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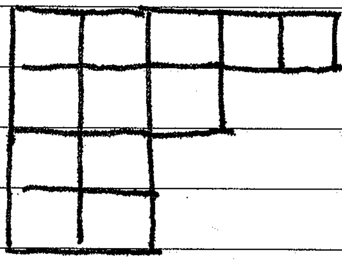
1. Partitions

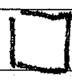
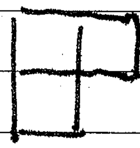
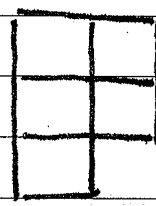
λ:  $(5,3,2,2) = 2^2 3^1 5^1$

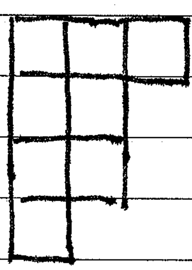
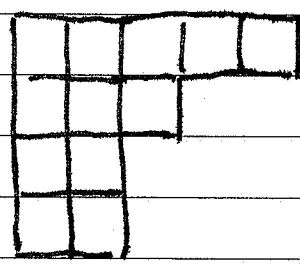
$|\lambda| = 12$. $\ell(\lambda) = 4$

(partial) ordering:

$\mu \le \lambda$ iff $\mu_1 + \mu_2 + \cdots + \mu_k \le \lambda_1 + \lambda_2 + \cdots + \lambda_k$ for all k.

$\mu \preceq \lambda$ iff λ is μ + a horizontal strip.

$\lambda^{(1)}$:  $\lambda^{(2)}$  $\lambda^{(3)}$ 

$\lambda^{(4)}$  $\lambda^{(5)} = \lambda$ 

$\phi \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \lambda^{(4)} \prec \lambda^{(5)} = \lambda$

Semi-standard Young tableau:

