\phi(z)/(z-b)^{3/2}$ is analytic at b, and then

$$f_b(z) = \left(-\frac{3}{4}\phi(z)\right)^{\frac{2}{3}} \tag{3.41}$$

is a conformal mapping in a neighbourhood $D(b, \epsilon)$ around b satisfying (1.49), where $\epsilon > 0$ is a small enough constant. Moreover, we also choose the shape of the contour Σ so that the image of $\Sigma \cap D(b, \epsilon)$ under the mapping f_b coincides with the jump contour Γ_{Ai} defined in (B.3) that is the jump contour of the RH problem B.1 for the Airy parametrix.

Let

$$g_1^{(b)}(z) = \frac{f'(z)/h(z)}{P_1^{(\infty)}(z)}, \qquad g_2^{(b)}(z) = \frac{z^\alpha}{P_2^{(\infty)}(z)}, \tag{3.42}$$

and define

$$\mathsf{P}^{(b)}(z) := \Psi^{(\mathrm{Ai})}(n^{\frac{2}{3}} f_b(z)) \begin{pmatrix} e^{-\frac{n}{2}\phi(z)} g_1^{(b)}(z) & 0 \\ 0 & e^{\frac{n}{2}\phi(z)} g_2^{(b)}(z) \end{pmatrix}, \quad z \in D(b, \epsilon) \setminus \Sigma. \tag{3.43}$$

From (3.42) and RH problem 3.4 satisfied by $P^{(\infty)}(z)$, we have

$$g_{1,+}^{(b)}(x) = -\, g_{2,-}^{(b)}(x), \qquad g_{2,+}^{(b)}(x) = g_{1,-}^{(b)}(x), \qquad \text{for } x \in (b-\epsilon, b), \tag{3.44}$$

$$g_1^{(b)}(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad g_2^{(b)}(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad \text{for } z \to b. \tag{3.45}$$

Then we have the following RH problem satisfied by $\mathsf{P}^{(b)}(z)$:

RH Problem 3.7.

1. *$\mathsf{P}^{(b)}(z)$ is a 2×2 matrix-valued function analytic for $z \in D(b, \epsilon) \setminus \Sigma$.*
2. *For $z \in \Sigma \cap D(b, \epsilon)$, we have*

$$\mathsf{P}_+^{(b)}(z) = \mathsf{P}_-^{(b)}(z) J_Q(z), \tag{3.46}$$

where $J_Q(z)$ is defined in (3.35)*.*

3.

$$(\mathsf{P}^{(b)})_{ij}(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \quad ((\mathsf{P}^{(b)})^{-1})_{ij}(z) = \mathcal{O}((z-b)^{-\frac{1}{4}}), \quad \text{as } z \to b, \quad i, j = 1, 2. \tag{3.47}$$

4. *For $z \in \partial D(b, \epsilon)$, we have, as $n \to \infty$,*

$$E^{(b)}(z) \mathsf{P}^{(b)}(z) = I + \mathcal{O}(n^{-1}), \tag{3.48}$$

where

$$E^{(b)}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} g_1^{(b)}(z) & 0 \\ 0 & g_2^{(b)}(z) \end{pmatrix}^{-1} e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n^{\frac{1}{6}} f_b(z)^{\frac{1}{4}} & 0 \\ 0 & n^{-\frac{1}{6}} f_b(z)^{-\frac{1}{4}} \end{pmatrix}. \tag{3.49}$$

It is straightforward to see that $E^{(b)}(z)$ defined in (3.49) is analytic on $D(b, \epsilon) \setminus (b - \epsilon, b]$, and for $x \in (b - \epsilon, b)$

$$E_+^{(b)}(x) E_-^{(b)}(x)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.50}$$

and as $z \to b$,

$$E^{(b)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}}) \\ \mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}}) \end{pmatrix}, \qquad E^{(b)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(z^{\frac{1}{2}}) & \mathcal{O}(z^{\frac{1}{2}}) \end{pmatrix}. \tag{3.51}$$

Then we define a 2×2 matrix-valued function

$$P^{(b)}(z) = E^{(b)}(z) \mathsf{P}^{(b)}(z), \quad z \in D(b, \epsilon) \setminus \Sigma, \tag{3.52}$$

where $\mathsf{P}^{(b)}$ is given in (3.43). Then we have the following RH problem satisfied by $P^{(b)}(z)$:

RH Problem 3.8.

1. *$P^{(b)}(z)$ is analytic in $D(b,\epsilon)\setminus\Sigma$.*

2. *For $z\in\Sigma\cap D(b,\epsilon)$, we have*

$$P_+^{(b)}(z)=\begin{cases}P_-^{(b)}(z)J_Q(z), & z\in\Sigma\cap D(b,\epsilon)\setminus(b-\epsilon,b],\\ \begin{pmatrix}0&1\\1&0\end{pmatrix}P_-^{(b)}(z)J_Q(z), & z\in(b-\epsilon,b).\end{cases}\tag{3.53}$$

3.

$$(P^{(b)})_{ij}(z)=\mathcal{O}((z-b)^{-\frac{1}{4}}),\quad ((P^{(b)})^{-1})_{ij}(z)=\mathcal{O}((z-b)^{-\frac{1}{4}}),\quad \text{as } z\to b,\quad i,j=1,2.\tag{3.54}$$

4. *For z on the boundary $\partial D(b,\epsilon)$, we have, as $n\to\infty$, $P^{(b)}(z)=I+\mathcal{O}(n^{-1})$.*

At last, we define a vector-valued function $V^{(b)}$ by

$$V^{(b)}(z)=Q(z)P^{(b)}(z)^{-1},\quad z\in D(b,\epsilon)\setminus\Sigma,\tag{3.55}$$

where $Q(z)$ is defined in (3.33). We find that $V^{(b)}(z)$ has only the trivial jump on $(\Sigma_1\cup\Sigma_2\cup[b,b+\epsilon))\cap D(b,\epsilon)$, so $V^{(b)}(z)$ can be defined by continuation on $D(b,\epsilon)\setminus(b-\epsilon,b]$. It satisfies the following RH problem:

RH Problem 3.9.

1. *$V^{(b)}=(V_1^{(b)},V_2^{(b)})$ is analytic in $D(b,\epsilon)\setminus(b-\epsilon,b]$.*

2. *For $x\in(b-\epsilon,b)$, we have*

$$V_+^{(b)}(x)=V_-^{(b)}(x)\begin{pmatrix}0&1\\1&0\end{pmatrix}.\tag{3.56}$$

3.

$$V_1^{(b)}(z)=\mathcal{O}(1),\qquad V_2^{(b)}(z)=\mathcal{O}(1),\qquad \text{as } z\to b.\tag{3.57}$$

4. *For $z\in\partial D(b,\epsilon)$, we have, as $n\to\infty$, $V^{(b)}(z)=Q(z)(I+\mathcal{O}(n^{-1}))$.*

3.7 Construction of local parametrix near 0

Using Part 1 of Lemma 2.2, we have that $\lim_{z\to 0 \text{ in } \mathbb{C}_\pm}\phi(z)=\pm\pi i$. We define in the vicinity of 0 that

$$\phi^L(z)=\begin{cases}\phi(z)-\pi i, & z\in\mathbb{C}_+,\\ \phi(z)+\pi i, & z\in\mathbb{C}_-,\end{cases}\tag{3.58}$$

Then by Part 3 of Lemma 2.2 and Part 5 of Requirement 1, we obtain the local behaviour for ϕ (or equivalently ϕ^L) at 0 that (ψ_0 is the positive constant defined in Part 5 of Requirement 1)

$$\phi^L(z)=4\pi\psi_0(-z)^{1/2}+\mathcal{O}((-z)^{3/2}),\tag{3.59}$$

$\phi^L(z)/(-z)^{1/2}$ is analytic at 0, and then

$$f_0(z):=\frac{\phi^L(z)^2}{16}\tag{3.60}$$

is a conformal mapping in a neighbourhood $D(0,\epsilon)$ around 0 satisfying (1.50), where $\epsilon > 0$ is a small enough constant. Moreover, we also choose the shape of the contour Σ so that the image of $\Sigma \cap D(0,\epsilon)$ under the mapping f_0 coincides with the jump contour Γ_{Be} defined in (B.9) that is the jump contour of the RH problem B.2 for the Bessel parametrix.

Let

$$g_1^{(0)}(z) = \frac{(-z)^{-\alpha/2} f'(z)/h(z)}{P_1^{(\infty)}(z)}, \qquad g_2^{(0)}(z) = \frac{(-z)^{\alpha/2}}{P_2^{(\infty)}(z)}, \tag{3.61}$$

where the $(-z)^{\pm\alpha/2}$ takes the principal branch, and define

$$\mathsf{P}^{(0)}(z) = \Phi^{(\mathrm{Be})}(n^2 f_0(z)) \begin{pmatrix} e^{-\frac{n}{2}\phi(z)} g_1^{(0)}(z) & 0 \\ 0 & e^{\frac{n}{2}\phi(z)} g_2^{(0)}(z) \end{pmatrix}, \quad z \in D(0,\epsilon) \setminus \Sigma. \tag{3.62}$$

From (3.61) and RH problem 3.4 satisfied by $P^{(\infty)}(z)$, we have

$$g_{1,+}^{(0)}(x) = -\, g_{2,-}^{(0)}(x), \qquad g_{2,+}^{(0)}(x) = g_{1,-}^{(0)}(x), \qquad \text{for } x \in (0,\epsilon), \tag{3.63}$$

$$g_1^{(0)}(z) = \mathcal{O}(z^{\frac{1}{4}}), \qquad g_2^{(0)}(z) = \mathcal{O}(z^{\frac{1}{4}}), \qquad \text{for } z \to 0. \tag{3.64}$$

Then we have the following RH problem satisfied by $\mathsf{P}^{(0)}(z)$:

RH Problem 3.10.

1. *$\mathsf{P}^{(0)}(z)$ is a 2×2 matrix-valued function analytic for $z \in D(0,\epsilon) \setminus \Sigma$.*
2. *For $z \in \Sigma \cap D(0,\epsilon)$, we have*

$$\mathsf{P}_+^{(0)}(z) = \mathsf{P}_-^{(0)}(z) J_Q(z), \tag{3.65}$$

where $J_Q(z)$ is defined in (3.35).

3. *As $z \in \partial D(0,\epsilon)$, we have*

$$E^{(0)}(z)\mathsf{P}^{(0)}(z) = (I + \mathcal{O}(n^{-1})), \tag{3.66}$$

where

$$E^{(0)}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} g_1^{(0)}(z) & 0 \\ 0 & g_2^{(0)}(z) \end{pmatrix}^{-1} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} n^{\frac{1}{2}} f_0(z)^{\frac{1}{4}} & 0 \\ 0 & n^{-\frac{1}{2}} f_0(z)^{-\frac{1}{4}} \end{pmatrix} (2\pi)^{\frac{1}{2}\sigma_3}. \tag{3.67}$$

4. *As $z \to 0$, if $\alpha \in (-1,0)$, then*

$$\mathsf{P}^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}) \end{pmatrix}, \quad \mathsf{P}^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \tag{3.68}$$

if $\alpha = 0$, then

$$\mathsf{P}^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{1}{4}}\log z) & \mathcal{O}(z^{\frac{1}{4}}\log z) \\ \mathcal{O}(z^{\frac{1}{4}}\log z) & \mathcal{O}(z^{\frac{1}{4}}\log z) \end{pmatrix}, \quad \mathsf{P}^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{-\frac{1}{4}}\log z) & \mathcal{O}(z^{-\frac{1}{4}}\log z) \\ \mathcal{O}(z^{-\frac{1}{4}}\log z) & \mathcal{O}(z^{-\frac{1}{4}}\log z) \end{pmatrix}, \tag{3.69}$$

and if $\alpha > 0$, then outside the lens

$$\mathsf{P}^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}) \end{pmatrix}, \quad \mathsf{P}^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \tag{3.70}$$

and inside the lens

$$\mathsf{P}^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}) \end{pmatrix}, \quad \mathsf{P}^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \tag{3.71}$$

It is straightforward to see that $E^{(0)}(z)$ is analytic on $D(0,\epsilon) \setminus [0,\epsilon)$, and for $x \in (0,\epsilon)$,

$$E_+^{(0)}(x)E_-^{(0)}(x)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.72}$$

and as $z \to 0$,

$$E^{(0)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}}) \\ \mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}}) \end{pmatrix}, \qquad E^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(z^{\frac{1}{2}}) & \mathcal{O}(z^{\frac{1}{2}}) \end{pmatrix}. \tag{3.73}$$

Then we define a 2×2 matrix-valued function

$$P^{(0)}(z) = E^{(0)}(z)\mathsf{P}^{(0)}(z), \quad z \in D(0,\epsilon) \setminus \Sigma, \tag{3.74}$$

where $\mathsf{P}^{(0)}$ is given in (3.62). Hence, we have the following RH problem satisfied by $P^{(0)}(z)$:

RH Problem 3.11.

1. *$P^{(0)}(z)$ is analytic in $D(0,\epsilon) \setminus \Sigma$.*
2. *For $z \in \Sigma \cap D(0,\epsilon)$, we have*

$$P_+^{(0)}(z) = \begin{cases} P_-^{(0)}(z)J_Q(z), & z \in \Sigma \cap D(0,\epsilon) \setminus [0,\epsilon), \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P_-^{(0)}(z)J_Q(z), & z \in (0,\epsilon). \end{cases} \tag{3.75}$$

3. *As $z \to 0$, if $\alpha \in (-1,0)$, then*

$$P^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \quad P^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \tag{3.76}$$

if $\alpha = 0$, then

$$P^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{-\frac{1}{4}}\log z) & \mathcal{O}(z^{-\frac{1}{4}}\log z) \\ \mathcal{O}(z^{-\frac{1}{4}}\log z) & \mathcal{O}(z^{-\frac{1}{4}}\log z) \end{pmatrix}, \quad P^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{-\frac{1}{4}}\log z) & \mathcal{O}(z^{-\frac{1}{4}}\log z) \\ \mathcal{O}(z^{-\frac{1}{4}}\log z) & \mathcal{O}(z^{-\frac{1}{4}}\log z) \end{pmatrix}, \tag{3.77}$$

and if $\alpha > 0$, then outside the lens

$$P^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \quad P^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}. \tag{3.78}$$

and inside the lens

$$P^{(0)}(z) = \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \quad P^{(0)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) & \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}) \end{pmatrix}, \tag{3.79}$$

4. For z on the boundary $\partial D(0,\epsilon)$, we have, as $n \to \infty$, $P^{(0)}(z) = I + \mathcal{O}(n^{-1})$.

Consider the vector-valued function

$$U(z) = (U_1(z), U_2(z)) := Q(z)\mathsf{P}^{(0)}(z)^{-1}, \quad \in D(0,\epsilon) \setminus \Sigma, \tag{3.80}$$

where $Q(z)$ is defined in (3.33), and then define the vector-valued function $V^{(0)}$ on $D(0,\epsilon) \setminus \Sigma$ by

$$V^{(0)}(z) = (V_1^{(0)}(z), V_2^{(0)}(z)) := Q(z)P^{(0)}(z)^{-1} = U(z)E^{(0)}(z)^{-1}. \tag{3.81}$$

Due to the jump conditions (3.34) and (3.46), we have that $U(z)$ can be extended analytically to $D(0,\epsilon) \setminus \{0\}$. Furthermore, as $z \to 0$, from part 4 of RH problem 3.6 satisfied by $Q(z)$ and part 4 of RHP 3.10 satisfied by $\mathsf{P}^{(0)}(z)$, we have that

$$(U_1(z), U_2(z)) = \begin{cases} (\mathcal{O}(1), \mathcal{O}(1)), & \alpha > 0 \text{ and } z \text{ is outside the lens}, \\ (\mathcal{O}(z^{-\alpha}), \mathcal{O}(z^{-\alpha})), & \alpha > 0 \text{ and } z \text{ is inside the lens}, \\ (\mathcal{O}((\log z)^2), \mathcal{O}((\log z)^2)), & \alpha = 0, \\ (\mathcal{O}(z^{\alpha}), \mathcal{O}(z^{\alpha})), & \alpha \in (-1, 0). \end{cases} \tag{3.82}$$

Since 0 is an isolated singular point of $U_1(z)$ and $U_2(z)$, the estimates above implies that 0 is a removable singular point of $U_1(z)$ and $U_2(z)$, or equivalently, these two functions can be extended analytically to $D(0,\epsilon)$. Hence, $V^{(0)}(z)$ satisfies the following RH problem:

RH Problem 3.12.

