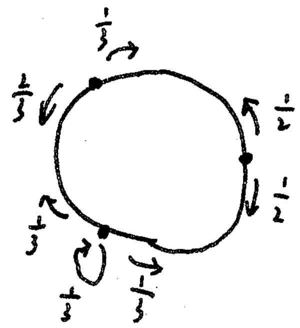
1. A general construction of multivariate Markov chains.

Let $(S_1, \dots, S_n)$ be an $n$-tuple discrete sets, and $P_1, \dots, P_n$ be stochastic matrices defining Markov chains $S_k \to S_k$.

( e.g. $S = \{e^0, e^{2\pi i/3}, e^{4\pi i/3}\}$.

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$



)

Let $\Lambda_1^2, \dots, \Lambda_{n-1}^n$ be stochastic links between these sets:

$$\Lambda_{k-1}^k : S_k \times S_{k-1} \to [0, 1], \qquad \sum_{y \in S_{k-1}} \Lambda_{k-1}^k (x, y) = 1.$$

Assume that they satisfy the commutation relations

$$\Delta_{k-1}^k = \Lambda_{k-1}^k P_{k-1} = P_k \Lambda_{k-1}^k .$$

or equivalently,

$$\Delta_{k-1}^k (x, y) = \sum_{z \in S_{k-1}} \Lambda_{k-1}^k (x, z) P_{k-1} (z, y) = \sum_{w \in S_k} P_k(x, w) \Lambda_{k-1}^k (w, y).$$

Then define the multivariate Markov chain on

$$S^{(n)} = \{(x_1, \dots, x_n) \in S_1 \times \dots \times S_n \mid \prod_{k=2}^n \Lambda_{k-1}^k (x_k, x_{k-1}) \neq 0\}$$

with the transition matrix $P^{(n)}$ on $S^{(n)}$ as

$$P^{(n)}(X_n, Y_n) = P_1 (x_1, y_1) \prod_{k=2}^n \frac{P_k(x_k, y_k) \Lambda_{k-1}^k (y_k, y_{k-1})}{\Delta_{k-1}^k (X_k, y_{k-1})}$$

We understand the dynamics on $S^{(n)}$ as a sequential update from $S_1$ to $S_n$. First, $x_1$ moves to $y_1$ with transition probability $P_1(x_1, y_1)$; after $x_1, \dots, x_{k-1}$ having moved to $y_1, \dots, y_{k-1}$, $x_k$ moves to $y_k$ under the condition that $\Lambda^k_{k-1}(y_k, y_{k-1}) \neq 0$, with transition probability const. $P_k(x_k, y_k)\, \Lambda^k_{k-1}(y_k, y_{k-1})$.

Proposition: Let $m_n$ be a probability measure on $S_n$. Let $m^{(n)}$ be a probability measure on $S^{(n)}$ defined by

$$m^{(n)}(X_n) = m_n(x_n)\, \Lambda^n_{n-1}(x_n, x_{n-1}) \cdots \Lambda^2_1(x_2, x_1).$$

Set $\tilde{m}_n = m_n P_n$ ($m_n$ evolves by one step) and let

$$\tilde{m}^{(n)}(X_n) = \tilde{m}_n(x_n)\, \Lambda^n_{n-1}(x_n, x_{n-1}) \cdots \Lambda^2_1(x_2, x_1)$$

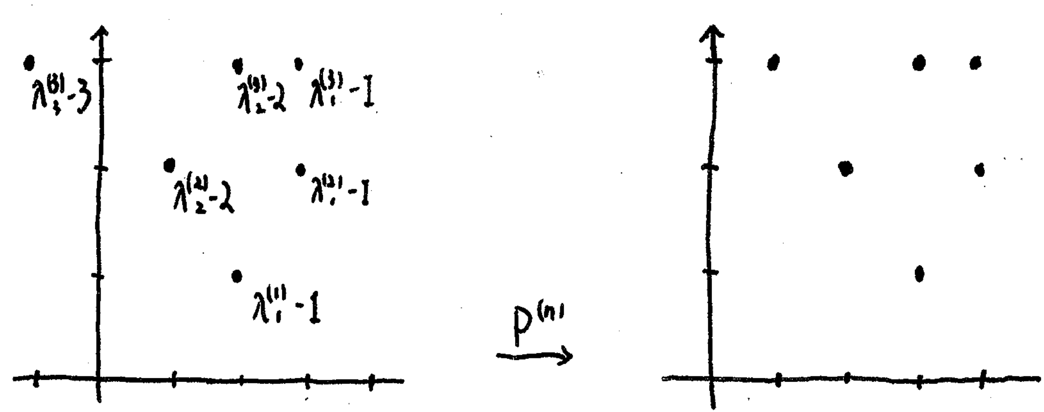
Then $m^{(n)} P^{(n)} = \tilde{m}^{(n)}$.