1	2	4	5	5
2	3	5		
3	4			
4	5			

row: weakly increasing

column: strictly increasing

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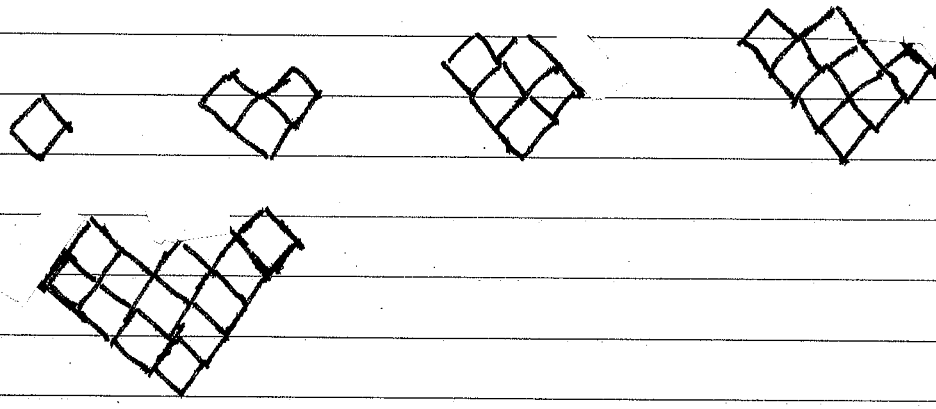
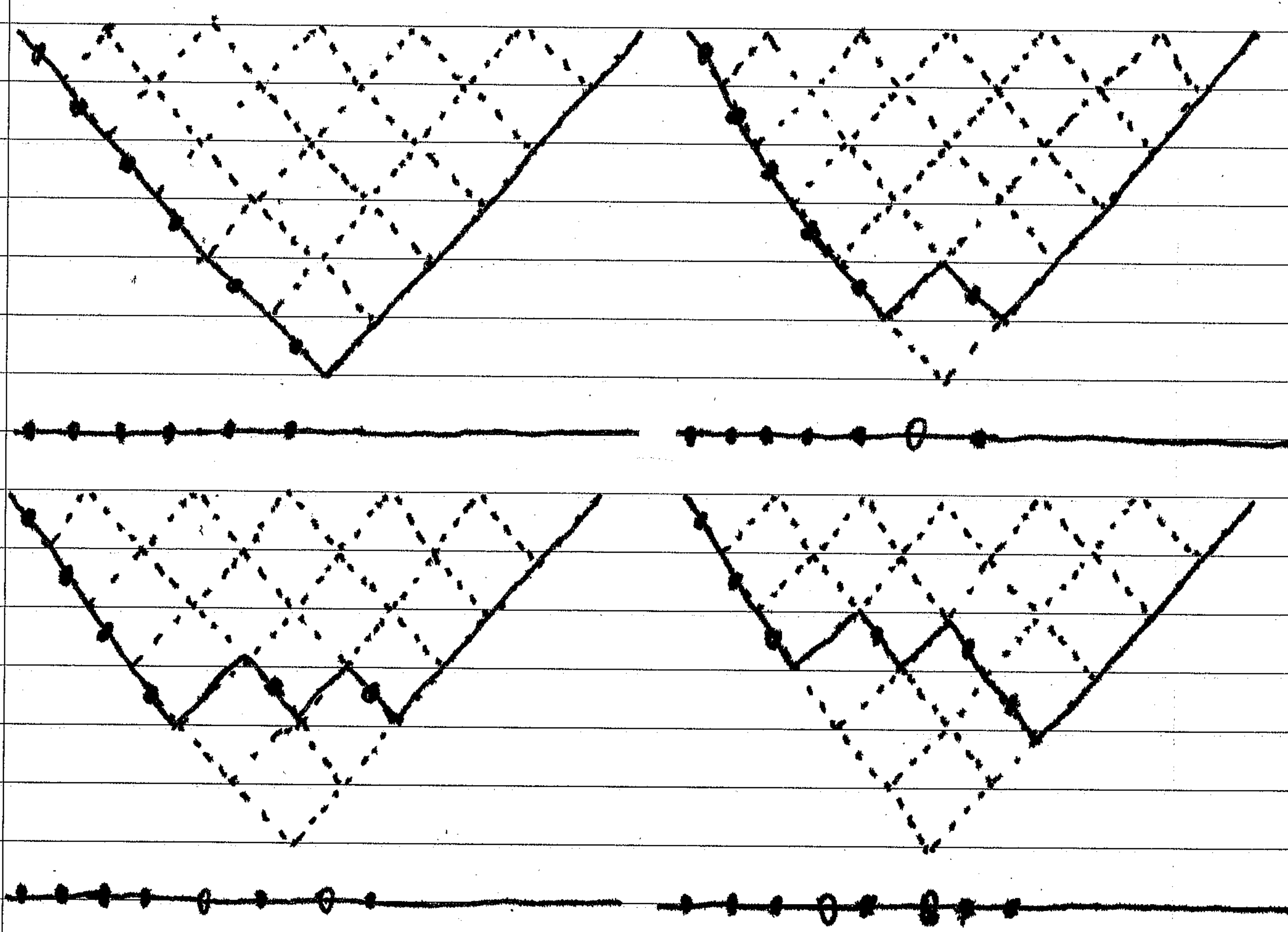
Growth model :



Particle model :



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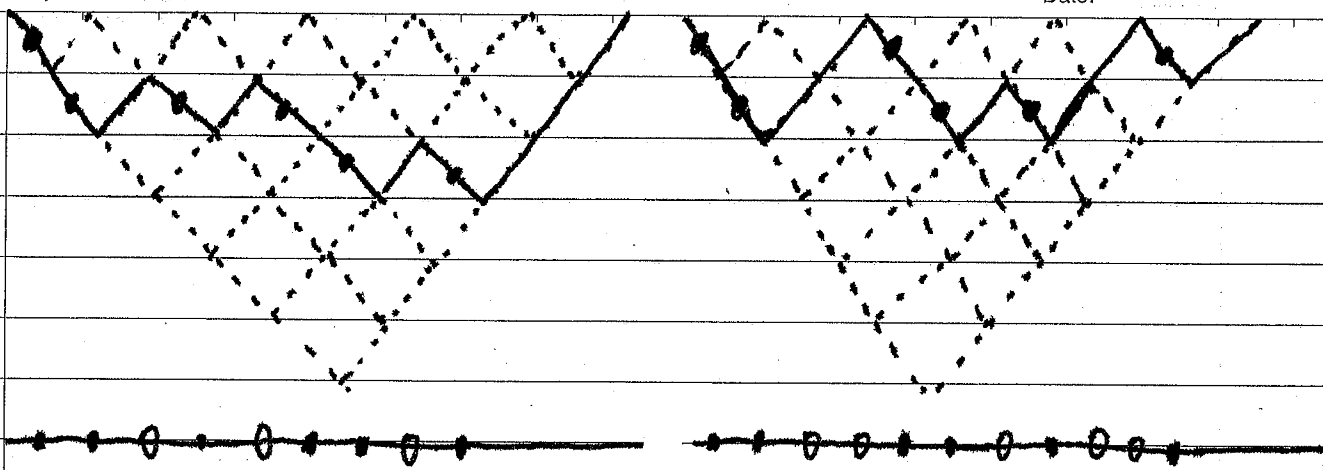