1. *$V^{(0)}(z) = (V_1^{(0)}(z), V_2^{(0)}(z))$ is analytic in $D(0,\epsilon) \setminus [0,\epsilon)$.*
2. *For $x \in (0,\epsilon)$, we have*

$$V_+^{(0)}(x) = V_-^{(0)}(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.83}$$

3. *As $z \to 0$, we have $V(z) = (\mathcal{O}(1), \mathcal{O}(1))$.*
4. *As $z \in \partial D(0,\epsilon)$, $V^{(0)}(z) = Q(z)(I + \mathcal{O}(n^{-1}))$.*

3.8 Final transformation

We define $R(z) = (R_1(z), R_2(z))$ as

$$R_1(z) = \begin{cases} V_1^{(b)}(z), & z \in D(b,\epsilon) \setminus (b-\epsilon, b], \\ V_1^{(0)}(z), & z \in D(0,\epsilon) \setminus [0,\epsilon), \\ Q_1(z), & z \in \mathbb{C} \setminus (\overline{D(b,\epsilon)} \cup \overline{D(0,\epsilon)} \cup \Sigma), \end{cases} \tag{3.84}$$

$$R_2(z) = \begin{cases} V_2^{(b)}(z), & z \in D(b,\epsilon) \setminus (b-\epsilon, b], \\ V_2^{(0)}(z), & z \in D(0,\epsilon) \setminus [0,\epsilon), \\ Q_2(z), & z \in \mathbb{P} \setminus (\overline{D(b,\epsilon)} \cup \overline{D(0,\epsilon)} \cup \Sigma). \end{cases} \tag{3.85}$$

We set

$$\Sigma^R := [0,b] \cup [b+\epsilon, \infty) \cup \partial D(0,\epsilon) \cup \partial D(b,\epsilon) \cup \Sigma_1^R \cup \Sigma_2^R, \tag{3.86}$$

where

$$\Sigma_i^R := \Sigma_i \setminus \{D(0,\epsilon) \cup D(b,\epsilon)\}, \quad i = 1, 2. \tag{3.87}$$

See Figure 6 for an illustration and the orientation of the arcs. Here we can fix the shape of Σ_i^R (and so finally fix the shape of Σ_i) by letting Σ_i be a continuous arc and $\Re\phi(z) < 0$ on Σ_i^R. It is straightforward to check that R satisfies the following RH problem:

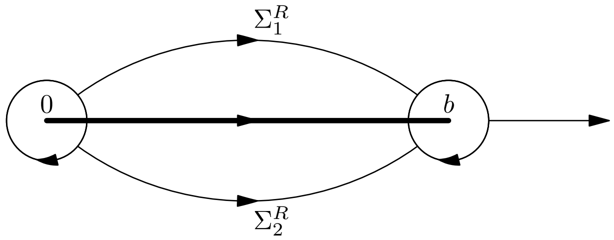

Figure 6: Contour Σ^R.

RH Problem 3.13.

1. $R(z) = (R_1(z), R_2(z))$, *where* $R_1(z)$ *is analytic in* $\mathbb{C}\setminus\Sigma^R$, *and* $R_2(z)$ *is analytic in* $\mathbb{P}\setminus\Sigma^R$.

2. $R(z)$ *satisfies the following jump conditions:*

$$R_+(z) = R_-(z)\begin{cases} J_Q(z), & z\in\Sigma_1^R\cup\Sigma_2^R\cup(b+\epsilon,+\infty),\\ P^{(b)}(z), & z\in\partial D(b,\epsilon),\\ P^{(0)}(z), & z\in\partial D(0,\epsilon),\\ \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, & z\in(0,b)\setminus\{\epsilon, b-\epsilon\}.\end{cases} \tag{3.88}$$

3.

$$R_1(z) = 1+\mathcal{O}(z^{-1}) \quad \text{as } z\to\infty \text{ in } \mathbb{C}, \qquad R_2(z) = \mathcal{O}(1) \quad \text{as } f(z)\to\infty \text{ in } \mathbb{P}. \tag{3.89}$$

4.

$$R_1(z) = \mathcal{O}(1), \qquad R_2(z) = \mathcal{O}(1), \qquad \text{as } z\to 0, \tag{3.90}$$

$$R_1(z) = \mathcal{O}(1), \qquad R_2(z) = \mathcal{O}(1), \qquad \text{as } z\to b. \tag{3.91}$$

5. At $z\in\rho\cup\{-\pi^2/4\}\cup\bar{\rho}$, $R_2(z)$ *satisfies the same boundary condition as* $Y_2(z)$ *in* (3.7).

Similar to the idea used in the construction of global parametrix, to estimate R for large n, we now transform the RH problem for R to a scalar one on the complex plane by defining

$$\mathcal{R}(s) = \begin{cases} R_1(\mathbf{J}_c(s)), & s\in\mathbb{C}\setminus\overline{D} \text{ and } s\notin\mathbf{I}_1(\Sigma^R),\\ R_2(\mathbf{J}_c(s)), & s\in D\setminus[0,1] \text{ and } s\notin\mathbf{I}_2(\Sigma^R),\end{cases} \tag{3.92}$$

where we recall that D is the region bounded by the curves γ_1 and γ_2, $\mathbf{I}_1 : \mathbb{C}\setminus[0,b]\to\mathbb{C}\setminus\overline{D}$ and $\mathbf{I}_2 : \mathbb{P}\setminus[0,b]$ are defined in (1.26) and (1.27), respectively.

We are now at the stage of describing the RH problem for $\mathcal{R}$. For this purpose, we define

$$\begin{aligned} &\boldsymbol{\Sigma}^{(1)} := \mathbf{I}_1(\Sigma_1^R\cup\Sigma_2^R)\subseteq\mathbb{C}\setminus\overline{D}, && \boldsymbol{\Sigma}^{(1')} := \mathbf{I}_2(\Sigma_1^R\cup\Sigma_2^R)\subseteq D,\\ &\boldsymbol{\Sigma}^{(2)} := \mathbf{I}_1((b+\epsilon,+\infty))\subseteq\mathbb{C}\setminus\overline{D}, && \boldsymbol{\Sigma}^{(2')} := \mathbf{I}_2((b+\epsilon,+\infty))\subseteq D,\\ &\boldsymbol{\Sigma}^{(3)} := \mathbf{I}_1(\partial D(b,\epsilon))\subseteq\mathbb{C}\setminus\overline{D}, && \boldsymbol{\Sigma}^{(3')} := \mathbf{I}_2(\partial D(b,\epsilon))\subseteq D,\\ &\boldsymbol{\Sigma}^{(4)} := \mathbf{I}_1(\partial D(0,\epsilon))\subseteq\mathbb{C}\setminus\overline{D}, && \boldsymbol{\Sigma}^{(4')} := \mathbf{I}_2(\partial D(0,\epsilon))\subseteq D,\end{aligned} \tag{3.93}$$

and set

$$\boldsymbol{\Sigma} := \boldsymbol{\Sigma}^{(1)}\cup\boldsymbol{\Sigma}^{(1')}\cup\boldsymbol{\Sigma}^{(2)}\cup\boldsymbol{\Sigma}^{(2')}\cup\boldsymbol{\Sigma}^{(3)}\cup\boldsymbol{\Sigma}^{(3')}\cup\boldsymbol{\Sigma}^{(4)}\cup\boldsymbol{\Sigma}^{(4')}. \tag{3.94}$$

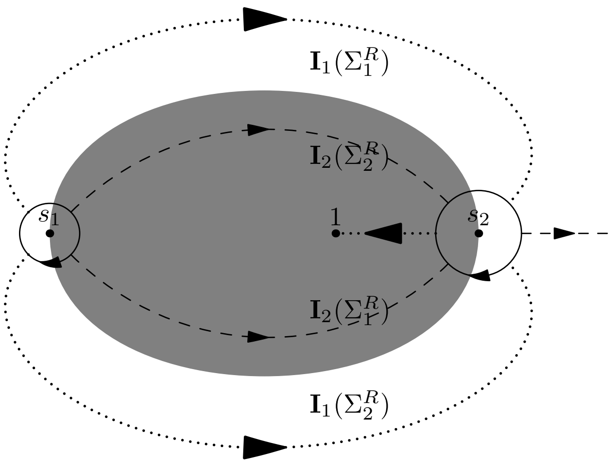

Figure 7: Contour $\boldsymbol{\Sigma}$. (It is also $\omega(\tilde{\boldsymbol{\Sigma}})$.) The solid and the dotted curves are the non-trivial jump contour for the RH problem for $\mathcal{R}$. (The solid and dashed curves, upon the mapping ω, are the non-trivial jump contour for the RH problem for $\tilde{\mathcal{R}}$.)

See Figure 7 for an illustration. We also define the following functions on each curve constituting $\boldsymbol{\Sigma}$: (Below $z = \mathbf{J}_c(s)$)

$$J_{\boldsymbol{\Sigma}^{(1)}}(s) = (J_Q)_{21}(z), \qquad s \in \boldsymbol{\Sigma}^{(1)}, \tag{3.95}$$
$$J_{\boldsymbol{\Sigma}^{(2')}}(s) = (J_Q)_{12}(z), \qquad s \in \boldsymbol{\Sigma}^{(2')}, \tag{3.96}$$
$$J^1_{\boldsymbol{\Sigma}^{(3)}}(s) = (P^{(b)})_{11}(z) - 1, \quad J^2_{\boldsymbol{\Sigma}^{(3)}}(s) = (P^{(b)})_{21}(z), \qquad s \in \boldsymbol{\Sigma}^{(3)}, \tag{3.97}$$
$$J^1_{\boldsymbol{\Sigma}^{(3')}}(s) = (P^{(b)})_{22}(z) - 1, \quad J^2_{\boldsymbol{\Sigma}^{(3')}}(s) = (P^{(b)})_{12}(z), \qquad s \in \boldsymbol{\Sigma}^{(3')}, \tag{3.98}$$
$$J^1_{\boldsymbol{\Sigma}^{(4)}}(s) = (P^{(0)})_{11}(z) - 1, \quad J^2_{\boldsymbol{\Sigma}^{(4)}}(s) = (P^{(0)})_{21}(z), \qquad s \in \boldsymbol{\Sigma}^{(4)}, \tag{3.99}$$
$$J^1_{\boldsymbol{\Sigma}^{(4')}}(s) = (P^{(0)})_{22}(z) - 1, \quad J^2_{\boldsymbol{\Sigma}^{(4')}}(s) = (P^{(0)})_{12}(z), \qquad s \in \boldsymbol{\Sigma}^{(4')}, \tag{3.100}$$

where $z = \mathbf{J}_c(s)$ is in $\Sigma_1^R \cup \Sigma_2^R$ in (3.95); in $(b+\epsilon, +\infty)$ in (3.96); in $\partial D(b, \epsilon)$ in (3.97) and (3.98); in $\partial D(0, \epsilon)$ in (3.99) and (3.100). With the aid of these functions, we further define an operator $\Delta_{\boldsymbol{\Sigma}}$ such that for any complex-valued function $f(s)$ defined on $\boldsymbol{\Sigma}$, $\Delta_{\boldsymbol{\Sigma}}$ transforms it linearly into a function $\Delta_{\boldsymbol{\Sigma}} f$ that is also a complex-valued function defined on $\boldsymbol{\Sigma}$, with expression

$$(\Delta_{\boldsymbol{\Sigma}} f)(s) = \begin{cases} J_{\boldsymbol{\Sigma}^{(1)}}(s) f(\tilde{s}), & s \in \boldsymbol{\Sigma}^{(1)} \text{ and } \tilde{s} = \mathbf{I}_2(\mathbf{J}_c(s)) \in \boldsymbol{\Sigma}^{(1')}, \\ J_{\boldsymbol{\Sigma}^{(2')}}(s) f(\tilde{s}), & s \in \boldsymbol{\Sigma}^{(2')} \text{ and } \tilde{s} = \mathbf{I}_1(\mathbf{J}_c(s)) \in \boldsymbol{\Sigma}^{(2)}, \\ 0, & s \in \boldsymbol{\Sigma}^{(1')} \cup \boldsymbol{\Sigma}^{(2)}, \\ J^1_{\boldsymbol{\Sigma}^{(3)}}(s) f(s) + J^2_{\boldsymbol{\Sigma}^{(3)}}(s) f(\tilde{s}), & s \in \boldsymbol{\Sigma}^{(3)} \text{ and } \tilde{s} = \mathbf{I}_2(\mathbf{J}_c(s)) \in \boldsymbol{\Sigma}^{(3')}, \\ J^1_{\boldsymbol{\Sigma}^{(3')}}(s) f(s) + J^2_{\boldsymbol{\Sigma}^{(3')}}(s) f(\tilde{s}), & s \in \boldsymbol{\Sigma}^{(3')} \text{ and } \tilde{s} = \mathbf{I}_1(\mathbf{J}_c(s)) \in \boldsymbol{\Sigma}^{(3)}, \\ J^1_{\boldsymbol{\Sigma}^{(4)}}(s) f(s) + J^2_{\boldsymbol{\Sigma}^{(4)}}(s) f(\tilde{s}), & s \in \boldsymbol{\Sigma}^{(4)} \text{ and } \tilde{s} = \mathbf{I}_2(\mathbf{J}_c(s)) \in \boldsymbol{\Sigma}^{(4')}, \\ J^1_{\boldsymbol{\Sigma}^{(4')}}(s) f(s) + J^2_{\boldsymbol{\Sigma}^{(4')}}(s) f(\tilde{s}), & s \in \boldsymbol{\Sigma}^{(4')} \text{ and } \tilde{s} = \mathbf{I}_1(\mathbf{J}_c(s)) \in \boldsymbol{\Sigma}^{(4)}. \end{cases} \tag{3.101}$$

We note that all the functions $J_{\boldsymbol{\Sigma}^{(1)}}(s), \ldots, J^2_{\boldsymbol{\Sigma}^{(4')}}(s)$ that define $\Delta_{\boldsymbol{\Sigma}}$ in (3.101) are uniformly $\mathcal{O}(n^{-1})$. If we view $\Delta_{\boldsymbol{\Sigma}}$ as an operator from $L^2(\boldsymbol{\Sigma})$ to $L^2(\boldsymbol{\Sigma})$, then we have the estimate that for all large enough n, there is a constant $M_{\boldsymbol{\Sigma}} > 0$ such that

$$\|\Delta_{\boldsymbol{\Sigma}}\|_{L^2(\boldsymbol{\Sigma})} \le M_{\boldsymbol{\Sigma}} n^{-1}. \tag{3.102}$$

RH problem 3.13 entails a scalar shifted RH problem for $\mathcal{R}$:

RH Problem 3.14.

1. *$\mathcal{R}(s)$ is analytic in $\mathbb{C} \setminus \boldsymbol{\Sigma}$, where the contour $\boldsymbol{\Sigma}$ is defined in* (3.94).
2. *For $s \in \boldsymbol{\Sigma}$, we have*
$$\mathcal{R}_+(s) - \mathcal{R}_-(s) = (\Delta_{\boldsymbol{\Sigma}} \mathcal{R}_-)(s), \tag{3.103}$$
where $\Delta_{\boldsymbol{\Sigma}}$ s the operator defined in (3.94).
3. *As $s \to \infty$, we have*
$$\mathcal{R}(s) = 1 + \mathcal{O}(s^{-1}). \tag{3.104}$$
4. *As $s \to 0$, we have $\mathcal{R}(s) = \mathcal{O}(1)$.*

We have the following uniqueness result about the solution of the above RH problem.

Lemma 3.15. *The function $\mathcal{R}(s)$ defined in* (3.92) *is the unique solution of RH problem 3.14.*

Proof. Suppose $\mathcal{R}^{\mathrm{sol}}(s)$ is one solution to RH problem 3.14, then using (3.92) backwardly, we have a solution $R^{\mathrm{sol}}(z) = (R_1^{\mathrm{sol}}(z), R_2^{\mathrm{sol}}(z))$ to RH problem 3.13. From $R^{\mathrm{sol}}(s)$, we define the vector-valued function $U^{\mathrm{sol}}(z)$ on $D(0,\epsilon) \setminus [0,\epsilon)$ as (using (3.81) backwardly)
$$(U_1^{\mathrm{sol}}(z), U_2^{\mathrm{sol}}(z)) = (R_1^{\mathrm{sol}}(z), R_2^{\mathrm{sol}}(z)) E^{(0)}(z), \tag{3.105}$$
where $E^{(0)}(z)$ is defined in (3.73). Then $U_1^{\mathrm{sol}}(z)$ and $U_2^{\mathrm{sol}}(z)$ can be defined analytically in $D(0,\epsilon) \setminus \{0\}$, and at the isolated singularity 0 they may only blow up like inverse square root. Hence, $U_1^{\mathrm{sol}}(z)$ and $U_2^{\mathrm{sol}}(z)$ are actually analytic in $D(0,\epsilon)$. Next, we define $Q^{\mathrm{sol}}(z) = (Q_1^{\mathrm{sol}}(z), Q_2^{\mathrm{sol}}(z))$ by (using (3.84), (3.85) and (3.80) backwardly)
$$Q_1^{\mathrm{sol}}(z) = \begin{cases} R_1^{\mathrm{sol}}(z)(P^{(b)})_{11}(z) + R_2^{\mathrm{sol}}(z)(P^{(b)})_{21}(z), & z \in D(b,\epsilon) \setminus \Sigma, \\ U_1^{\mathrm{sol}}(z)(\mathsf{P}^{(0)})_{11}(z) + U_2^{\mathrm{sol}}(z)(\mathsf{P}^{(0)})_{21}(z), & z \in D(0,\epsilon) \setminus \Sigma, \\ R_1^{\mathrm{sol}}(z), & z \in \mathbb{C} \setminus (\overline{D(b,\epsilon)} \cup \overline{D(0,\epsilon)} \cup \Sigma), \end{cases} \tag{3.106}$$
$$Q_2^{\mathrm{sol}}(z) = \begin{cases} R_1^{\mathrm{sol}}(z)(P^{(b)})_{12}(z) + R_2^{\mathrm{sol}}(z)(P^{(b)})_{22}(z), & z \in D(b,\epsilon) \setminus \Sigma, \\ U_1^{\mathrm{sol}}(z)(\mathsf{P}^{(0)})_{12}(z) + U_2^{\mathrm{sol}}(z)(\mathsf{P}^{(0)})_{22}(z), & z \in D(0,\epsilon) \setminus \Sigma, \\ R_2^{\mathrm{sol}}(z), & z \in \mathbb{P} \setminus (\overline{D(b,\epsilon)} \cup \overline{D(0,\epsilon)} \cup \Sigma), \end{cases} \tag{3.107}$$
We find that $Q_1^{\mathrm{sol}}(z)$ and $Q_1^{\mathrm{sol}}(z)$ can be defined analytically on $\mathbb{C} \setminus \Sigma$ and $\mathbb{P} \setminus \Sigma$, respectively, and find that $Q^{\mathrm{sol}}(z)$ satisfies the variation of RH problem 3.6, such that in Item 4, the limit behaviour of $Q_1(z)$ as $z \to 0$ from outside of the lens is changed to $Q_1(z) = \mathcal{O}(z^{1/4} \log z)$, and in Item 6, the occurrences of $(z-b)^{\frac{1}{4}}$ in (3.39) are replaced by those of $(z-b)^{-\frac{1}{4}}$.

