1. q-releted formulas.

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots\cdots,$$

$$n_q! = \frac{(q;q)_n}{(1-q)^n} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q)\cdots(1-q)},$$

2. q-Whittaker functions.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{\ell+1})$. The Macdonald polynomial $P_\lambda(x_1, x_2, \cdots, x_{\ell+1}; q, t)$, as $t \to 0$, is the "q-deformed $gl_{\ell+1}$ Whittaker function", with the notational convention

$$\lim_{t\to 0} P_\lambda(x_1, \cdots, x_{\ell+1}; q, t) = \Psi_{x_1, \cdots, x_{\ell+1}}(\lambda_1, \cdots, \lambda_{\ell+1}).$$

Another definition of q-Whittaker functions:

$$\Psi_{x_1,\cdots,x_{\ell+1}}(\lambda^{(\ell+1)}) = \sum_{\substack{\lambda^{(\ell)}, \lambda^{(\ell-1)}, \cdots, \lambda^{(1)} \\ \text{in a G-T pattern}}} \prod_{k=1}^{\ell+1} x_k^{\sum_{i=1}^{k} \lambda_i^{(k)} - \sum_{i=1}^{k-1} \lambda_i^{(k-1)}}$$

$$\times \frac{\prod_{k=2}^{\ell} \prod_{i=1}^{k-1} (q;q)_{\lambda_i^{(k)} - \lambda_{i+1}^{(k)}}}{\prod_{k=1}^{\ell} \prod_{i=1}^{k} (q;q)_{\lambda_i^{(k+1)} - \lambda_i^{(k)}} (q;q)_{\lambda_i^{(k)} - \lambda_{i+1}^{(k+1)}}},$$

where $\lambda^{(k)} = (\lambda^{(k)}_1, \cdots, \lambda^{(k)}_k)$ $(k = 1, 2, \cdots, \ell, \ell+1)$ are in a Gelfand-Tsetlin pattern, that is,

$$\begin{array}{ccccccccc} \lambda^{(k+1)}_1 & & \lambda^{(k+1)}_2 & & \cdots & & & & \lambda^{(k+1)}_{k+1}. \\ & \geqq \lambda^{(k)}_1 \geqq & & \geqq \lambda^{(k)}_2 \geqq & & & \geqq \lambda^{(k)}_k \geqq & & \end{array}$$

Example: $\ell = 2$, $\lambda = \lambda^{(3)} = (2, 1, 1)$. There are only three Gelfand-Tsetlin patterns

$$\begin{array}{ccccc} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 2 & & \end{array} \qquad \begin{array}{ccccc} 2 & & 1 & & 1 \\ & 2 & & 1 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 2 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array}$$

Fix $q \in (0, 1)$ and let $t \to 0$, the Macdonald process and the Macdonald measure are still well defined, and are called the q-Whittaker process/measure.

3. Average of some functions on Macdonald measure and moments on q-Whittaker measure.

First consider the Macdonald measure with general q, t.

Suppose D is an operator on the space of symmetric polynomials such that

$$D P_\lambda = d_\lambda P_\lambda.$$

Recall the normalization constant (partition function)

$$\Pi(a_1, \cdots, a_N; b_1, \cdots b_M) = \sum_\lambda P_\lambda(a_1, \cdots, a_N) Q_\lambda(b_1, \cdots b_M).$$

We have

$$\langle d_\lambda \rangle_{MM(a_1, \cdots a_N; b_1, \cdots b_M)} = \frac{D\,\Pi(a_1, \cdots a_N; b_1, \cdots b_M)}{\Pi(a_1, \cdots a_N; b_1, \cdots, b_M)},$$

$$\langle d_\lambda^k \rangle_{MM(a_1, \cdots a_N; b_1, \cdots b_M)} = \frac{D^k\,\Pi(a_1, \cdots a_N; b_1, \cdots b_M)}{\Pi(a_1, \cdots a_N; b_1, \cdots b_M)}.$$

In particular, let D_n^1 be defined as

$$D_n^1 P_\lambda(x_1, \cdots, x_n) = e_1(q^{\lambda_1} t^{n-1}, q^{\lambda_2} t^{n-2}, \cdots, q^{\lambda_n}) P_\lambda(x_1, \cdots, x_n)$$

where

$$e_1(x_1, \cdots, x_n) = x_1 + x_2 + \cdots + x_n.$$

Then a theorem by Macdonald gives that for any

$$F(x) = F(x_1, \cdots, x_n) = f(x_1) \cdots f(x_n), \quad f \text{ holomorphic and } \neq 0.$$

$$(*)\quad \frac{((D_n^1)^k F)(x)}{F(x)} = \frac{(t-1)^k}{(2\pi i)^k} \oint \cdots \oint \prod_{1\le i<j\le k} \frac{(tz_i - qz_j)(z_i - z_j)}{(z_i - qz_j)(tz_i - z_j)}$$
$$\times \prod_{j=1}^{k} \left(\prod_{m=1}^{n} \frac{tz_j - x_m}{z_j - x_m} \right) \frac{f(qz_j)}{f(z_j)} \frac{dz_j}{z_j},$$

where the z_j contour contains $\{qz_{j+1}, \cdots, qz_k, x_1, \cdots x_k\}$ but no other singularities for all $j=1,\cdots k$.