Rudimentary example:

$$S_k : Y(k) = \{\text{Young tableaux with } k \text{ rows}\}$$
$$= \{(\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}$$

$$\Lambda^k_{k-1}((\lambda_1, \dots, \lambda_k), (\mu_1, \dots, \mu_{k-1})) = \begin{cases} 0 & \text{if } \mu \not\preceq \lambda, \text{ i.e. } \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n \\ & \text{is not satisfied.} \\ \dfrac{1}{\#\{\nu \in Y(k-1) \mid \nu \preceq \lambda\}} & \text{if } \mu \preceq \lambda \end{cases}$$

Dynamics:



2. Multivariate Markov chain defined by Macdonald polynomials

Let $\lambda, \mu \in \mathbb{Y}(k)$ and $\nu \in \mathbb{Y}(k-1)$. Define for any $a_1, \dots a_k, b$.

$$P^{\uparrow}_{\lambda\mu}(a_1, \dots, a_k ; b) = \frac{1}{\Pi(a_1, \dots, a_k ; b)} \frac{P_\mu(a_1, \dots a_k)}{P_\lambda(a_1, \dots a_k)} Q_{\mu/\lambda}(b).$$

$$P^{\downarrow}_{\lambda\nu}(a_1, \dots a_k) = \frac{P_\nu(a_1, \dots a_{k-1})}{P_\lambda(a_1, \dots a_k)} P_{\lambda/\nu}(a_k).$$

Then define the $\infty \times \infty$ matrices

$$P^{\uparrow}(a_1, \dots a_k ; b) = [P^{\uparrow}_{\lambda\mu}(a_1, \dots a_k ; b)]_{\lambda, \mu},$$
$$P^{\downarrow}(a_1, \dots a_k) = [P_{\lambda\nu}(a_1, \dots a_k)]_{\lambda, \nu}.$$

which are stochastic:

$$\sum_{\mu \in \mathbb{Y}(k)} P^{\uparrow}_{\lambda\mu}(a_1, \dots a_k ; b) = 1, \quad \sum_{\nu \in \mathbb{Y}(k-1)} P^{\downarrow}_{\lambda\nu}(a_1, \dots a_k) = 1.$$

Thus $[P^{\uparrow}_{\lambda\mu}]$ defines a transition matrix for $\mathbb{Y}(k)$, and $[P^{\downarrow}_{\lambda\nu}]$ defines a stochastic link between $\mathbb{Y}(k)$ and $\mathbb{Y}(k-1)$.

We can check that they satisfy the commutation relation

$$P^{\uparrow}(a_1, \dots a_k ; b) P^{\downarrow}(a_1, \dots a_k) = P^{\downarrow}(a_1, \dots a_k) P^{\uparrow}(a_1, \dots a_{k-1} ; b),$$

or more explicitly

$$\sum_{\mu \in \mathbb{Y}(k)} P^{\uparrow}_{\lambda\mu}(a_1, \dots a_k ; b) P^{\downarrow}_{\mu\nu}(a_1, \dots a_k) = \sum_{\mu \in \mathbb{Y}(k)} \frac{1}{\Pi(a_1, \dots a_k ; b)} \frac{P_\nu(a_1, \dots a_{k-1})}{P_\lambda(a_1, \dots a_k)} Q_{\mu/\lambda}(b) P_{\mu/\nu}(a_k)$$

$$\sum_{\mu \in \mathbb{Y}(k-1)} P^{\downarrow}_{\lambda\mu}(a_1, \dots a_k) P^{\uparrow}_{\mu\nu}(a_1, \dots a_{k-1} ; b) = \sum_{\mu \in \mathbb{Y}(k-1)} \frac{1}{\Pi(a_1, \dots a_{k-1}, b)} \frac{P_\nu(a_1, \dots a_{k-1})}{P_\lambda(a_1, \dots a_k)} Q_{\nu/\mu}(b) P_{\lambda/\mu}(a_k)$$

(It is equivalent to check $\dfrac{\sum_{\mu \in \mathbb{Y}(k)} Q_{\mu/\lambda}(b) P_{\mu/\nu}(a_k)}{\sum_{\mu \in \mathbb{Y}(k-1)} Q_{\mu/\nu}(b) P_{\lambda/\mu}(a_k)} = \dfrac{\Pi(a_1, \dots a_{k-1} ; b)}{\Pi(a_1, \dots a_k ; b)}$ )

Now we define $S_k = \mathbb{Y}(k)$, $P_k = P^{\uparrow}(a_1, \dots, a_k ; b)$, $\Lambda^k_{k-1} = P^{\downarrow}(a_1, \dots, a_k)$.