Furthermore, we do the transforms $Y \to T \to S \to Q$ backwardly, and find that from Q^{sol} we can construct $Y^{\mathrm{sol}}(z) = (Y_1^{\mathrm{sol}}(z), Y_2^{\mathrm{sol}}(z))$ that satisfies the variation of RH problem 3.1 such that Item 5 is replaced by Item 5'. in Section 3.1.1. By the argument in Section 3.1.1, we have that $Y^{\mathrm{sol}}(z)$ is unique, and then $\mathcal{R}^{\mathrm{sol}}(s)$ is unique. □

Finally, we show that

Lemma 3.16. *For all $s \in \mathbb{C} \setminus \boldsymbol{\Sigma}$, we have the uniform convergence*
$$\mathcal{R}(s) = 1 + \mathcal{O}(n^{-1}). \tag{3.108}$$

This lemma immediately yields that
$$R_1(z) = 1 + \mathcal{O}(n^{-1}) \quad \text{uniformly in } \mathbb{C} \setminus \Sigma^R, \quad R_2(z) = 1 + \mathcal{O}(n^{-1}) \quad \text{uniformly in } \mathbb{P} \setminus \Sigma^R. \tag{3.109}$$

Proof of Lemma 3.16. We use the strategy proposed in [13], and start with the claim that $\mathcal{R}$ satisfies the integral equation

$$\mathcal{R}(s) = 1 + \mathcal{C}(\Delta_{\boldsymbol{\Sigma}}\mathcal{R}_-)(s), \tag{3.110}$$

where $\mathcal{C}$ is the Cauchy transform on $\boldsymbol{\Sigma}$, such that for any $g(s)$ defined on $\boldsymbol{\Sigma}$,

$$\mathcal{C}g(s) = \frac{1}{2\pi i}\int_{\boldsymbol{\Sigma}} \frac{g(\xi)}{\xi - s}d\xi, \quad s \in \mathbb{C} \setminus \boldsymbol{\Sigma}. \tag{3.111}$$

To verify (3.110), by the uniqueness of RH problem 3.14, it suffices to show that the right-hand side of (3.110) satisfies RH problem 3.14, and it is straightforward.

(3.110) can be written as

$$\mathcal{R}(s) - 1 = \frac{1}{2\pi i}\int_{\boldsymbol{\Sigma}} \frac{\Delta_{\boldsymbol{\Sigma}}(\mathcal{R}_- - 1)(\xi)}{\xi - s}d\xi + \frac{1}{2\pi i}\int_{\boldsymbol{\Sigma}} \frac{\Delta_{\boldsymbol{\Sigma}}(1)(\xi)}{\xi - s}d\xi, \quad s \in \mathbb{C} \setminus \boldsymbol{\Sigma}. \tag{3.112}$$

Below we estimate the two terms on the right-hand side of the above formula.

By taking the limit where s approaches the minus side of $\boldsymbol{\Sigma}$, we obtain from (3.112) that

$$\mathcal{R}_-(s) - 1 = \mathcal{C}_{\Delta_{\boldsymbol{\Sigma}}}(\mathcal{R}_- - 1)(s) + \mathcal{C}_-(\Delta_{\boldsymbol{\Sigma}}(1))(s), \tag{3.113}$$

where $\mathcal{C}_-$ is the Hilbert-like transform defined on $\boldsymbol{\Sigma}$,

$$\mathcal{C}_-g(s) = \frac{1}{2\pi i}\lim_{s' \to s_-}\int_{\boldsymbol{\Sigma}} \frac{g(\xi)}{\xi - s'}d\xi, \qquad \text{and} \qquad \mathcal{C}_{\Delta_{\boldsymbol{\Sigma}}}f(s) = \mathcal{C}_-(\Delta_{\boldsymbol{\Sigma}}(f))(s), \tag{3.114}$$

such that the limit $s' \to s_-$ is taken when approaching the contour from the minus side. Since the Hilbert-like operator $\mathcal{C}_-$ is bounded on $L^2(\boldsymbol{\Sigma})$, we see from estimate (3.102) that the operator norm of $\mathcal{C}_{\Delta_{\boldsymbol{\Sigma}}}$ is also uniformly $\mathcal{O}(n^{-\frac{1}{2\theta+1}})$ as $n \to \infty$. Hence, if n is large enough, the operator $1 - \mathcal{C}_{\Delta_{\boldsymbol{\Sigma}}}$ is invertible, and we could rewrite (3.113) as

$$\mathcal{R}_-(s) - 1 = (1 - \mathcal{C}_{\Delta_{\boldsymbol{\Sigma}}})^{-1}(\mathcal{C}_-(\Delta_{\boldsymbol{\Sigma}}(1)))(s). \tag{3.115}$$

As one can check directly that

$$\|\Delta_{\boldsymbol{\Sigma}}(1)\|_{L^2(\boldsymbol{\Sigma})} = \mathcal{O}(n^{-1}), \tag{3.116}$$

combining the above two formulas gives us

$$\|\mathcal{R}_- - 1\|_{L^2(\boldsymbol{\Sigma})} = \mathcal{O}(n^{-1}). \tag{3.117}$$

By (3.112), we have that for any fixed $\delta > 0$, if $\operatorname{dist}(s, \boldsymbol{\Sigma}) > \delta$, then

$$|\mathcal{R}(s) - 1| \le \frac{1}{2\pi}\left(\|\Delta_{\boldsymbol{\Sigma}}(\mathcal{R}_- - 1)\|_{L^2(\boldsymbol{\Sigma})} + \|\Delta_{\boldsymbol{\Sigma}}(1)\|_{L^2(\boldsymbol{\Sigma})}\right) \cdot \|\frac{1}{\xi - s}\|_{L^2(\boldsymbol{\Sigma})} = \mathcal{O}(n^{-1}). \tag{3.118}$$

As a consequence, we conclude (3.108) holds uniformly in $\{s \in \mathbb{C} : \operatorname{dist}(s, \boldsymbol{\Sigma}) > \delta\}$. Since we can deform the contour Σ outside a neighbourhood of 1, say $D(1, \epsilon')$, by varying the value of ϵ in (3.84) and (3.85), choosing different shapes of Σ_1^R and Σ_2^R in (3.87), and deforming the jump contour (b, ∞) as in Remark 2, and we can then show that (3.108) holds uniformly in $\{s \in \mathbb{C} : s \notin \boldsymbol{\Sigma} \text{ and } |s - 1| \ge \epsilon'\}$. (We cannot freely deform $\boldsymbol{\Sigma}$ freely around 1, because $\boldsymbol{\Sigma}$ needs to connect to 1 as a vertex.) At last, in $D(1, \epsilon')$, $\mathcal{R}(s)$ satisfies a simple RH problem: its value on $\partial D(1, \epsilon')$ is uniformly $1 + \mathcal{O}(n^{-1})$, its limit at 1 is 1 and it has a jump along $[1, 1 + \epsilon')$, where the jump is given by $J_{\boldsymbol{\Sigma}^{(2')}}(s)$ that is exponentially small. Hence we also conclude that (3.108) holds uniformly in $\{s \in D(1, \epsilon') : s \notin \boldsymbol{\Sigma}\}$. □

4 Asymptotic analysis for $q_{n+k}^{(n)}(f(z))$

In this section we analyze $q_{n+k}^{(n)}(f(z))$ in the same way as we do $p_{n+k}^{(n)}(z)$ in Section 3. Since the method is parallel, we omit some detail. It is worth noting that the jump contours in this section can be taken the same as those in Section 3.

4.1 RH problem of the polynomials

Consider the following Cauchy transform of q_j:

$$\tilde{C}q_j(z) := \frac{1}{2\pi i}\int_{\mathbb{R}_+} \frac{q_j(f(x))}{x-z} W_\alpha^{(n)}(x)dx, \tag{4.1}$$

which is well defined for $z \in \mathbb{P} \setminus \mathbb{R}_+$. Since $W_\alpha^{(n)}(x)$ is real analytic and vanishes rapidly as $x \to +\infty$, we have the following asymptotic expansion for $\tilde{C}q_j(z)$ as $z \in \mathbb{P} \setminus \mathbb{R}_+$ and $\Re z \to +\infty$:

$$\begin{aligned}\tilde{C}q_j(z) &= \frac{-1}{2\pi i z}\int_{\mathbb{R}_+} \frac{q_j(f(x))}{1-x/z} W_\alpha^{(n)}(x)dx \\ &= \frac{-1}{2\pi i z}\sum_{k=0}^{M}\left(\int_{\mathbb{R}_+} q_j(f(x))x^k W_\alpha^{(n)}(x)dx\right) z^{-(k+1)} + \mathcal{O}(z^{-(M+2)}),\end{aligned} \tag{4.2}$$

for any $M \in \mathbb{N}$ and uniformly in $\Im z$. Thus due to the orthogonality,

$$\tilde{C}q_j(z) = \frac{-h^{(n)}}{2\pi i} z^{-(j+1)} + \mathcal{O}(z^{-(j+2)}), \tag{4.3}$$

where $h_j^{(n)}$ is given in (1.5).

Hence we conclude that if we define the array

$$\tilde{Y}(z) = \tilde{Y}^{(j,n)}(z) := (q_j(f(z)), Cq_j(z)), \tag{4.4}$$

then they satisfy the following conditions

RH Problem 4.1.

1. *$\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)$, where $\tilde{Y}_1$ is analytic on $\mathbb{P}$, and $\tilde{Y}_2$ is analytic on $\mathbb{C} \setminus \mathbb{R}_+$.*
2. *With the standard orientation of $\mathbb{R}_+$,*

$$\tilde{Y}_+(x) = \tilde{Y}_-(x)\begin{pmatrix}1 & W_\alpha^{(n)}(x)\\ 0 & 1\end{pmatrix}, \quad \text{for } x \in \mathbb{R}_+. \tag{4.5}$$

3. *As $f(z) \to \infty$ in $\mathbb{P}$ (i.e., $\Re z \to +\infty$), $\tilde{Y}_1(z) = f(z)^j + \mathcal{O}(f(z)^{j-1})$.*
4. *As $z \to \infty$ in $\mathbb{C}$, $\tilde{Y}_2(z) = \mathcal{O}(z^{-(j+1)})$.*
5. *As $z \to 0$ in $\mathbb{P}$ or $\mathbb{C}$,*

$$\tilde{Y}_1(z) = \mathcal{O}(1), \qquad \tilde{Y}_2(z) = \begin{cases}\mathcal{O}(1), & \alpha > 0,\\ \mathcal{O}(\log z), & \alpha = 0,\\ \mathcal{O}(z^\alpha), & \alpha \in (-1,0).\end{cases} \tag{4.6}$$

6. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, the limit $\tilde{Y}_1(z) := \lim_{w \to z \text{ in } \mathbb{P}} \tilde{Y}_1(w)$ exists and is continuous, and*

$$\tilde{Y}_1(z) = \tilde{Y}_1(\bar{z}). \tag{4.7}$$

Conversely, the RH problem for $\tilde{Y}$ has a unique solution given by (3.4). We omit the proof, since it is analogous to that of RH problem 3.1 given in Section 3.1.1.

Below we take $j = n + k$ where k is a constant integer, and our goal is to obtain the asymptotics for $\tilde{Y} = \tilde{Y}^{(n+k,n)}$ as $n \to \infty$.

4.2 First transformation $\tilde{Y} \mapsto \tilde{T}$

Analogous to (3.12), we denote $\tilde{Y} = \tilde{Y}^{(n+k,n)}$ and define $\tilde{T}$ as

$$\tilde{T}(z) = e^{-\frac{n\ell}{2}} \tilde{Y}(z) \begin{pmatrix} e^{-n\tilde{\mathbf{g}}(z)} & 0 \\ 0 & e^{n\mathbf{g}(z)} \end{pmatrix} e^{\frac{n\ell}{2}\sigma_3}. \tag{4.8}$$

Then $\tilde{T}$ satisfies a RH problem with the same domain of analyticity as $\tilde{Y}$, but with a different asymptotic behaviour and a different jump relation.

RH Problem 4.2.

1. *$\tilde{T} = (\tilde{T}_1, \tilde{T}_2)$, where $\tilde{T}_1$ is analytic in $\mathbb{P} \setminus \mathbb{R}_+$, and $\tilde{T}_2$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$.*
2. *$\tilde{T}$ satisfies the jump relation*

$$\tilde{T}_+(x) = \tilde{T}_-(x) J_{\tilde{T}}(x), \quad \text{for } x \in \mathbb{R}_+, \tag{4.9}$$

where

$$J_{\tilde{T}}(x) = \begin{pmatrix} e^{n(\tilde{\mathbf{g}}_-(x) - \tilde{\mathbf{g}}_+(x))} & x^\alpha h(x) e^{n(\mathbf{g}_-(x) + \tilde{\mathbf{g}}_+(x) - V(z) - \ell)} \\ 0 & e^{n(\mathbf{g}_+(x) - \mathbf{g}_-(x))} \end{pmatrix}. \tag{4.10}$$

3.

$$\tilde{T}_1(z) = f(z)^k + \mathcal{O}(f(z)^{k-1}) \quad \text{as } f(z) \to \infty \text{ in } \mathbb{P}, \quad \tilde{T}_2(z) = \mathcal{O}(z^{-(k+1)}) \quad \text{as } z \in \infty \text{ in } \mathbb{C}. \tag{4.11}$$

4. *As $z \to 0$ in $\mathbb{P}$ or in $\mathbb{C}$, $\tilde{T}(z)$ has the same limit behaviour as $\tilde{Y}(z)$ in (4.6).*
5.

$$\tilde{T}_1(z) = \mathcal{O}(1), \qquad \tilde{T}_2(z) = \mathcal{O}(1), \qquad \text{as } z \to b. \tag{4.12}$$

6. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $\tilde{T}_1(z)$ satisfies the same boundary condition as $\tilde{Y}_1(z)$ in (4.7).*

4.3 Second transformation $\tilde{T} \mapsto \tilde{S}$

Analogous to (3.17), for $x \in (0, b)$, we decompose the jump matrix $J_{\tilde{T}}(x)$ as

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{x^\alpha h(x)} e^{-n\phi_-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & x^\alpha h(x) e^{n(\tilde{\mathbf{g}}_-(x) + \mathbf{g}_+(x) - V(x) - \ell)} \\ -\frac{1}{x^\alpha h(x) e^{n(\tilde{\mathbf{g}}_-(x) + \mathbf{g}_+(x) - V(x) - \ell)}} & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 \\ \frac{1}{x^\alpha h(x)} e^{-n\phi_+(x)} & 1 \end{pmatrix}. \tag{4.13}$$

Then analogous to (3.20), define

$$\tilde{S}(z) := \begin{cases} \tilde{T}(z), & \text{outside of the lens,} \\ \tilde{T}(z) \begin{pmatrix} 1 & 0 \\ z^{-\alpha} h(z)^{-1} e^{-n\phi(z)} & 1 \end{pmatrix}, & \text{in the lower part of the lens,} \\ \tilde{T}(z) \begin{pmatrix} 1 & 0 \\ -z^{-\alpha} h(z)^{-1} e^{-n\phi(z)} & 1 \end{pmatrix}, & \text{in the upper part of the lens.} \end{cases} \tag{4.14}$$

From the definition of $\tilde{S}$, and decomposition of $J_{\tilde{T}}(x)$ in (3.17), we have analogous to RH problem 3.3 that

RH Problem 4.3.