Note that the partition function $\Pi(a_1,\cdots,a_N; b_1,\cdots,b_M)$ is in the form of $F(x)$ as a function of $a_1,\cdots,a_N$. It is expressed as

$$\Pi = \prod_{i=1}^{N} \underbrace{\prod_{j=1}^{M} \frac{(ta_ib_j; q)_\infty}{(a_ib_j; q)_\infty}}_{f(a_i)}.$$

So the average of $[e_1(q^{\lambda_1}t^{n-1}, \cdots, q^{\lambda_n})]^k$ can be evaluated in the Macdonald measure.

In the limit $t\to 0$, we have that

$$e_1(q^{\lambda_1}t^{n-1}, \cdots, q^{\lambda_n}) = q^{\lambda_n} + O(t).$$

So as a specialization of $(*)$,

$$\langle q^{k\lambda_N} \rangle_{MM_{t=0}(a_1,\cdots,a_N; b_1,\cdots,b_M)} = \frac{(D_N^1)^k\, \Pi(a_1,\cdots,a_N; b_1,\cdots,b_M)}{\Pi(a_1,\cdots,a_N; b_1,\cdots,b_M)}$$

$$(+) = \frac{(-1)^k}{(2\pi i)^k} q^{\frac{k(k-1)}{2}} \oint \cdots \oint \prod_{1 \le i < j \le k} \frac{z_i - z_j}{z_i - q z_j} \prod_{j=1}^{k} \left(\prod_{m=1}^{N} \frac{a_m}{a_m - z_j} \right) \frac{f(q z_j)}{f(z_j)} \frac{dz_j}{z_j}.$$

Now we consider a special choice of $b_1, \ldots b_M$. The <u>Plancherel</u> specialization is that

$$Q_{(n)}(b_1, \ldots, b_M) = \frac{\gamma^n}{n!}, \qquad \gamma > 0.$$

(Actually, no finite number of b_i's can make the identity above hold for all n. We need to take $M = \infty$. But later we do not need the specific values of b_i anyway.)

The Plancherel specialization implies that (represent $b_1, b_2, \cdots$ by ρ)

$$\prod(a_1, \ldots, a_N; \rho) = \prod_{j=1}^{N} e^{\gamma a_j},$$

and then the term $f(q z_j) / f(z_j)$ in (+) becomes

$$\frac{f(q z_j)}{f(z_j)} = e^{(q-1)\gamma z_j}.$$

4. Fredholm determinant.

Let X be an integral domain (like $[0,1]$ in $\int_0^1 x^2 dx$), and $K(x,y)$ be an integral operator on X. The <u>Fredholm determinant</u> of $I+K$ is, formally without consideration of convergence,

$$\det(I+K) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \cdots \int_X \det[K(x_i, x_j)]_{i,j=1}^{n} \, dx_1 \cdots dx_n.$$

Here X can be a contour in the complex plane.

Technical lemma: Let

$$\mu_k = \frac{(-1)^k}{(2\pi i)^k} q^{\frac{k(k-1)}{2}} \oint \cdots \oint \prod_{1 \le i < j \le k} \frac{z_i - z_j}{z_i - q z_j} \prod_{j=1}^{k} \frac{g(z_j)}{z_j} \, dz_1 \cdots dz_k,$$

where $g(z)$ is a meromorphic function with poles $A = \{a_1, \dots, a_N\}$ such that $q^m A$ is disjoint from A for all $m \geq 1$, and the z_i contour contains $\{q z_j\}_{j>i} \cup A$, but not 0. Then

$$\sum_{k \geq 0} \mu_k \frac{\zeta^k}{k_q!} = \det(I+K) \qquad \text{formally},$$

where K is defined on the domain $\mathbb{Z}_{>0} \times C_w$, with

$$K(n_1, w_1; n_2, w_2) = \frac{(1-q)^{n_1} \zeta^{n_1} g(w_1) g(q w_1) \cdots g(q^{n_1 - 1} w_1)}{q^{n_1} w_1 - w_2},$$

and the contour C_w satisfies that it encloses all points in A but no other poles.

The proof of the lemma and the condition that the "formally true" identity is meaningful analytically is omitted.

Take

$$g(z) := \left(\prod_{m=1}^{N} \frac{a_m}{a_m - z} \right) e^{(q-1)\gamma z},$$

and use the identity that

$$\frac{1}{((1-q)x ; q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{k_q!},$$

we obtain from (+) and the technical lemma that on the q-Whittaker measure.

$$\left\langle \frac{1}{((1-q)\zeta q^{\lambda_N} ; q)_\infty} \right\rangle_{MM} = \det(1+K).$$

(A simple change of variable can remove $(1-q)$ both on the left-hand side of the formula above and in the formula of K.)