Then $S^{(n)}$ is the set of Gelfond-Tsetlin patterns

$$S^{(n)} = \{(\lambda^{(1)}, \dots, \lambda^{(n)}) \mid \lambda^{(1)} \preceq \lambda^{(2)} \preceq \cdots \preceq \lambda^{(n)}\}.$$

and the transition matrix $P^{(n)}$ is

$$P^{(n)}((\lambda^{(1)}, \dots, \lambda^{(n)}), (\mu^{(1)}, \dots, \mu^{(n)})) = P^{\uparrow}_{\lambda^{(1)}\mu^{(1)}}(a_1 ; b) \prod_{k=2}^{n} \frac{P^{\uparrow}_{\lambda^{(k)}\mu^{(k)}}(a_1, \dots, a_k ; b)\, P^{\downarrow}_{\mu^{(k)}\mu^{(k-1)}}(a_1, \dots, a_k)}{\sum_{\nu \in \mathbb{Y}(k)} P^{\uparrow}_{\lambda^{(k)}\nu}(a_1, \dots, a_k ; b)\, P^{\downarrow}_{\nu, \mu^{(k-1)}}(a_1, \dots, a_k)}$$

$$= \prod_{k=1}^{n} P_{a_k, b}(\mu^{(k)} \,\|\, \mu^{(k-1)}, \lambda^{(k)}).$$

where

$$P_{a_k, b}(\nu \| \lambda, \mu) = \begin{cases} P^{\uparrow}_{\mu\nu}(a_1 ; b) = \text{const} \cdot P_\nu(a_k)\, Q_\nu(b) & \text{if } k = 1 \\ \dfrac{P^{\uparrow}_{\mu\nu}(a_1, \dots, a_k ; b)\, P^{\downarrow}_{\nu\lambda}(a_1, \dots, a_k)}{\sum_{\kappa \in \mathbb{Y}(k)} P^{\uparrow}_{\mu\kappa}(a_1, \dots, a_k ; b)\, P^{\downarrow}_{\kappa\lambda}(a_1, \dots, a_k)} = \text{const} \cdot P_{\nu/\lambda}(a_k)\, Q_{\nu/\mu}(b) & \text{otherwise}. \end{cases}$$

Furthermore, let $m_n$ be the Macdonald measure on $S_n = \mathbb{Y}(n)$:

$$m_n(\lambda^{(n)}) = P_{\lambda^{(n)}}(a_1, \dots, a_n)\, Q_{\lambda^{(n)}}(b_1, \dots, b_m) \big/ \Pi(a_1, \dots, a_n ; b_1, \dots, b_m).$$

We have that

$$\tilde{m}_n(\lambda^{(n)}) = \sum_{\mu \in \mathbb{Y}(n)} m_n(\mu)\, P^{\uparrow}_{\mu\lambda^{(n)}}(a_1, \dots, a_k ; b),$$

$$= P_{\lambda^{(n)}}(a_1, \dots, a_n)\, Q_{\lambda^{(n)}}(b_1, \dots, b_m, b) \big/ \Pi(a_1, \dots, a_n ; b_1, \dots, b_m, b).$$

is also a Macdonald measure, with ~~the~~ more $b$-parameter. We can also check that

$$m^{(n)}(\lambda^{(1)}, \dots, \lambda^{(n)}) = P_{\lambda^{(1)}}(a_1)\, P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots P_{\lambda^{(n)}/\lambda^{(n-1)}}(a_n)\, Q(b_1, \dots, b_m) / \Pi(a_1, \dots, a_n ; b_1, \dots, b_m)$$

So by the Proposition,

$$\tilde{m}^{(n)}(\lambda^{(1)}, \ldots, \lambda^{(n)}) = m^{(n)} p^{(n)}$$

$$= P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots P_{\lambda^{(n)}/\lambda^{(n-1)}}(a_n) Q(b_1, \ldots b_m, b) \Big/ \Pi(a_1, \ldots a_n; b_1, \ldots b_m, b)$$

is again the Macdonald process.