1. *$\tilde{S} = (\tilde{S}_1, \tilde{S}_2)$, where $\tilde{S}_1$ is analytic in $\mathbb{P} \setminus \Sigma_S$, and $\tilde{S}_2$ is analytic in $\mathbb{C} \setminus \Sigma_S$.*
2. *We have*

$$\tilde{S}_+(z) = \tilde{S}_-(z) J_{\tilde{S}}(z), \quad \textit{for } z \in \Sigma_S, \tag{4.15}$$

where

$$J_{\tilde{S}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-\alpha} h(z)^{-1} e^{-n\phi(z)} & 1 \end{pmatrix}, & \textit{for } z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & z^{\alpha} h(z) \\ -z^{-\alpha} h(z)^{-1} & 0 \end{pmatrix}, & \textit{for } z \in (0, b), \\ \begin{pmatrix} 1 & z^{\alpha} h(z) e^{n\phi(z)} \\ & 1 \end{pmatrix}, & \textit{for } z \in (b, \infty). \end{cases} \tag{4.16}$$

3. *As $z \to \infty$ in $\mathbb{P}$ or $\mathbb{C}$, $\tilde{S}(z)$ has the same limit behaviour as $\tilde{T}(z)$ in (4.11).*
4. *As $z \to 0$ in $\mathbb{P} \setminus \Sigma$, we have*

$$\tilde{S}_1(z) = \begin{cases} \mathcal{O}(z^{-\alpha}), & \alpha > 0 \textit{ and } z \textit{ inside the lens}, \\ \mathcal{O}(\log z), & \alpha = 0 \textit{ and } z \textit{ inside the lens}, \\ \mathcal{O}(1), & z \textit{ outside the lens or } -1 < \alpha < 0. \end{cases} \tag{4.17}$$

5. *As $z \to 0$ in $\mathbb{C}$, $\tilde{S}_2$ has the same behaviour as $\tilde{Y}_2(z)$ in (4.6).*
6. *As $z \to b$, $\tilde{S}(z)$ has the same limit behaviour as $\tilde{T}(z)$ in (4.12).*
7. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $\tilde{S}_1(z)$ satisfies the same boundary condition as $\tilde{Y}_1(z)$ in (4.7).*

4.4 Construction of the global parametrix

Analogous to RH problem 3.4, we construct the following

RH Problem 4.4.

1. *$\tilde{P}^{(\infty)} = (\tilde{P}_1^{(\infty)}, \tilde{P}_2^{(\infty)})$, where $\tilde{P}_1^{(\infty)}$ is analytic in $\mathbb{P} \setminus [0, b]$, and $\tilde{P}_2^{(\infty)}$ is analytic in $\mathbb{C} \setminus [0, b]$.*
2. *For $x \in (0, b)$, we have*

$$\tilde{P}_+^{(\infty)}(x) = \tilde{P}_-^{(\infty)}(x) \begin{pmatrix} 0 & z^{\alpha} h(z) \\ -z^{-\alpha} h(z)^{-1} & 0 \end{pmatrix}.$$

3. *As $z \to \infty$ in $\mathbb{P}$ or $\mathbb{C}$, $\tilde{P}^{(\infty)}(z)$ has the same limit behaviour as $\tilde{T}(z)$ in (4.11).*

4. *At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $\tilde{P}_1^{(\infty)}(z)$ satisfies the same boundary condition as $\tilde{Y}_1(z)$ in (4.7).*

To construct a solution to the above RH problem, we set, analogous to (3.25),

$$\tilde{\mathcal{P}}(s) := \begin{cases} \tilde{P}_2^{(\infty)}(\mathbf{J}_c(s)), & s \in \mathbb{C} \setminus \overline{D}, \\ \tilde{P}_1^{(\infty)}(\mathbf{J}_c(s)), & s \in D \setminus [0,1]. \end{cases} \tag{4.18}$$

Like $\mathcal{P}(s)$ in (3.25), $\tilde{\mathcal{P}}$ is well defined onto $[0,1)$ by continuation. Analogous to RH problem 3.5 for $\mathcal{P}$, $\tilde{\mathcal{P}}$ satisfies the following:

RH Problem 4.5.

1. *$\tilde{\mathcal{P}}$ is analytic in $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2 \cup \{1\})$.*

2. *For $s \in \gamma_1 \cup \gamma_2$, $\tilde{\mathcal{P}}_+(s) = \tilde{\mathcal{P}}_-(s) J_{\tilde{\mathcal{P}}}(s)$, where*

$$J_{\tilde{\mathcal{P}}}(s) = \begin{cases} \mathbf{J}_c^{\alpha}(s) h(\mathbf{J}_c(s)), & s \in \gamma_1, \\ -\mathbf{J}_c^{-\alpha}(s) h(\mathbf{J}_c(s))^{-1}, & s \in \gamma_2. \end{cases} \tag{4.19}$$

3. *As $s \to \infty$, $\tilde{\mathcal{P}}(s) = \mathcal{O}(s^{-(k+1)})$.*

4. *As $s \to 1$, $\tilde{\mathcal{P}}(s) = e^{kc}(s-1)^{-k} + \mathcal{O}((s-1)^{-(k+1)})$.*

A solution $\tilde{\mathcal{P}}$ to the above RH problem is explicitly given by

$$\tilde{\mathcal{P}}(s) = \begin{cases} \tilde{G}_k(s), & s \in D \setminus \{1\}, \\ \frac{(1-s_1)^{\alpha+\frac{1}{2}} \sqrt{s_2 - 1}\, i \tilde{D}(1)^{-1} e^{kc} \mathbf{J}_c(s)^{\alpha}}{(s-s_1)^{\alpha}(s-1)^k \sqrt{(s-s_1)(s-s_2)}} D(s)^{-1}, & \in \mathbb{C} \setminus \overline{D}, \end{cases} \tag{4.20}$$

where the square root is taken in $\mathbb{C} \setminus \gamma_2$ with $\sqrt{(s-s_1)(s-s_2)} \sim s$ as $s \to \infty$. Later in this paper we take (4.20) as the definition of $\tilde{\mathcal{P}}(s)$.

Based on this solution, we construct the solution to RH problem 3.4 as

$$\tilde{P}_2^{(\infty)}(z) = \tilde{\mathcal{P}}(\mathbf{I}_1(z)), \qquad z \in \mathbb{C} \setminus [0,b], \tag{4.21}$$

$$\tilde{P}_1^{(\infty)}(z) = \tilde{\mathcal{P}}(\mathbf{I}_2(z)) = \tilde{G}_k(\mathbf{I}_2(z)), \qquad z \in \mathbb{P} \setminus [0,b]. \tag{4.22}$$

By direct calculation in Section A, we have, analogous to (3.31) and (3.32),

$$\tilde{P}_1^{(\infty)}(z) = \mathcal{O}(z^{-\frac{\alpha}{2}-\frac{1}{4}}), \qquad \tilde{P}_2^{(\infty)}(z) = \mathcal{O}(z^{\frac{\alpha}{2}-\frac{1}{4}}), \qquad \text{as } z \to 0, \tag{4.23}$$

$$\tilde{P}_1^{(\infty)}(z) = \mathcal{O}(z^{-\frac{1}{4}}), \qquad \tilde{P}_2^{(\infty)}(z) = \mathcal{O}(z^{-\frac{1}{4}}), \qquad \text{as } z \to b. \tag{4.24}$$

4.5 Third transformation $\tilde{S} \mapsto \tilde{Q}$

Noting that $\tilde{P}_1^{(\infty)}(z) \neq 0$ for all $z \in \mathbb{P} \setminus [0,b]$ and $\tilde{P}_2^{(\infty)}(z) \neq 0$ for all $z \in \mathbb{C} \setminus [0,b]$, we define analogous to (3.33)

$$\tilde{Q}(z) = (\tilde{Q}_1(z), \tilde{Q}_2(z)) = \left(\frac{\tilde{S}_1(z)}{\tilde{P}_1^{\infty}(z)}, \frac{\tilde{S}_2(z)}{\tilde{P}_2^{\infty}(z)} \right). \tag{4.25}$$

Analogous to RH problem 3.6, $\tilde{Q}$ satisfies the following RH problem:

RH Problem 4.6.

1. $\tilde{Q} = (\tilde{Q}_1, \tilde{Q}_2)$, *where* $\tilde{Q}_1$ *is analytic in* $\mathbb{P} \setminus \Sigma$, *and* $\tilde{Q}_2$ *is analytic in* $\mathbb{C} \setminus \Sigma$.

2. For $z \in \Sigma$, *we have*

$$\tilde{Q}_+(z) = \tilde{Q}_-(z) J_{\tilde{Q}}(z), \tag{4.26}$$

where

$$J_{\tilde{Q}}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-\alpha} h(z)^{-1} \frac{\tilde{P}_2^{(\infty)}(z)}{\tilde{P}_1^{(\infty)}(z)} e^{-n\phi(z)} & 1 \end{pmatrix}, & z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & z \in (0, b), \\ \begin{pmatrix} 1 & z^{\alpha} h(z) \frac{\tilde{P}_1^{(\infty)}(z)}{\tilde{P}_2^{(\infty)}(z)} e^{-n\phi(z)} \\ 0 & 1 \end{pmatrix}, & z \in (b, \infty). \end{cases} \tag{4.27}$$

3.

$$\tilde{Q}_1(z) = 1 + \mathcal{O}(f(z)^{-1}), \quad \text{as } f(z) \to \infty \text{ in } \mathbb{P}, \quad \tilde{Q}_2(z) = \mathcal{O}(z^{-1}), \quad \text{as } z \to \infty \text{ in } \mathbb{C}. \tag{4.28}$$

4. As $z \to 0$ *in* $\mathbb{P} \setminus \Sigma$, *we have*

$$\tilde{Q}_1(z) = \begin{cases} \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}), & \alpha > 0 \text{ and } z \text{ inside the lens}, \\ \mathcal{O}(z^{\frac{1}{4}} \log z), & \alpha = 0 \text{ and } z \text{ inside the lens}, \\ \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}), & z \text{ outside the lens or } -1 < \alpha < 0. \end{cases} \tag{4.29}$$

5. As $z \to 0$ *in* $\mathbb{C}$, *we have*

$$\tilde{Q}_2(z) = \begin{cases} \mathcal{O}(z^{-\frac{\alpha}{2}+\frac{1}{4}}), & \alpha > 0, \\ \mathcal{O}(z^{\frac{1}{4}} \log z), & \alpha = 0, \\ \mathcal{O}(z^{\frac{\alpha}{2}+\frac{1}{4}}), & \alpha \in (-1, 0). \end{cases} \tag{4.30}$$

6.

$$\tilde{Q}_1(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad \tilde{Q}_2(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad \text{as } z \to b. \tag{4.31}$$

7. At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $\tilde{Q}_1(z)$ *satisfies the same boundary condition as* $\tilde{Y}_1(z)$ *in (4.7).*

4.6 Construction of local parametrix near b

Let, analogous to (3.42),

$$\tilde{g}_1^{(b)}(z) = \frac{h(z)^{-1}}{\tilde{P}_1^{(\infty)}(z)}, \qquad \tilde{g}_2^{(b)}(z) = \frac{z^{\alpha}}{\tilde{P}_2^{(\infty)}(z)}, \tag{4.32}$$

and define analogous to (3.43)

$$\tilde{\mathsf{P}}^{(b)}(z) := \Psi^{(\mathrm{Ai})}(n^{\frac{2}{3}} f_b(z)) \begin{pmatrix} e^{-\frac{n}{2}\phi(z)} \tilde{g}_1^{(b)}(z) & 0 \\ 0 & e^{\frac{n}{2}\phi(z)} \tilde{g}_2^{(b)}(z) \end{pmatrix}, \quad z \in D(b, \epsilon) \setminus \Sigma. \tag{4.33}$$

From (4.32) and RH problem 4.4 satisfied by $\tilde{P}^{(\infty)}(z)$, we have, analogous to (3.44) and (3.45),

$$\tilde{g}_{1,+}^{(b)}(x) = -\tilde{g}_{2,-}^{(b)}(x), \qquad \tilde{g}_{2,+}^{(b)}(x) = \tilde{g}_{1,-}^{(b)}(x), \qquad \text{for } x \in (b-\epsilon, b), \tag{4.34}$$

$$\tilde{g}_{1}^{(b)}(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad \tilde{g}_{2}^{(b)}(z) = \mathcal{O}((z-b)^{\frac{1}{4}}), \qquad \text{for } z \to b. \tag{4.35}$$

Then analogous to RH problem 3.7, we have the following RH problem satisfied by $\tilde{\mathsf{P}}^{(b)}(z)$:

RH Problem 4.7.

1. *$\tilde{\mathsf{P}}^{(b)}(z)$ is a 2×2 matrix-valued function analytic for $z \in D(b,\epsilon) \setminus \Sigma$.*
2. *For $z \in \Sigma \cap D(b,\epsilon)$, we have*

$$\tilde{\mathsf{P}}_{+}^{(b)}(z) = \tilde{\mathsf{P}}_{-}^{(b)}(z) J_{\tilde{Q}}(z), \tag{4.36}$$

   where $J_{\tilde{Q}}(z)$ is defined in (4.27).

3. *As $z \to b$, the limit behaviour of $\tilde{\mathsf{P}}^{(b)}(z)$ and $(\tilde{\mathsf{P}}^{(b)})^{-1}(z)$ is the same as that of $\mathsf{P}^{(b)}(z)$ and $(\mathsf{P}^{(b)})^{-1}(z)$ in (3.47).*
4. *For $z \in \partial D(b,\epsilon)$, we have, as $n \to \infty$,*

$$\tilde{E}^{(b)}(z)\tilde{\mathsf{P}}^{(b)}(z) = I + \mathcal{O}(n^{-1}), \tag{4.37}$$

   where

$$\begin{aligned}\tilde{E}^{(b)}(z) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{g}_1^{(b)}(z) & 0 \\ 0 & \tilde{g}_2^{(b)}(z) \end{pmatrix}^{-1} e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n^{\frac{1}{6}} f_b(z)^{\frac{1}{4}} & 0 \\ 0 & n^{-\frac{1}{6}} f_b(z)^{-\frac{1}{4}} \end{pmatrix} \tilde{\mathsf{P}}^{(b)}(z) \\ &= I + \mathcal{O}(n^{-1}).\end{aligned} \tag{4.38}$$

We now define a 2×2 matrix-valued function

$$\tilde{P}^{(b)}(z) = \tilde{E}^{(b)}(z)\tilde{\mathsf{P}}^{(b)}(z) \quad z \in D(b,\epsilon) \setminus \Sigma, \tag{4.39}$$

where $\tilde{\mathsf{P}}^{(b)}$ is given in (4.33) and $\tilde{E}^{(b)}(z)$ is defined in (4.38), like $E^{(b)}(z)$ defined in (3.49), it is straightforward to see that $\tilde{E}^{(b)}(z)$ is analytic on $D(b,\epsilon) \setminus (b-\epsilon, b]$, and for $x \in (b-\epsilon, b)$

$$\tilde{E}_{+}^{(b)}(x)\tilde{E}_{-}^{(b)}(x)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.40}$$

and as $z \to b$,

$$\tilde{E}^{(b)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}}) \\ \mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}}) \end{pmatrix}, \qquad \tilde{E}^{(b)}(z)^{-1} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(z^{\frac{1}{2}}) & \mathcal{O}(z^{\frac{1}{2}}) \end{pmatrix}. \tag{4.41}$$

Hence, analogous to RH problem 3.8, we have the following RH problem satisfied by $\tilde{P}^{(b)}(z)$:

RH Problem 4.8.

1. *$\tilde{P}^{(b)}(z)$ is analytic in $D(b,\epsilon) \setminus \Sigma$.*

2. For $z \in \Sigma \cap D(b,\epsilon)$, we have

$$\tilde{P}_+^{(b)}(z) = \begin{cases} \tilde{P}_-^{(b)}(z) J_{\tilde{Q}}(z), & z \in \Sigma \cap D(b,\epsilon) \setminus (b-\epsilon, b], \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{P}_-^{(b)}(z) J_{\tilde{Q}}(z), & z \in (b-\epsilon, b). \end{cases} \tag{4.42}$$

3. As $z \to b$, the limit behaviour of $\tilde{P}^{(b)}(z)$ and $(\tilde{P}^{(b)})^{-1}(z)$ is the same as that of $P^{(b)}(z)$ and $(P^{(b)})^{-1}(z)$ in (3.54).

4. For z on the boundary $\partial D(b,\epsilon)$, we have, as $n \to \infty$, $\tilde{P}^{(b)}(z) = I + \mathcal{O}(n^{-1})$.

At last, analogous to (3.55), we define a vector-valued function $\tilde{V}^{(b)}$ by

$$\tilde{V}^{(b)}(z) = \tilde{Q}(z) \tilde{P}^{(b)}(z)^{-1}, \quad z \in D(b,\epsilon) \setminus \Sigma, \tag{4.43}$$

where $\tilde{Q}(z)$ is defined in (4.25). It satisfies the following RH problem that is analogous to RH problem 3.9:

RH Problem 4.9.