2. Symmetric polynomials. (indexed by partition).

Let $x = (x_1, \cdots, x_n)$.

Monomial symmetric polynomials:

$$m_{(3,2,1)}(x_1, x_2, x_3) = \underbrace{x_1^3 x_2^2 x_3^1 + x_1^3 x_2^1 x_3^2 + \cdots}_{6 \text{ terms}}$$

$$m_{(3,2,1)}(x_1, x_2, x_3, x_4) = \underbrace{x_1^3 x_2^2 x_3^1 + \cdots}_{6 \text{ terms}} + \underbrace{x_1^3 x_2^2 x_4^1 + \cdots}_{6 \text{ terms}} + \cdots$$

+ totally 24 terms.

Power sum symmetric polynomials:

$$p_{(k)}(x) = x_1^k + x_2^k + \cdots + x_n^k$$

$$p_\lambda(x) = p_{\lambda_1}(x)\, p_{\lambda_2}(x) \cdots p_{\lambda_{\ell(\lambda)}}(x).$$

Schur polynomials:

$$s_\lambda(x) = \sum_{\mu \le \lambda} K_{\lambda\mu}\, m_\mu(x).$$

$K_{\lambda\mu}$: Kostka number, the number of ways of putting

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μ_1's 1, μ_2's 2, $\cdots$ into the Young diagram of shape λ to make a semi-standard Young tableau.

$K_{(3,2)\,(2,1,1,1)} = 3$: $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$ $\quad$ $\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$ $\quad$ $\begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$

Jacobi - Trudi formula.

$$s_\lambda(x_1, x_2, \cdots x_n) = \frac{\begin{vmatrix} x_1^{n-1+\lambda_1} & \cdots & x_n^{n-1+\lambda_1} \\ x_1^{n-2+\lambda_2} & \cdots & x_n^{n-2+\lambda_2} \\ \vdots & & \vdots \\ x_1^{1+\lambda_{n-1}} & \cdots & x_n^{1+\lambda_{n-1}} \\ x_1^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & \vdots \\ x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{vmatrix}}$$

Macdonald polynomials.

Let $q, t \in (0,1)$.

$$P_\lambda(x) = P_\lambda(x; q, t) = m_\lambda + \sum_{\mu < \lambda} R_{\lambda\mu}(q,t) P_\mu(x),$$

defined by the orthogonality that $\langle P_\lambda, P_\mu \rangle = 0$ if $\lambda \neq \mu$.

The inner product is defined by

$$\langle P_\lambda, P_\mu \rangle = \langle P_\lambda, P_\mu \rangle_{q,t} = \delta_{\lambda\mu}\, z_\lambda(q,t)$$

where

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$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$, $z_\lambda = \prod_{i\geq 1} i^{m_i}(m_i)!$ $(\lambda = 1^{m_1} 2^{m_2} \cdots)$.

also define

$$Q_\lambda = \frac{P_\lambda}{\langle P_\lambda, P_\lambda \rangle}.$$

Special cases of Macdonald polynomials

$t = q^2$ and $q \to 1$: Zonal polynomials.

$q = t$: Schur polynomials.

note that when $q = t$, $\langle P_\lambda, P_\lambda \rangle = 1$ and then $s_\lambda = P_\lambda = Q_\lambda$.