1. $\tilde{V}^{(b)} = (\tilde{V}_1^{(b)}, \tilde{V}_2^{(b)})$ is analytic in $D(b,\epsilon) \setminus (b-\epsilon, b]$.

2. For $x \in (b-\epsilon, b)$, we have

$$\tilde{V}_+^{(b)}(x) = \tilde{V}_-^{(b)}(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.44}$$

3.

$$\tilde{V}_1^{(b)}(z) = \mathcal{O}(1), \qquad \tilde{V}_2^{(b)}(z) = \mathcal{O}(1), \qquad \text{as } z \to b. \tag{4.45}$$

4. For $z \in \partial D(b,\epsilon)$, we have, as $n \to \infty$, $\tilde{V}^{(b)}(z) = Q(z)(I + \mathcal{O}(n^{-1}))$.

4.7 Construction of local parametrix near 0

Let, analogous to (3.61)

$$\tilde{g}_1^{(0)}(z) = \frac{(-z)^{-\alpha/2}/h(z)}{\tilde{P}_1^{(\infty)}(z)}, \qquad \tilde{g}_2^{(0)}(z) = \frac{(-z)^{\alpha/2}}{\tilde{P}_2^{(\infty)}(z)}, \tag{4.46}$$

where the $(-z)^{\pm\alpha/2}$ takes the principal branch, and define, analogous to (3.62),

$$\mathsf{P}^{(0)}(z) = \Phi^{(\mathrm{Be})}(n^2 f_0(z)) \begin{pmatrix} e^{-\frac{n}{2}\phi(z)} \tilde{g}_1^{(0)}(z) & 0 \\ 0 & e^{\frac{n}{2}\phi(z)} \tilde{g}_2^{(0)}(z) \end{pmatrix}, \quad z \in D(0,\epsilon) \setminus \Sigma. \tag{4.47}$$

Analogous to (3.63) and (3.64), we have

$$\tilde{g}_{1,+}^{(0)}(x) = -\tilde{g}_{2,-}^{(0)}(x), \qquad \tilde{g}_{2,+}^{(0)}(x) = \tilde{g}_{1,-}^{(0)}(x), \qquad \text{for } x \in (0,\epsilon), \tag{4.48}$$

$$\tilde{g}_1^{(0)}(z) = \mathcal{O}(z^{\frac{1}{4}}), \qquad \tilde{g}_2^{(0)}(z) = \mathcal{O}(z^{\frac{1}{4}}), \qquad \text{for } z \to 0. \tag{4.49}$$

Analogous to RH problem 3.10, we have the following RH problem satisfied by $\tilde{\mathsf{P}}^{(0)}(z)$:

RH Problem 4.10.

1. $\tilde{\mathsf{P}}^{(0)}(z)$ *is a* 2×2 *matrix-valued function analytic for* $z\in D(0,\epsilon)\setminus\Sigma$.

2. *For* $z\in\Sigma\cap D(0,\epsilon)$, *we have*

$$\tilde{\mathsf{P}}^{(0)}_+(z)=\tilde{\mathsf{P}}^{(0)}_-(z)J_{\tilde{Q}}(z), \tag{4.50}$$

   where $J_{\tilde{Q}}(z)$ *is defined in* (4.27).

3. *As* $z\in\partial D(0,\epsilon)$, *we have*

$$\tilde{E}^{(0)}(z)\tilde{\mathsf{P}}^{(0)}(z)=(I+\mathcal{O}(n^{-1})), \tag{4.51}$$

   where

$$\tilde{E}^{(0)}(z)=\frac{1}{\sqrt{2}}\begin{pmatrix}\tilde{g}^{(0)}_1(z) & 0\\ 0 & \tilde{g}^{(0)}_2(z)\end{pmatrix}^{-1}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}n^{\frac{1}{2}}f_0(z)^{\frac{1}{4}} & 0\\ 0 & n^{-\frac{1}{2}}f_0(z)^{-\frac{1}{4}}\end{pmatrix}(2\pi)^{\frac{1}{2}\sigma_3}. \tag{4.52}$$

4. *As* $z\to 0$, *the limit behaviour of* $\tilde{\mathsf{P}}^{(0)}(z)$ *and* $(\tilde{\mathsf{P}}^{(0)})^{-1}(z)$ *is the same as that of* $\mathsf{P}^{(0)}(z)$ *and* $(\mathsf{P}^{(0)})^{-1}(z)$ *in* (3.68)–(3.71).

Like $E^{(0)}(z)$, it is straightforward to see that $\tilde{E}^{(0)}(z)$ is analytic on $D(0,\epsilon)\setminus[0,\epsilon)$, and for $x\in(0,\epsilon)$, like (3.72)

$$\tilde{E}^{(0)}_+(x)\tilde{E}^{(0)}_-(x)^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{4.53}$$

and as $z\to 0$, like (3.73)

$$\tilde{E}^{(0)}(z)=\begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}})\\ \mathcal{O}(1) & \mathcal{O}(z^{-\frac{1}{2}})\end{pmatrix},\qquad \tilde{E}^{(0)}(z)^{-1}=\begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(1)\\ \mathcal{O}(z^{\frac{1}{2}}) & \mathcal{O}(z^{\frac{1}{2}})\end{pmatrix}. \tag{4.54}$$

Then we define a 2×2 matrix-valued function

$$\tilde{P}^{(0)}(z)=\tilde{E}^{(0)}(z)\tilde{\mathsf{P}}^{(0)}(z),\quad z\in D(0,\epsilon)\setminus\Sigma, \tag{4.55}$$

where $\tilde{\mathsf{P}}^{(0)}$ is given in (4.47). Hence, we have the following RH problem satisfied by $\tilde{P}^{(0)}(z)$:

RH Problem 4.11.

1. $\tilde{P}^{(0)}(z)$ *is analytic in* $D(0,\epsilon)\setminus\Sigma$.

2. *For* $z\in\Sigma\cap D(0,\epsilon)$, *we have*

$$\tilde{P}^{(0)}_+(z)=\begin{cases}\tilde{P}^{(0)}_-(z)J_Q(z), & z\in\Sigma\cap D(0,\epsilon)\setminus[0,\epsilon),\\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\tilde{P}^{(0)}_-(z)J_Q(z), & z\in(0,\epsilon).\end{cases} \tag{4.56}$$

3. *As* $z\to 0$, *the limit behaviour of* $\tilde{P}^{(0)}(z)$ *and* $(\tilde{P}^{(0)})^{-1}(z)$ *is the same as that of* $P^{(0)}(z)$ *and* $(P^{(0)})^{-1}(z)$ *in* (3.76)–(3.79).

4. *For* z *on the boundary* $\partial D(0,\epsilon)$, *we have, as* $n\to\infty$, $\tilde{P}^{(0)}(z)=I+\mathcal{O}(n^{-1})$.

Consider the vector-valued function

$$\tilde{U}(z) = (\tilde{U}_1(z), \tilde{U}_2(z)) := \tilde{Q}(z)\tilde{\mathsf{P}}^{(0)}(z)^{-1}, \quad \in D(0,\epsilon) \setminus \Sigma, \tag{4.57}$$

where $\tilde{Q}(z)$ is defined in (4.25), and then define the vector-valued function $\tilde{V}^{(0)}$ on $D(0,\epsilon) \setminus \Sigma$ by

$$\tilde{V}^{(0)}(z) = (\tilde{V}_1^{(0)}(z), \tilde{V}_2^{(0)}(z)) := \tilde{Q}(z)P^{(0)}(z)^{-1} = \tilde{U}(z)\tilde{E}^{(0)}(z)^{-1}. \tag{4.58}$$

Like $U(z)$ defined in (3.80), we can show that 0 is a removable singular point of $\tilde{U}_1(z)$ and $\tilde{U}_2(z)$. Hence, like RH problem 3.12, $\tilde{V}^{(0)}(z)$ satisfies the following RH problem:

RH Problem 4.12.

1. *$\tilde{V}^{(0)}(z) = (\tilde{V}_1^{(0)}(z), \tilde{V}_2^{(0)}(z))$ is analytic in $D(0,\epsilon) \setminus [0,\epsilon)$.*
2. *For $x \in (0,\epsilon)$, we have*

$$\tilde{V}_+^{(0)}(x) = \tilde{V}_-^{(0)}(x) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.59}$$

3. *As $z \to 0$, we have $\tilde{V}(z) = (\mathcal{O}(1), \mathcal{O}(1))$.*
4. *As $z \in \partial D(0,\epsilon)$, $\tilde{V}^{(0)}(z) = \tilde{Q}(z)(I + \mathcal{O}(n^{-1}))$.*

4.8 Final transformation

Analogous to (3.84) and (3.85), we define $\tilde{R}(z) = (\tilde{R}_1(z), \tilde{R}_2(z))$ as

$$\tilde{R}_1(z) = \begin{cases} \tilde{V}_1^{(b)}(z), & z \in D(b,\epsilon) \setminus (b-\epsilon, b], \\ \tilde{V}_1^{(0)}(z), & z \in D(0,\epsilon) \setminus [0,\epsilon), \\ \tilde{Q}_1(z), & z \in \mathbb{P} \setminus (\overline{D(b,\epsilon)} \cup \overline{D(0,\epsilon)} \cup \Sigma), \end{cases} \tag{4.60}$$

$$\tilde{R}_2(z) = \begin{cases} \tilde{V}_2^{(b)}(z), & z \in D(b,\epsilon) \setminus (b-\epsilon, b], \\ \tilde{V}_2^{(0)}(z), & z \in D(0,\epsilon) \setminus [0,\epsilon), \\ \tilde{Q}_2(z), & z \in \mathbb{C} \setminus (\overline{D(b,\epsilon)} \cup \overline{D(0,\epsilon)} \cup \Sigma). \end{cases} \tag{4.61}$$

Then $\tilde{R}$ satisfies the following RH problem that is analogous to RH problem 3.13:

RH Problem 4.13.

1. *$\tilde{R}(z) = (\tilde{R}_1(z), \tilde{R}_2(z))$, where $\tilde{R}_1(z)$ is analytic in $\mathbb{P} \setminus \Sigma^R$, and $\tilde{R}_2(z)$ is analytic in $\mathbb{C} \setminus \Sigma^R$.*
2. *$\tilde{R}(z)$ satisfies the following jump conditions:*

$$\tilde{R}_+(z) = \tilde{R}_-(z) \begin{cases} J_{\tilde{Q}}(z), & z \in \Sigma_1^R \cup \Sigma_2^R \cup (b+\epsilon, +\infty), \\ \tilde{P}^{(b)}(z), & z \in \partial D(b,\epsilon), \\ \tilde{P}^{(0)}(z), & z \in \partial D(0,\epsilon), \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & z \in (0,b) \setminus \{\epsilon, b-\epsilon\}. \end{cases} \tag{4.62}$$

3.

$$\tilde{R}_1(z) = 1 + \mathcal{O}(f(z)^{-1}) \quad \text{as } f(z) \to \infty \text{ in } \mathbb{P}, \quad \tilde{R}_2(z) = \mathcal{O}(1) \quad \text{as } z \to \infty \text{ in } \mathbb{C}. \tag{4.63}$$

4.

$$\tilde{R}_1(z) = \mathcal{O}(1), \qquad \tilde{R}_2(z) = \mathcal{O}(1), \qquad \text{as } z \to 0, \tag{4.64}$$

$$\tilde{R}_1(z) = \mathcal{O}(1), \qquad \tilde{R}_2(z) = \mathcal{O}(1), \qquad \text{as } z \to b. \tag{4.65}$$

5. At $z \in \rho \cup \{-\pi^2/4\} \cup \bar{\rho}$, $\tilde{R}_1(z)$ satisfies the same boundary condition as $\tilde{Y}_1(z)$ in (4.7).

Similar to the idea used in the construction of global parametrix, to estimate R for large n, we now transform the RH problem for R to a scalar one on the complex s-plane. Let $\omega : z \to \omega(z)$ be a mapping from $\mathbb{C} \setminus \{1\}$ to $\mathbb{C} \setminus \{1\}$ as $\omega(z) = 1 + 1/(z-1)$. Then let $\tilde{\mathbf{J}}_c(s)$ be the mapping

$$\tilde{\mathbf{J}}_c(s) = \mathbf{J}_c(\omega(s)). \tag{4.66}$$

by defining

$$\tilde{\mathcal{R}}(s) = \begin{cases} \tilde{R}_1(\tilde{\mathbf{J}}_c(s)), & s \in \omega(\mathbb{C} \setminus \overline{D}) \text{ and } s \notin \omega(\mathbf{I}_1(\Sigma^R)), \\ \tilde{R}_2(\tilde{\mathbf{J}}_c(s)), & s \in \omega(D \setminus [0,1]) \text{ and } s \notin \omega(\mathbf{I}_2(\Sigma^R)), \end{cases} \tag{4.67}$$

where we recall that D is the region bounded by the curves γ_1 and γ_2, $\mathbf{I}_1 : \mathbb{C} \setminus [0,b] \to \mathbb{C} \setminus \overline{D}$ and $\mathbf{I}_2 : \mathbb{P} \setminus [0,b]$ are defined in (1.26) and (1.27), respectively.

We are now at the stage of describing the RH problem for $\mathcal{R}$. For this purpose, we define, analogous to (3.94) and set

$$\tilde{\boldsymbol{\Sigma}} := \tilde{\boldsymbol{\Sigma}}^{(1)} \cup \tilde{\boldsymbol{\Sigma}}^{(1')} \cup \tilde{\boldsymbol{\Sigma}}^{(2)} \cup \tilde{\boldsymbol{\Sigma}}^{(2')} \cup \tilde{\boldsymbol{\Sigma}}^{(3)} \cup \tilde{\boldsymbol{\Sigma}}^{(3')} \cup \tilde{\boldsymbol{\Sigma}}^{(4)} \cup \tilde{\boldsymbol{\Sigma}}^{(4')}, \quad \text{where} \quad \tilde{\boldsymbol{\Sigma}}^{(*)} = \omega(\boldsymbol{\Sigma}^{(*)}), \tag{4.68}$$

for $* = 1, 1', 2, 2', 3, 3', 4, 4'$ respectively.

We also define the following functions on each curve constituting $\tilde{\boldsymbol{\Sigma}}$: (Below $z = \tilde{\mathbf{J}}_c(s)$)

$$J_{\tilde{\boldsymbol{\Sigma}}^{(1')}}(s) = (J_{\tilde{Q}})_{21}(z), \qquad s \in \tilde{\boldsymbol{\Sigma}}^{(1')}, \tag{4.69}$$

$$J_{\tilde{\boldsymbol{\Sigma}}^{(2)}}(s) = (J_{\tilde{Q}})_{12}(z), \qquad s \in \tilde{\boldsymbol{\Sigma}}^{(2)}, \tag{4.70}$$

$$J^1_{\tilde{\boldsymbol{\Sigma}}^{(3)}}(s) = (\tilde{P}^{(b)})_{22}(z) - 1, \quad J^2_{\tilde{\boldsymbol{\Sigma}}^{(3)}}(s) = (\tilde{P}^{(b)})_{12}(z), \qquad s \in \tilde{\boldsymbol{\Sigma}}^{(3)}, \tag{4.71}$$

$$J^1_{\tilde{\boldsymbol{\Sigma}}^{(3')}}(s) = (\tilde{P}^{(b)})_{11}(z) - 1, \quad J^2_{\tilde{\boldsymbol{\Sigma}}^{(3')}}(s) = (\tilde{P}^{(b)})_{21}(z), \qquad s \in \tilde{\boldsymbol{\Sigma}}^{(3')}, \tag{4.72}$$

$$J^1_{\tilde{\boldsymbol{\Sigma}}^{(4)}}(s) = (\tilde{P}^{(0)})_{22}(z) - 1, \quad J^2_{\tilde{\boldsymbol{\Sigma}}^{(4)}}(s) = (\tilde{P}^{(0)})_{12}(z), \qquad s \in \tilde{\boldsymbol{\Sigma}}^{(4)}, \tag{4.73}$$

$$J^1_{\tilde{\boldsymbol{\Sigma}}^{(4')}}(s) = (\tilde{P}^{(0)})_{11}(z) - 1, \quad J^2_{\tilde{\boldsymbol{\Sigma}}^{(4')}}(s) = (\tilde{P}^{(0)})_{21}(z), \qquad s \in \tilde{\boldsymbol{\Sigma}}^{(4')}, \tag{4.74}$$

where $z = \tilde{\mathbf{J}}_c(s)$ is in $\Sigma_1^R \cup \Sigma_2^R$ in (4.69); in $(b+\epsilon, +\infty)$ in (4.70); in $\partial D(b,\epsilon)$ in (4.71) and (4.72); in $\partial D(0,\epsilon)$ in (4.73) and (4.74). With the aid of these functions, we further define an operator $\Delta_{\tilde{\boldsymbol{\Sigma}}}$ such that for any complex-valued function $f(s)$ defined on $\tilde{\boldsymbol{\Sigma}}$, $\Delta_{\tilde{\boldsymbol{\Sigma}}}$ transforms it linearly into a function $\Delta_{\tilde{\boldsymbol{\Sigma}}} f$ that is also a complex-valued function defined on $\tilde{\boldsymbol{\Sigma}}$, with expression

$$(\Delta_{\tilde{\boldsymbol{\Sigma}}} f)(s) = \begin{cases} J_{\tilde{\boldsymbol{\Sigma}}^{(1')}}(s) f(\tilde{s}), & s \in \tilde{\boldsymbol{\Sigma}}^{(1)} \text{ and } \tilde{s} = \mathbf{I}_1(\mathbf{J}_c(s)) \in \tilde{\boldsymbol{\Sigma}}^{(1')}, \\ J_{\tilde{\boldsymbol{\Sigma}}^{(2)}}(s) f(\tilde{s}), & s \in \tilde{\boldsymbol{\Sigma}}^{(2)} \text{ and } \tilde{s} = \mathbf{I}_2(\mathbf{J}_c(s)) \in \tilde{\boldsymbol{\Sigma}}^{(2)}, \\ 0, & s \in \tilde{\boldsymbol{\Sigma}}^{(1)} \cup \tilde{\boldsymbol{\Sigma}}^{(2')}, \\ J^1_{\tilde{\boldsymbol{\Sigma}}^{(3)}}(s) f(s) + J^2_{\tilde{\boldsymbol{\Sigma}}^{(3)}}(s) f(\tilde{s}), & s \in \tilde{\boldsymbol{\Sigma}}^{(3)} \text{ and } \tilde{s} = \mathbf{I}_2(\mathbf{J}_c(s)) \in \tilde{\boldsymbol{\Sigma}}^{(3')}, \\ J^1_{\tilde{\boldsymbol{\Sigma}}^{(3')}}(s) f(s) + J^2_{\tilde{\boldsymbol{\Sigma}}^{(3')}}(s) f(\tilde{s}), & s \in \tilde{\boldsymbol{\Sigma}}^{(3')} \text{ and } \tilde{s} = \mathbf{I}_1(\mathbf{J}_c(s)) \in \tilde{\boldsymbol{\Sigma}}^{(3)}, \\ J^1_{\tilde{\boldsymbol{\Sigma}}^{(4)}}(s) f(s) + J^2_{\tilde{\boldsymbol{\Sigma}}^{(4)}}(s) f(\tilde{s}), & s \in \tilde{\boldsymbol{\Sigma}}^{(4)} \text{ and } \tilde{s} = \mathbf{I}_2(\mathbf{J}_c(s)) \in \tilde{\boldsymbol{\Sigma}}^{(4')}, \\ J^1_{\tilde{\boldsymbol{\Sigma}}^{(4')}}(s) f(s) + J^2_{\tilde{\boldsymbol{\Sigma}}^{(4')}}(s) f(\tilde{s}), & s \in \tilde{\boldsymbol{\Sigma}}^{(4')} \text{ and } \tilde{s} = \mathbf{I}_1(\mathbf{J}_c(s)) \in \tilde{\boldsymbol{\Sigma}}^{(4)}. \end{cases} \tag{4.75}$$

We note that all the functions $J_{\tilde{\boldsymbol{\Sigma}}^{(1')}}(s)$, ..., $J^2_{\tilde{\boldsymbol{\Sigma}}^{(4')}}(s)$ that define $\Delta_{\tilde{\boldsymbol{\Sigma}}}$ in (4.75) are uniformly $\mathcal{O}(n^{-1})$. If we view $\Delta_{\tilde{\boldsymbol{\Sigma}}}$ as an operator from $L^2(\tilde{\boldsymbol{\Sigma}})$ to $L^2(\tilde{\boldsymbol{\Sigma}})$, then we have the estimate that for all large enough n, there is a constant $M_{\tilde{\boldsymbol{\Sigma}}} > 0$ such that

$$\|\Delta_{\tilde{\boldsymbol{\Sigma}}}\|_{L^2(\tilde{\boldsymbol{\Sigma}})} \leq M_{\tilde{\boldsymbol{\Sigma}}} n^{-1}. \tag{4.76}$$

RH problem 4.13 entails a scalar shifted RH problem for $\mathcal{R}$:

RH Problem 4.14.

1. *$\tilde{\mathcal{R}}(s)$ is analytic in $\mathbb{C} \setminus \tilde{\boldsymbol{\Sigma}}$, where the contour $\tilde{\boldsymbol{\Sigma}}$ is defined in* (4.68).
2. *For $s \in \tilde{\boldsymbol{\Sigma}}$, we have*
$$\tilde{\mathcal{R}}_+(s) - \tilde{\mathcal{R}}_-(s) = (\Delta_{\tilde{\boldsymbol{\Sigma}}} \tilde{\mathcal{R}}_-)(s), \tag{4.77}$$
where $\Delta_{\tilde{\boldsymbol{\Sigma}}}$ s the operator defined in (4.75).
3. *As $s \to \infty$, we have*
$$\tilde{\mathcal{R}}(s) = 1 + \mathcal{O}(s^{-1}). \tag{4.78}$$
4. *As $s \to 0$, we have $\tilde{\mathcal{R}}(s) = \mathcal{O}(1)$.*

Like Lemma 3.15, we have

Lemma 4.15. *The function $\tilde{\mathcal{R}}(s)$ defined in* (4.67) *is the unique solution of RH problem 4.14 after trivial analytical extension.*

The proof is the same as that of Lemma 3.15, and we omit it.
Finally, we have analogous to Lemma 3.16 that

Lemma 4.16. *For all $s \in \mathbb{C} \setminus \tilde{\boldsymbol{\Sigma}}$, we have the uniform convergence*

$$\tilde{\mathcal{R}}(s) = 1 + \mathcal{O}(n^{-1}). \tag{4.79}$$

The proof of this lemma is analogous to that of Lemma 3.16, and we omit it. This lemma immediately yields that

$$\tilde{R}_1(z) = 1 + \mathcal{O}(n^{-1}) \quad \text{uniformly in } \mathbb{P} \setminus \Sigma^R, \quad \tilde{R}_2(z) = 1 + \mathcal{O}(n^{-1}) \quad \text{uniformly in } \mathbb{C} \setminus \Sigma^R. \tag{4.80}$$

5 Proof of main results

In this section we prove Theorem 1.11. In the statement of Theorem 1.11, $\mathbb{C}_+ \cup \mathbb{R}$ is divided into A_δ, B_δ, C_δ and D_δ. We can consider the limit of $p^{(n)}_{n+k}(z)$ and and $q^{(n)}_{n+k}(z)$ in the hard edge region $D(0, \epsilon)$, the soft edge region $D(b, \epsilon)$, the bulk region enclosed by $\Sigma^R_1, \Sigma^R_2, \partial D(0, \epsilon), \partial D(b, \epsilon)$, and the outside region that is the complement of these three, since by deforming the shape of contour Σ^R, these four regions cover $C_\delta, D_\delta, B_\delta, A_\delta$ respectively.

5.1 Outside region

For $z \in \mathbb{C}$ that is out of the lens and out of $D(0,\epsilon)$ and $D(b,\epsilon)$, we have, by RH problems 3.1, 3.2, 3.3, 3.6 and 3.13

$$\begin{aligned} p^{(n)}_{n+k}(z) = Y^{(n+k,n)}_1(z) = T_1(z)e^{n\mathbf{g}(z)} = S_1(z)e^{n\mathbf{g}(z)} = Q_1(z)P^{(\infty)}_1(z)e^{n\mathbf{g}(z)} \\ = R_1(z)P^{(\infty)}_1(z)e^{n\mathbf{g}(z)}. \end{aligned} \tag{5.1}$$

Since $R_1(z) = 1 + \mathcal{O}(n^{-1})$ uniformly by (3.109) and $P^{(\infty)}_1(z) = G_k(\mathbf{I}_1(z))$ by (3.29), we prove (1.59).

Similarly, for $z \in \mathbb{P}$ that is out of the lens and out of $D(0,\epsilon)$ and $D(b,\epsilon)$, we consider the counterpart of (5.1) for $q^{(n)}_{n+k}(f(z))$. For later use in Section 5.5, we also consider $\tilde{C}q^{(n)}_{n+k}(f(z))$ for $z \in \mathbb{C}$ that is out of the lens and out of $D(0,\epsilon)$ and $D(b,\epsilon)$. We have, by RH problems 4.1, 4.2, 4.3, 4.6 and 4.13, like (5.1),

$$q^{(n)}_{n+k}(f(z)) = \tilde{R}_1(z)\tilde{P}^{(\infty)}_1(z)e^{n\tilde{\mathbf{g}}(z)}, \qquad \tilde{C}q^{(n)}_{n+k}(z) = e^{n\ell}\tilde{R}_2(z)\tilde{P}^{(\infty)}_2(z)e^{-n\mathbf{g}(z)}. \tag{5.2}$$

Since $\tilde{R}_1(z) = 1 + \mathcal{O}(n^{-1})$ and $\tilde{R}_2(z) = 1 + \mathcal{O}(n^{-1})$ uniformly by (4.80), $\tilde{P}^{(\infty)}_1(z) = \tilde{G}_k(\mathbf{I}_2(z))$ by (4.22), and $\tilde{P}^{(\infty)}_2(z)$ has the expression by (4.21) and (4.20) we prove (1.60) and get

$$\tilde{C}q^{(n)}_{n+k}(z) = \frac{(1-s_1)^{\alpha+\frac{1}{2}}\sqrt{s_2-1}\,i\tilde{D}(1)^{-1}e^{kc}z^{\alpha}e^{n\ell}e^{-n\mathbf{g}(z)}}{(\mathbf{I}_1(z)-s_1)^{\alpha}(\mathbf{I}_1(z)-1)^{k}\sqrt{(\mathbf{I}_1(z)-s_1)(\mathbf{I}_1(z)-s_2)}D(\mathbf{I}_1(z))}(1+\mathcal{O}(n^{-1})). \tag{5.3}$$

5.2 Bulk region

Similar to (5.1), we have that for z in the upper lens and out of $D(0,\epsilon)$ and $D(b,\epsilon)$,

$$\begin{aligned} p^{(n)}_{n+k}(z) &= Y^{(n+k,n)}_1(z) = T_1(z)e^{n\mathbf{g}(z)} = S_1(z)e^{n\mathbf{g}(z)} + z^{-\alpha}h(z)^{-1}f'(z)S_2(z)e^{n(V(z)-\tilde{\mathbf{g}}(z)+\ell)} \\ &= Q_1(z)P^{(\infty)}_1(z)e^{n\mathbf{g}(z)} + z^{-\alpha}h(z)^{-1}f'(z)Q_2(z)P^{(\infty)}_2(z)e^{n(V(z)-\tilde{\mathbf{g}}(z)+\ell)} \\ &= R_1(z)P^{(\infty)}_1(z)e^{n\mathbf{g}(z)} + z^{-\alpha}h(z)^{-1}f'(z)R_2(z)P^{(\infty)}_2(z)e^{n(V(z)-\tilde{\mathbf{g}}(z)+\ell)}. \end{aligned} \tag{5.4}$$

Like in (5.1), $R_1(z) = 1 + \mathcal{O}(n^{-1})$ and $R_2(z) = 1 + \mathcal{O}(n^{-1})$ uniformly by (3.109). By (3.29), (3.30), (3.27), (1.55), (1.56) and (1.53), we have $P^{(\infty)}_1(z) = G_k(\mathbf{I}_1(z))$ as in the outside region and

$$z^{-\alpha}h(z)^{-1}f'(z)P^{(\infty)}_2(z) = G_k(\mathbf{I}_2(z)). \tag{5.5}$$

Hence we prove (1.61).

In particular, if $x \in (\epsilon, b-\epsilon)$ and $z \to x$ from above, we have by (2.12) that $\lim_{z\to x}\mathbf{g}(z) = \mathbf{g}_+(x)$ and $\lim_{z\to x} V(z) - \tilde{\mathbf{g}}(z) + \ell = V(x) - \tilde{\mathbf{g}}_+(x) + \ell = \mathbf{g}_-(x) = \overline{\mathbf{g}_+(x)}$. From the definition (1.47) of $\mathbf{g}(z)$, we have $\mathbf{g}_\pm(x) = \int \log|x-y|d\mu(y) \pm \mu([x,b])i$. On the other hand, as $z \to x$ from above, by (1.28) and (1.29), $\mathbf{I}_1(z)$ and $\mathbf{I}_2(z)$ converges to $\mathbf{I}_+(x)$ and $\mathbf{I}_-(x)$ respectively. Noting that $\mathbf{I}_-(x) = \overline{\mathbf{I}_+(x)}$, and then $\lim_{z\to x} G_k(\mathbf{I}_2(z)) = G_{k,+}(\mathbf{I}_-(x)) = \overline{G_{k,+}(\mathbf{I}_+(x))} = \overline{\lim_{z\to x} G_k(\mathbf{I}_1(z))}$ and $\lim_{z\to x} R_2(z) = \mathcal{R}(\mathbf{I}_-(x)) = \overline{\mathcal{R}(\mathbf{I}_+(x))} = \overline{\lim_{z\to x} R_1(z)}$, we have

$$\begin{aligned} p^{(n)}_{n+k}(x) = \lim_{z\to x \text{ in } \mathbb{C}_+} p^{(n)}_{n+k}(z) &= 2\Re\left[\mathcal{R}(\mathbf{I}_+(x))G_{k,+}(\mathbf{I}_+(x))e^{n\mathbf{g}_+(x)}\right] \\ &= 2\Re\left((1+\mathcal{O}(n^{-1}))G_{k,+}(\mathbf{I}_+(x))e^{n\mathbf{g}_+(x)}\right), \end{aligned} \tag{5.6}$$

which implies (1.63).

Analogous to (5.4), we have that for z in the upper lens and out of $D(0,\epsilon)$ and $D(b,\epsilon)$,

$$q^{(n)}_{n+k}(f(z)) = \tilde{R}_1(z)\tilde{P}^{(\infty)}_1(z)e^{n\tilde{\mathbf{g}}(z)} + z^{-\alpha}h(z)^{-1}\tilde{R}_2(z)\tilde{P}^{(\infty)}_2(z)e^{n(V(z)-\mathbf{g}(z)+\ell)}. \tag{5.7}$$

Like in (5.2), $\tilde{R}_1(z) = 1+\mathcal{O}(n^{-1})$ and $\tilde{R}_2(z) = 1+\mathcal{O}(n^{-1})$ uniformly by (4.80). By (4.22), (4.21) and (4.20), (1.56) and (1.53), we have $\tilde{P}^{(\infty)}_1(z) = \tilde{G}_k(\mathbf{I}_2(z))$ and

$$z^{-\alpha}h(z)^{-1}\tilde{P}^{(\infty)}_2(z) = \tilde{G}_k(\mathbf{I}_1(z)). \tag{5.8}$$

Hence we prove (1.62).

In particular, if $x\in(\epsilon, b-\epsilon)$ and $z\to x$ from above, we have by (2.12) that $\lim_{z\to x}\tilde{\mathbf{g}}(z) = \tilde{\mathbf{g}}_+(x)$ and $\lim_{z\to x} V(z)-\mathbf{g}(z)+\ell = V(x)-\mathbf{g}_+(x)+\ell = \tilde{\mathbf{g}}_-(x) = \overline{\tilde{\mathbf{g}}_+(x)}$. From the definition (1.47) of $\tilde{\mathbf{g}}(z)$, we have $\tilde{\mathbf{g}}_\pm(x) = \int\log|f(x)-f(y)|d\mu(y) \pm \mu([x,b])i$. Like (5.6), we have

$$q^{(n)}_{n+k}(f(x)) = \lim_{z\to x \text{ in } \mathbb{C}_+} q^{(n)}_{n+k}(f(z)) = 2\Re\left((1+\mathcal{O}(n^{-1}))\tilde{G}_{k,+}(\mathbf{I}_+(x))e^{n\tilde{\mathbf{g}}_+(x)}\right), \tag{5.9}$$

which implies (1.63).