Skew Macdonald symmetric polynomials

$P_{\lambda/\mu}(x) = \sum_\nu f^\lambda_{\mu\nu} P_\nu(x)$, so that $\langle P_{\lambda/\mu}, Q_\nu \rangle = \langle P_\lambda, Q_\mu Q_\nu \rangle$.

Similarly we define $Q_{\lambda/\mu}$ by the relation that

$\langle Q_{\lambda/\mu}, P_\nu \rangle = \langle Q_\lambda, P_\mu P_\nu \rangle$.

How to compute $P_{\lambda/\mu}(x)$ in the simplest case that

① $\mu \prec \lambda$ ($\lambda - \mu$ is a horizontal strip).

② $x = (x_1)$ (only one variable)?

Pieri's formula

$$Q_\mu Q_{(r)} = \sum_\lambda \varphi_{\lambda/\mu} Q_\lambda.$$

where λ is a partition such that $|\lambda| = |\mu| + r$ and $\mu \prec \lambda$.

and

$$\psi_{\lambda/\mu} = \prod_{1 \le i \le j \le \ell(\lambda)} \frac{f(q^{\lambda_i - \lambda_{j+1}} t^{j-i})\, f(q^{\mu_i - \mu_j} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i})\, f(q^{\mu_i - \lambda_{j+1}} t^{j-i})},$$

where

$$f(u) = \frac{(tu;q)_\infty}{(qu;q)_\infty} \quad \text{where } (a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots$$

Note that if $q = t$, then

$$f(u) = 1 \quad \text{and} \quad \varphi_{\lambda/\mu} = 1.$$

Thus if $\mu \prec \lambda$, we write $r = |\lambda| - |\mu|$, and then

$$P_{\lambda/\mu}(x_1) = f^{\lambda}_{\mu(r)} P_{(r)}(x_1) + \sum_{\ell(\nu) > 1} f^{\lambda}_{\mu\nu} P_\nu(x_1),$$

and

$$f^{\lambda}_{\mu(r)} = \langle P_\lambda, Q_\mu Q_{(r)} \rangle = \psi_{\lambda/\mu}.$$

The definition of P_ν implies that

$$P_\nu(x_1) = \begin{cases} x_1^{\,r} & \text{if } \nu = (r) \\ 0 & \text{if } \ell(\nu) > 1. \end{cases}$$

We conclude that

$$P_{\lambda/\mu}(x_1) = \begin{cases} x_1^{\,r} & \text{if } \mu \preceq \lambda \text{ and } |\lambda| - |\mu| = r \\ 0 & \text{otherwise.} \end{cases}$$

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3. Macdonald processes.

Given the two specialisations

$$x = (a_1, \dots, a_N), \qquad y = (b_1, \dots, b_M),$$

we define the Macdonald measure of partitions

$$(*) \quad MM(\lambda) = \frac{P_\lambda(a_1, \dots, a_N)\, Q_\lambda(b_1, \dots, b_M)}{\Pi(a_1, \dots, a_N ; b_1, \dots, b_M)}$$

where the normalisation constant is given by the Cauchy identity

$$\Pi(a_1, \dots, a_N ; b_1, \dots, b_M) = \prod_{\substack{i=1,\dots,N \\ j=1,\dots,M}} \frac{(t a_i b_j ; q)_\infty}{(a_i b_j ; q)_\infty}$$

$$= \prod_{\substack{i=1,\dots,N \\ j=1,\dots,M}} (1 - a_i b_j) \qquad \text{if } q = t.$$

It is more interesting to consider the growth process of partitions $0 \preceq \lambda^{(1)} \preceq \lambda^{(2)} \preceq \cdots \preceq \lambda^{(N)}$ with the measure of Macdonald ascending measure

$$M(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(N)}) = \frac{1}{C} P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)$$

$$\times Q_{\lambda^{(N)}}(b_1, \dots, b_M).$$

It turns out that $C = \Pi(a_1, \dots, a_N : b_1, \dots, b_M)$ and the

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marginal distribution of $\lambda^{(N)}$ is given by $(*)$.