5.3 Soft edge region

Like (5.1) and (5.4), we have, for $z\in D(b,\epsilon)\cap\mathbb{C}_+$,

$$\begin{aligned} p^{(n)}_{n+k}(z) = Y^{(n+k,n)}_1(z) = T_1(z)e^{n\mathbf{g}(z)} = S_1(z)e^{n\mathbf{g}(z)} = P^{(\infty)}_1(z)Q_1(z)e^{n\mathbf{g}(z)} \\ + \begin{cases} 0, & \text{outside the lens,} \\ z^{-\alpha}h(z)^{-1}f'(z)P^{(\infty)}_2(z)Q_2(z)e^{n(V(z)-\tilde{\mathbf{g}}(z)+\ell}, & \text{inside the upper lens.}\end{cases} \end{aligned} \tag{5.10}$$

and then by (3.55), (3.52), (3.49), (3.43), (3.42) and (1.48) (also (B.6) if z is inside the upper lens)

$$\begin{aligned} p^{(n)}_{n+k}(z) = \sqrt{\pi}\Bigg[n^{\frac{1}{6}}f_b^{\frac{1}{4}}(z)\left(P^{(\infty)}_1(z)V^{(b)}_1(z) - i\frac{f'(z)}{z^\alpha h(z)}P^{(\infty)}_2(z)V^{(b)}_2(z)\right)\mathrm{Ai}(n^{\frac{2}{3}}f_b(z)) \\ -n^{-\frac{1}{6}}f_b^{-\frac{1}{4}}(z)\left(P^{(\infty)}_1(z)V^{(b)}_1(z) + i\frac{f'(z)}{z^\alpha h(z)}P^{(\infty)}_2(z)V^{(b)}_2(z)\right)\mathrm{Ai}'(n^{\frac{2}{3}}f_b(z))\Bigg]e^{\frac{n}{2}(\mathbf{g}(z)-\tilde{\mathbf{g}}(z)+V(z)+\ell)}, \end{aligned} \tag{5.11}$$

By (3.84) and (3.85), we have that $V_1(z) = R_1(z)$ and $V_2(z) = R_2(z)$, so they are both uniformly $1+\mathcal{O}(n^{-1})$ by (3.109). Also we still have (3.29) and (5.5) for $P^{(\infty)}_1, P^{(\infty)}_2$ like in the outside and bulk regions. Hence we have (1.65).

For $z\in\mathbb{C}_+$ in the vicinity of b, we have the limits of $\mathbf{I}_1(z)$ and $\mathbf{I}_2(z)$ by (A.14) and (A.15) respectively, and for s in the vicinity of s_2, we have by (1.55),

$$G_k(s) = \left(\frac{c^2(s_2-s_1)}{4b}\right)^{\alpha+\frac{1}{2}}\left(\frac{c}{2}\right)^k s_2^{\frac{1}{2}}D(s_2)(s-s_2)^{-\frac{1}{2}}(1+\mathcal{O}(s-s_2)). \tag{5.12}$$

where $(s-s_2)^{-1/2}$ is positive on $(s_2,+\infty)$ and has the branch along γ_1. Hence we derive (1.67).

Like (5.10), we have, for $z\in D(b,\epsilon)\cap\mathbb{C}_+$,

$$\begin{aligned} q^{(n)}_{n+k}(f(z)) = \tilde{P}^{(\infty)}_1(z)\tilde{Q}_1(z)e^{n\tilde{\mathbf{g}}(z)} \\ + \begin{cases} 0, & \text{outside the lens,} \\ z^{-\alpha}h(z)^{-1}\tilde{P}^{(\infty)}_2(z)\tilde{Q}_2(z)e^{n(V(z)-\mathbf{g}(z)+\ell}, & \text{inside the upper lens.}\end{cases} \end{aligned} \tag{5.13}$$

and then by (4.43), (4.39), (4.38), (4.33), (4.32) and (1.48) (also (B.6) if z is inside the upper lens)

$$q_{n+k}^{(n)}(f(z)) = \sqrt{\pi}\left[n^{\frac{1}{6}} f_b^{\frac{1}{4}}(z)\left(\tilde{P}_1^{(\infty)}(z)\tilde{V}_1^{(b)}(z) - i\frac{1}{z^\alpha h(z)}\tilde{P}_2^{(\infty)}(z)\tilde{V}_2^{(b)}(z)\right)\mathrm{Ai}(n^{\frac{2}{3}} f_b(z))\right.$$
$$\left. -n^{-\frac{1}{6}} f_b^{-\frac{1}{4}}(z)\left(\tilde{P}_1^{(\infty)}(z)\tilde{V}_1^{(b)}(z) + i\frac{1}{z^\alpha h(z)}\tilde{P}_2^{(\infty)}(z)\tilde{V}_2^{(b)}(z)\right)\mathrm{Ai}'(n^{\frac{2}{3}} f_b(z))\right] e^{\frac{n}{2}(\tilde{\mathbf{g}}(z)-\mathbf{g}(z)+V(z)+\ell)}, \tag{5.14}$$

By (4.60) and (4.61), we have that $\tilde{V}_1(z) = \tilde{R}_1(z)$ and $\tilde{V}_2(z) = \tilde{R}_2(z)$, so they are both uniformly $1 + \mathcal{O}(n^{-1})$ by (4.80). Also we still have (4.22) and (5.8) for $\tilde{P}_1^{(\infty)}, \tilde{P}_2^{(\infty)}$ like in the outside and bulk regions. Hence we have (1.66).

Similar to (5.12), we use the limits of $\mathbf{I}_1(z)$ and $\mathbf{I}_2(z)$, and have for s in the vicinity of s_2

$$\tilde{G}_k(s) = \left(\frac{1-s_1}{s_2-s_1}\right)^{\alpha+\frac{1}{2}}\left(\frac{c}{2}\right)^{k-\frac{1}{2}} e^{kc}\frac{\tilde{D}(s_2)}{\tilde{D}(1)} i(s-s_2)^{-\frac{1}{2}}(1+\mathcal{O}(s-s_2)), \tag{5.15}$$

where $(s-s_2)^{-1/2}$ is positive on $(s_2, +\infty)$ and has the branch along γ_2. Hence we prove (1.68) in the same way as (1.67).

5.4 Hard edge region

For $z \in D(0,\epsilon)$, (5.10), still holds, and then by (3.81), (3.74), (3.67), (3.62), (3.61), and (1.48) (also (B.16) if z is inside the upper lens)

$$p_{n+k}^{(n)}(z) = \sqrt{\pi}\left[n^{\frac{1}{2}} f_0^{\frac{1}{4}}(z)\left(P_1^{(\infty)}(z)V_1^{(0)}(z) + i\frac{f'(z)}{(-z)^\alpha h(z)}P_2^{(\infty)}(z)V_2^{(0)}(z)\right)I_\alpha(2n\sqrt{f_0(z)})\right.$$
$$\left. + n^{-\frac{1}{2}} f_0^{-\frac{1}{4}}(z)\left(P_1^{(\infty)}(z)V_1^{(0)}(z) - i\frac{f'(z)}{(-z)^\alpha h(z)}P_2^{(\infty)}(z)V_2^{(0)}(z)\right)I'_\alpha(2n\sqrt{f_0(z)})\right]$$
$$\times e^{\frac{n}{2}(\mathbf{g}(z)-\tilde{\mathbf{g}}(z)+V(z)+\ell)}, \tag{5.16}$$

where the $(-z)^\alpha$ factor takes the principal branch as $\arg(-z) \in (-\pi, \pi)$. Like in the soft edge region, $V_1^{(0)}(z)$ and $V_2^{(0)}(z)$ are both uniformly $1+\mathcal{O}(n^{-1})$ by (3.109), and $P_1^{(\infty)}(z)$ and $P_2^{(\infty)}(z)$ are still given by (3.29), (3.30), (3.27), (1.55) and (1.53). Then we have (1.69).

For $z \in \mathbb{C}_+$ in the vicinity of 0, we have the limits of $\mathbf{I}_1(z)$ and $\mathbf{I}_2(z)$ by (A.9) and (A.10) respectively, and for s in the vicinity of s_1, we have by (1.55),

$$G_k(s) = \left(\frac{4(1-s_1)^2(-s_1)}{(s_2-s_1)^2}\right)^{\alpha+\frac{1}{2}}\left(\frac{c^2}{4}(s_1-1)\right)^k\sqrt{\frac{-s_1}{s_2-s_1}}D(s_1)(s_1-s)^{-\alpha-\frac{1}{2}}(1+\mathcal{O}(s-s_1)), \tag{5.17}$$

where $(s_1-s)^{-\alpha-1/2}$ is positive on $(-\infty, s_1)$ and has the branch cut along γ_1. Hence we derive, if $z \in D(0,\epsilon) \cap \mathbb{C}_+$ and $z = f_0'(0)^{-1}n^{-2}t$, with t bounded, then uniformly

$$n^{-\frac{1}{2}} e^{-\frac{n}{2}(\mathbf{g}(z)-\tilde{\mathbf{g}}(z)+V(z)+\ell)}(-z)^{\frac{\alpha}{2}} p_{n+k}^{(n)}(z) =$$
$$2\sqrt{\pi}\sqrt{\frac{-s_1}{s_2-s_1}}\left(\frac{c(1-s_1)\sqrt{-s_1}}{s_2-s_1}\right)^{\alpha+\frac{1}{2}}\left(\frac{c^2}{4}(s_1-1)\right)^k D(s_1)(-f_0(0))^{\frac{1}{4}}\left(I_\alpha(2\sqrt{t}) + \mathcal{O}(n^{-1})\right), \tag{5.18}$$

which implies (1.71) on $D(0,\epsilon) \cap (\mathbb{C}_+ \cup \mathbb{R})$ by changing I_α into J_α as [35, 10.27.6].

Like (5.16), we have, for $z \in D(0,\epsilon) \cap \mathbb{C}_+$,

$$q^{(n)}_{n+k}(f(z)) = \sqrt{\pi}\left[n^{\frac{1}{2}} f_0^{\frac{1}{4}}(z)\left(\tilde{P}^{(\infty)}_1(z)\tilde{V}^{(0)}_1(z) + i\frac{1}{(-z)^\alpha h(z)}\tilde{P}^{(\infty)}_2(z)\tilde{V}^{(0)}_2(z)\right) I_\alpha(2n\sqrt{f_0(z)}) \right.$$
$$\left. + n^{-\frac{1}{2}} f_0^{-\frac{1}{4}}(z)\left(\tilde{P}^{(\infty)}_1(z)\tilde{V}^{(0)}_1(z) - i\frac{1}{(-z)^\alpha h(z)}\tilde{P}^{(\infty)}_2(z)\tilde{V}^{(0)}_2(z)\right) I'_\alpha(2n\sqrt{f_0(z)})\right]$$
$$\times e^{\frac{n}{2}(\tilde{\mathbf{g}}(z) - \mathbf{g}(z) + V(z) + \ell)}. \quad (5.19)$$

By (4.60) and (4.61), we have that $\tilde{V}_1(z) = \tilde{R}_1(z)$ and $\tilde{V}_2(z) = \tilde{R}_2(z)$, so they are both uniformly $1 + \mathcal{O}(n^{-1})$ by (4.80). Also we still have (5.8) for $\tilde{P}^{(\infty)}_1, \tilde{P}^{(\infty)}_2$ like in the bulk region and the soft edge region. Hence we have (1.70).

Similar to (5.17), we also have for s in the vicinity of s_1

$$\tilde{G}_k(s) = (1-s_1)^{\alpha+\frac{1}{2}}(s_1-1)^{-k} e^{kc}\sqrt{\frac{s_2-1}{s_2-s_1}}\frac{\tilde{D}(s_1)}{\tilde{D}(1)}(s-s_1)^{-\alpha-\frac{1}{2}}(1+\mathcal{O}(s-s_1)). \quad (5.20)$$

Hence we prove (1.72) in the same way as (1.71).

5.5 Computation of $h^{(n)}_{n+k}$

$h^{(n)}_{n+k}$ is defined in (1.5). By the limiting formulas (1.59) for $p^{(n)}_{n+k}(z)$ and (1.60) for $q^{(n)}_{n+k}(f(z))$, and the regularity condition (2.13), we have that if $\epsilon > 0$ is a small constant, then there is $\delta = \delta(\epsilon) > 0$, such that for any $R > b + \delta$,

$$\left|h^{(n)}_{n+k} - h^{(n)}_{n+k}(R)\right| = oe^{(1-\epsilon)n\ell}, \quad \text{where} \quad h^{(n)}_{n+k}(R) = \int_0^R p^{(n)}_{n+k}(x) q^{(n)}_{n+k}(f(x)) W^{(n)}_\alpha(x)dx. \quad (5.21)$$

Hence to prove (1.73), we only need to compute

$$h^{(n)}_{n+k}(R) = -\oint_{C_R} p^{(n)}_{n+k}(z)\tilde{C}q^{(n)}_{n+k}(z)dz, \quad (5.22)$$

where C_R is the circular contour with positive orientation, centred at 0 with radius R. With the help of (1.59) and (5.3), we have

$$\begin{aligned} h^{(n)}_{n+k}(R) &= e^{n\ell}\left(-\oint_{C_R}\frac{G_k(\mathbf{I}_1(z))(1-s_1)^{\alpha+\frac{1}{2}}\sqrt{s_2-1}\,i\tilde{D}(1)^{-1}e^{kc}z^\alpha D(\mathbf{I}_1(z))^{-1}}{(\mathbf{I}_1(z)-s_1)^\alpha(\mathbf{I}_1(z)-1)^k\sqrt{(\mathbf{I}_1(z)-s_1)(\mathbf{I}_1(z)-s_2)}}dz + \mathcal{O}(n^{-1})\right) \\ &= e^{n\ell}\left(-\frac{\sqrt{s_2-1}\,i}{\tilde{D}(1)}\left(\frac{c^2(1-s_1)}{4}\right)^{\alpha+\frac{1}{2}}\left(\frac{c^2}{4}\right)^k e^{kc}\oint_{C_R}\frac{\mathbf{I}_1(z)^{1/2}}{z^{1/2}(\mathbf{I}_1(z)-s_2)}dz + \mathcal{O}(n^{-1})\right) \\ &= e^{n\ell}\left(-\frac{\sqrt{s_2-1}\,i}{\tilde{D}(1)}\left(\frac{c^2(1-s_1)}{4}\right)^{\alpha+\frac{1}{2}}\left(\frac{c^2}{4}\right)^k e^{kc}(c\pi i) + \mathcal{O}(n^{-1})\right), \end{aligned} \quad (5.23)$$

where in the last step we take $R \to \infty$ and use $\mathbf{I}_1(z) = 4c^{-2}z + \mathcal{O}(1)$ as $z \to \infty$. Hence we derive (1.73).

A Properties of $\mathbf{J}_x(s)$ and related functions

Proof of Part 4 of Lemma 1.6 Below we give a constructive description of $\gamma_1(x)$. Any $s \in \mathbb{C}_+$ can be represented as

$$s = \frac{\cosh(u+iv)+1}{\cosh(u+iv)-1}, \quad u \in (0,\infty) \text{ and } v \in (-\pi, 0). \tag{A.1}$$

Then the condition $s \in \gamma_1(x) \subseteq \mathbb{C}_+$ is equivalent to $\Im J_x(s) = 0$, which can be expressed as $x \sin v/(\cosh u - \cos v) - v = 0$, or equivalently,

$$\cosh u = x\frac{\sin v}{v} + \cos v. \tag{A.2}$$

By direct computation, we see that the right-hand side of (A.2) is an increasing on $(-\pi, 0)$, and its limits at $-\pi$ and 0 are -1 and $x+1$ respectively. Hence, (A.2) has a solution only if $v \in (v^*, 0)$, where $v^* \in (-\pi, 0)$ is the solution to $1 = x\sin(v)/v + \cos v$. We then construct the curve

$$\gamma_1'(x) = \{\frac{\cosh(u(v)+iv)+1}{\cosh(u(v)+iv)-1} : v \in [v^*, 0]\}, \quad \text{where} \quad u(v) = \operatorname{arcosh}\left(x\frac{\sin v}{v} + \cos v\right). \tag{A.3}$$

which lies in $\mathbb{C}_+$ and connects $\gamma_1'(v^*) = (\cosh(iv^*)+1)/(\cosh(iv^*)-1) = s_1(x)$ and $\gamma_1'(0) = 1 + 2/x = s_2(x)$.

By expressing s in u, v in (A.1) and parametrizing u by v as in (A.3), we have that $J_x(s)\mid_{s\in\gamma_1'(x)}$ is parametrized by $v \in (v^*, 0)$. Then we can compute

$$\frac{d}{dv}J_x(s(u(v),v)) = -\frac{x+1-\cosh(u(v)+iv)}{\cosh(u(v)+iv)-1}(u'(v)+i). \tag{A.4}$$

Since by the construction, we know that the $J_x(s) \in \mathbb{R}$ if $s \in \gamma_1'(x)$, we know that the derivative above is real valued wherever it is well defined on $(v^*, 0)$. On the other hand, we have that $x + 1 - \cosh(u(v) + iv) \neq 0$ and $u'(v) + i \neq 0$ on $(v^*, 0)$. Hence the derivative in (A.4) is real, non-vanishing, and continuous on $(v^*, 0)$, and then it has to be always positive or always negative there. By comparing $J_x(s_1(x))$ and $J_x(s_2(x))$, we conclude that the derivative is always positive on $(v^*, 0)$.

Now we see that the curve $\gamma_1'(x)$ satisfies the properties of $\gamma_1(x)$ described in Part 4 of Lemma 1.6, and it is the only candidate of $\gamma_1(x)$. So we let $\gamma_1(x)$ be $\gamma_1'(x)$ constructed in (A.3), and prove constructively Part 4 of Lemma 1.6.

Proof of Lemma 1.7 Both $\mathbb{C}_+$ and $\mathbb{C}_+ \setminus D$ are simply connected regions. From Parts 1, 2 and 4 of Lemma 1.6, we have that $\mathbf{J}_x(s)$ maps $(-\infty, s_1(x)] \cup \gamma_1(x) \cup [s_2(x), \infty)$, the boundary of $\mathbb{C}_+ \setminus D$, homeomorphically to $\mathbb{R}$, the boundary of $\mathbb{C}_+$. Also we have $J[x](s) = (x^2/4)s + \mathcal{O}(1)$. For any large enough $R \in \mathbb{R}_+$, we let C_R be the region enclosed by the semicircle $S_R := \{Re^{i\theta} : \theta \in (0, \pi)\}J[x](s)$ and the lower boundary $(-R, s_1(s)] \cup \gamma_1(x) \cup [s_2(x), R)$. We then have that $\mathbf{J}_x(s)$ maps the boundary of C_R homeomorphically to its image lying in $\mathbb{C}_+ \cup \mathbb{R}$. Hence by standard argument in complex analysis, $\mathbf{J}_x(s)$ maps C_R analytically and bijectively to its image lying in $\mathbb{C}_+$. Letting $R \to \infty$, we have that $\mathbf{J}_x(s)$ maps $\mathbb{C}_+ \setminus$ analytically and bijectively to $\mathbb{C}_+$.

Similarly, we can prove that $\mathbf{J}_x(s)$ maps $\mathbb{C}_+ \cap D$ bijectively to $\mathbb{C}_- \cap \mathbb{P}$.

At last, since $\mathbf{J}_x(\bar{s}) = \overline{\mathbf{J}_x(s)}$, the results above can be extended to $\mathbb{C} \setminus \bar{D}$ and $\mathbb{C} \cap D$. Hence we finish the proof of Lemma 1.7.

Limiting shape of $\gamma_1(x)$ as $x \to \infty$ and $x \to 0$ With the help of expression (A.3) of $\gamma_1(x) = \gamma_1'(x)$, we have the following results by direct calculation.

1. As $x \to \infty$, $s_1(x) = -(\frac{\pi}{x})^2(1 + \mathcal{O}(x^{-1}))$ and $b(x) = \frac{x^2}{4}(1 + \mathcal{O}(x^{-1}))$. Given $\epsilon > 0$, we have

$$\mathbf{I}_{x,+}(y) = \frac{4}{x^2}\left(y(1 + \mathcal{O}(x^{-1})) + i\pi\sqrt{y}(1 + \mathcal{O}(x^{-1}))\right), \quad y \in \left(\frac{\epsilon x^2}{4}, \frac{(1-\epsilon)x^2}{4}\right), \tag{A.5}$$

where the two $\mathcal{O}(x^{-1})$ terms are uniform for $y \in (\epsilon x^2/4, (1-\epsilon)x^2/4)$.

2. As $x \to 0_+$, $s_1(x) = -\frac{2}{x}(1 + \mathcal{O}(x))$ and $b(x) = 2x(1 + \mathcal{O}(x))$. Given $\epsilon > 0$, we have

$$\mathbf{I}_{x,+}(y) = \frac{2}{x^2}\left((y - x) + \sqrt{y(2x - y)}i\right)(1 + \mathcal{O}(x)), \quad y \in (\epsilon x, (2-\epsilon)x), \tag{A.6}$$

where the $1 + \mathcal{O}(x)$ term is uniform for $y \in (\epsilon x, (2-\epsilon)x)$.

Limit behaviour of $\mathbf{J}_x(s)$ around $s_1(x)$ and $s_2(x)$, and its inverse functions around 0 and $b(x)$ By direct computation, we have the follows.

1. In the vicinity of $s_1(x)$

$$\mathbf{J}_x(s) = \frac{x^2(s_2(x) - s_1(x))^2}{16 s_1(x)(1 - s_1(x))^2}(s - s_1(x))^2 + \mathcal{O}((s - s_1(x))^3), \tag{A.7}$$

and around 0, for a small enough $\epsilon > 0$,

$$\begin{aligned}
\mathbf{I}_{x,\pm}(y) &= s_1(x) \pm e_1(x)\sqrt{y}i + \mathcal{O}(y), && y \in (0, \epsilon), && \text{(A.8)}\\
\mathbf{I}_{x,1}(z) &= s_1(x) - e_1(x)(-z)^{\frac{1}{2}} + \mathcal{O}(z), && z \in N(0, \epsilon) \setminus [0, \epsilon), && \text{(A.9)}\\
\mathbf{I}_{x,2}(z) &= s_1(x) + e_1(x)(-z)^{\frac{1}{2}} + \mathcal{O}(z), && z \in N(0, \epsilon) \setminus [0, \epsilon). && \text{(A.10)}
\end{aligned}$$

where

$$e_1(x) = \frac{4\sqrt{-s_1(x)}(1 - s_1(x))}{x(s_2(x) - s_1(x))}. \tag{A.11}$$

2. In the vicinity of $s_2(x)$

$$\mathbf{J}_x(s) = b(x) + \frac{x^2 b(x)^{1/2}}{8 s_2(x)^{1/2}}(s - s_2(x))^2 + \mathcal{O}((s - s_2(x))^3), \tag{A.12}$$

and around $b(x)$, for a small enough $\epsilon > 0$,

$$\begin{aligned}
\mathbf{I}_{x,\pm}(y) &= s_2(x) \pm d_1(x)\sqrt{b(x) - y}i + \mathcal{O}(b(x) - y), && y \in (b(x) - \epsilon, b(x)), && \text{(A.13)}\\
\mathbf{I}_{x,1}(z) &= s_2(x) + d_1(x)\sqrt{z - b(x)} + \mathcal{O}(b(x) - z), && z \in N(b(x), \epsilon) \setminus [b(x) - \epsilon, b(x)), && \text{(A.14)}\\
\mathbf{I}_{x,2}(z) &= s_2(x) - d_1(x)\sqrt{z - b(x)} + \mathcal{O}(b(x) - z), && z \in N(b(x), \epsilon) \setminus [b(x) - \epsilon, b(x)). && \text{(A.15)}
\end{aligned}$$

where

$$d_1(x) = \frac{2\sqrt{2} s_2(x)^{1/4}}{x b(x)^{1/4}}. \tag{A.16}$$

B Local universal parametrices

B.1 The Airy parametrix

In this subsection, let y_0, y_1 and y_2 be the functions defined by

$$y_0(\zeta) = \sqrt{2\pi} e^{-\frac{\pi i}{4}} \operatorname{Ai}(\zeta), \quad y_1(\zeta) = \sqrt{2\pi} e^{-\frac{\pi i}{4}} \omega \operatorname{Ai}(\omega\zeta), \quad y_2(\zeta) = \sqrt{2\pi} e^{-\frac{\pi i}{4}} \omega^2 \operatorname{Ai}(\omega^2\zeta), \tag{B.1}$$

where Ai is the usual Airy function (cf. [35, Chapter 9]) and $\omega = e^{2\pi i/3}$. We then define a 2×2 matrix-valued function $\Psi^{(\mathrm{Ai})}$ by

$$\Psi^{(\mathrm{Ai})}(\zeta) = \begin{cases} \begin{pmatrix} y_0(\zeta) & -y_2(\zeta) \\ y_0'(\zeta) & -y_2'(\zeta) \end{pmatrix}, & \arg\zeta \in (0, \frac{2\pi}{3}), \\ \begin{pmatrix} -y_1(\zeta) & -y_2(\zeta) \\ -y_1'(\zeta) & -y_2'(\zeta) \end{pmatrix}, & \arg\zeta \in (\frac{2\pi}{3}, \pi), \\ \begin{pmatrix} -y_2(\zeta) & y_1(\zeta) \\ -y_2'(\zeta) & y_1'(\zeta) \end{pmatrix}, & \arg\zeta \in (-\pi, -\frac{2\pi}{3}), \\ \begin{pmatrix} y_0(\zeta) & y_1(\zeta) \\ y_0'(\zeta) & y_1'(\zeta) \end{pmatrix}, & \arg\zeta \in (-\frac{2\pi}{3}, 0). \end{cases} \tag{B.2}$$

It is well-known that $\det(\Psi^{(\mathrm{Ai})}(z)) = 1$ and $\Psi^{(\mathrm{Ai})}(\zeta)$ is the unique solution of the following 2×2 RH problem; cf. [17, Section 7.6].

RH Problem B.1.

1. *$\Psi(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_{\mathrm{Ai}}$, where the contour Γ_{Ai} is defined in*

$$\Gamma_{\mathrm{Ai}} := e^{-\frac{2\pi i}{3}}[0, +\infty) \cup \mathbb{R} \cup e^{\frac{2\pi i}{3}}[0, +\infty) \tag{B.3}$$

with the orientation shown in Figure 8.

2. *For $z \in \Gamma_{\mathrm{Ai}}$, we have*

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \arg\zeta = 0, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \arg\zeta = \pm\frac{2\pi}{3}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \arg\zeta = \pi. \end{cases} \tag{B.4}$$

3. *As $\zeta \to \infty$, we have*

$$\Psi(\zeta) = \zeta^{-\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3} (I + \mathcal{O}(\zeta^{-\frac{3}{2}})) e^{-\frac{2}{3}\zeta^{\frac{3}{2}}\sigma_3}. \tag{B.5}$$

4. *As $\zeta \to 0$, we have $\Psi_{i,j}(\zeta) = \mathcal{O}(1)$, where $i, j = 1, 2$.*

We note that the jump condition (B.4) can be derived from the identity [17, Equation (7.116)]

$$\operatorname{Ai}(\zeta) + \omega \operatorname{Ai}(\omega\zeta) + \omega^2 \operatorname{Ai}(\omega^2\zeta) = 0. \tag{B.6}$$

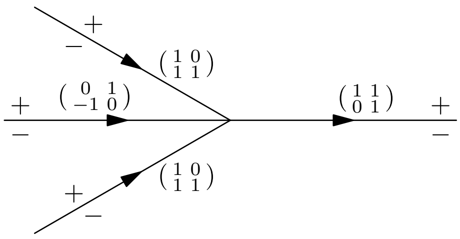

Figure 8: The jump contour Γ_{Ai} for the RH problem B.1 for $\Psi^{(\mathrm{Ai})}$.

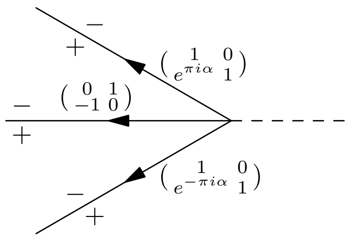

Figure 9: The jump contour Γ_{Be} for the RH problem B.1 for $\Psi_\alpha^{(\mathrm{Be})}$.

B.2 The Bessel parametrix

The Bessel kernel used in our paper is essentially the $\tilde{\Psi}$ constructed in [24, Equation (6.51)]. In this subsection, let w_0, w_1, w_2 and w_3 be the functions defined by

$$w_0(\zeta) = I_\alpha(2\zeta^{\frac{1}{2}}), \quad w_1(\zeta) = \frac{-i}{\pi} K_\alpha(2\zeta^{\frac{1}{2}}), \quad w_2(\zeta) = \frac{1}{2} H_\alpha^{(1)}(2(-\zeta)^{\frac{1}{2}}), \quad w_3(\zeta) = \frac{1}{2} H_\alpha^{(2)}(2(-\zeta)^{\frac{1}{2}}), \tag{B.7}$$

where all $\zeta^{1/2}$ are defined by the principal branch on the sector $\arg\zeta \in (-\pi, \pi)$, I_α and K_α are modified Bessel functions of order α, and $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ are Hankel functions of order α of the first and second kind, respectively. We then define a 2×2 matrix-valued function $\Psi_\alpha^{(\mathrm{Be})}$ by

$$\Psi_\alpha^{(\mathrm{Be})}(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & -2\pi i \zeta \end{pmatrix} \times \begin{cases} \begin{pmatrix} w_0(\zeta) & w_1(\zeta) \\ w_0'(\zeta) & w_1'(\zeta) \end{pmatrix}, & \arg\zeta \in (-\frac{2\pi}{3}, \frac{2\pi}{3}), \\ \begin{pmatrix} w_2(\zeta) & -w_3(\zeta) \\ w_2'(\zeta) & -w_3'(\zeta) \end{pmatrix} e^{\frac{\alpha}{2}\pi i \sigma_3}, & \arg\zeta \in (\frac{2\pi}{3}, \pi), \\ \begin{pmatrix} w_3(\zeta) & w_2(\zeta) \\ w_3'(\zeta) & w_2'(\zeta) \end{pmatrix} e^{-\frac{\alpha}{2}\pi i \sigma_3}, & \arg\zeta \in (-\pi, -\frac{2\pi}{3}). \end{cases} \tag{B.8}$$

It is well known that $\det(\Psi_\alpha^{(\mathrm{Be})}) = 1$ and $\Psi_\alpha^{(\mathrm{Be})}$ is the unique solution of the following RH problem:

RH Problem B.2.

1. *$\Psi(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma_{\mathrm{Be}}$, where the contour Γ_{Be} is defined in*

$$\Gamma_{\mathrm{Be}} := e^{-\frac{2\pi i}{3}}[0, +\infty) \cup \mathbb{R}_- \cup e^{\frac{2\pi i}{3}}[0, +\infty) \tag{B.9}$$

with the orientation shown in Figure 9.

2. *For $z \in \Gamma_{\mathrm{Be}}$, we have*

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & \arg\zeta = \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \arg\zeta = -\frac{2\pi}{3}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \arg\zeta = \pi. \end{cases} \tag{B.10}$$

3. As $\zeta \to \infty$*, we have*

$$\Psi(\zeta) = (2\pi)^{-\frac{1}{2}\sigma_3}\zeta^{-\frac{1}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}(I + \mathcal{O}(\zeta^{-\frac{1}{2}}))e^{2\zeta^{\frac{1}{2}}\sigma_3}. \tag{B.11}$$

4. As $\zeta \to 0$*, if* $\alpha \in (-1, 0)$*, then*

$$\Psi(\zeta) = \begin{pmatrix} \mathcal{O}(\zeta^{\alpha/2}) & \mathcal{O}(\zeta^{\alpha/2}) \\ \mathcal{O}(\zeta^{\alpha/2}) & \mathcal{O}(\zeta^{\alpha/2}) \end{pmatrix}, \qquad \Psi(\zeta)^{-1} = \begin{pmatrix} \mathcal{O}(\zeta^{\alpha/2}) & \mathcal{O}(\zeta^{\alpha/2}) \\ \mathcal{O}(\zeta^{\alpha/2}) & \mathcal{O}(\zeta^{\alpha/2}) \end{pmatrix}, \tag{B.12}$$

if $\alpha = 0$*, then*

$$\Psi(\zeta) = \begin{pmatrix} \mathcal{O}(\log\zeta) & \mathcal{O}(\log\zeta) \\ \mathcal{O}(\log\zeta) & \mathcal{O}(\log\zeta) \end{pmatrix}, \qquad \Psi(\zeta)^{-1} = \begin{pmatrix} \mathcal{O}(\log\zeta) & \mathcal{O}(\log\zeta) \\ \mathcal{O}(\log\zeta) & \mathcal{O}(\log\zeta) \end{pmatrix}, \tag{B.13}$$

and if $\alpha > 0$*, then in the sector* $\arg\zeta \in (-2\pi/3, 2\pi/3)$*,*

$$\Psi(\zeta) = \begin{pmatrix} \mathcal{O}(\zeta^{\alpha/2}) & \mathcal{O}(\zeta^{-\alpha/2}) \\ \mathcal{O}(\zeta^{\alpha/2}) & \mathcal{O}(\zeta^{-\alpha/2}) \end{pmatrix}, \qquad \Psi(\zeta)^{-1} = \begin{pmatrix} \mathcal{O}(\zeta^{-\alpha/2}) & \mathcal{O}(\zeta^{-\alpha/2}) \\ \mathcal{O}(\zeta^{\alpha/2}) & \mathcal{O}(\zeta^{\alpha/2}) \end{pmatrix}, \tag{B.14}$$

and in the sector $\arg\zeta \in (2\pi/3, \pi)$ *or in the sector* $\arg\zeta \in (-\pi, -2\pi/3)$*,*

$$\Psi(\zeta) = \begin{pmatrix} \mathcal{O}(\zeta^{-\alpha/2}) & \mathcal{O}(\zeta^{-\alpha/2}) \\ \mathcal{O}(\zeta^{-\alpha/2}) & \mathcal{O}(\zeta^{-\alpha/2}) \end{pmatrix}, \qquad \Psi(\zeta)^{-1} = \begin{pmatrix} \mathcal{O}(\zeta^{-\alpha/2}) & \mathcal{O}(\zeta^{-\alpha/2}) \\ \mathcal{O}(\zeta^{-\alpha/2}) & \mathcal{O}(\zeta^{-\alpha/2}) \end{pmatrix}. \tag{B.15}$$

Like (B.6), the jump condition (B.10) can be derived from the identity [24, Proof of Theorem 6.3]

$$w_2(\zeta) + w_3(\zeta) = w_0(\zeta). \tag{B.16}$$

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