3. Probabilistic meaning of a simple model.

Consider the simplest case of the Macdonald process: $q = t$, so both $P_\lambda$ and $Q_\lambda$ are the Schur polynomial $s_\lambda$. Let $a_1 = \cdots = a_n = 1$ and $b = p \in (0,1)$. Note that

$$S_{\lambda/\mu}(x) = \begin{cases} 0 & \text{if } \mu \not\preceq \lambda, \\ x^{|\lambda| - |\mu|} & \text{if } \mu \preceq \lambda. \end{cases}$$

The movement of the Gelfand-Tsetlin pattern $(\lambda^{(1)}, \ldots, \lambda^{(n)})$ to $(\mu^{(1)}, \ldots, \mu^{(n)})$ can be described as follows, equivalent to $p^{(n)}$:

First, $\lambda^{(1)}_1$ jumps right to $\mu^{(1)}_1 \geq \lambda^{(1)}_1$, with probability $\text{const} \cdot p^{\mu^{(1)}_1 - \lambda^{(1)}_1}$, where $\text{const} = 1 - p$.

Second, $\lambda^{(2)}_1$ jumps right to $\mu^{(2)}_1 \geq \max(\lambda^{(2)}_1, \mu^{(1)}_1)$ with probability $\text{const} \cdot p^{\mu^{(2)}_1 - \lambda^{(2)}_1}$, and $\lambda^{(2)}_2$ jumps right to $\mu^{(2)}_2$ that satisfies $\lambda^{(2)}_2 \leq \mu^{(2)}_2 \leq \mu^{(1)}_1$, with probability $\text{const} \cdot p^{\mu^{(2)}_2 - \lambda^{(2)}_2}$.

Here the two constants depend on $\lambda^{(2)}, \mu^{(1)}$ but not $\mu^{(2)}$.

Third. $\lambda_1^{(3)}$ jumps right to $\mu_1^{(3)} \geq \max(\lambda_1^{(3)}, \mu_1^{(2)})$,

$\lambda_2^{(3)}$ jumps right to $\mu_2^{(3)}$ such that $\max(\lambda_2^{(3)}, \mu_2^{(2)}) \leq \mu_2^{(3)} \leq \mu_1^{(2)}$

$\lambda_3^{(3)}$ jumps right to $\mu_3^{(3)}$ such that $\lambda_3^{(3)} \leq \mu_3^{(3)} \leq \mu_2^{(2)}$.

with probabilities $\text{const} \cdot p^{\mu_i^{(3)} - \lambda_i^{(3)}}$, $i = 1, 2, 3$. where the constants depend on $\lambda^{(3)}$ and $\mu^{(2)}$ but not $\mu^{(3)}$

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It is clear that the movement of $\lambda_1^{(1)}, \lambda_2^{(2)}, \cdots, \lambda_n^{(n)}$ is independent to other entries in the Gelfond-Tsetlin pattern. Suppose $x_k = \lambda_k^{(k)} - k$ are positions of particles on $\mathbb{Z}$. $x_1 > x_2 > \cdots > x_n$. Then their movement, in one step, is:

First. $x_1$ jumps to the right $k_1$ units in probability $(1-p)p^{k_1}$. ($k_1 = 0, 1, 2, \cdots$).

Second. $x_2$ jumps to the right $k_2$ units in probability $\frac{1-p}{1-p^{x_1+k_1-x_2}} p^{k_2}$. ($k_2 = 0, 1, \cdots, x_1 + k_1 - x_2 - 1$)

Third. $x_3$ jumps to the right $k_3$ units in probability $\frac{1-p}{1-p^{x_2+k_2-x_3}} p^{k_3}$. ($k_3 = 0, 1, \cdots, x_2 + k_2 - x_3 - 1$)

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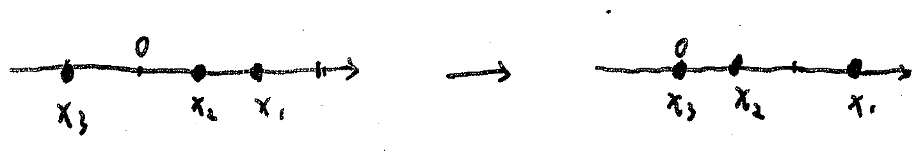
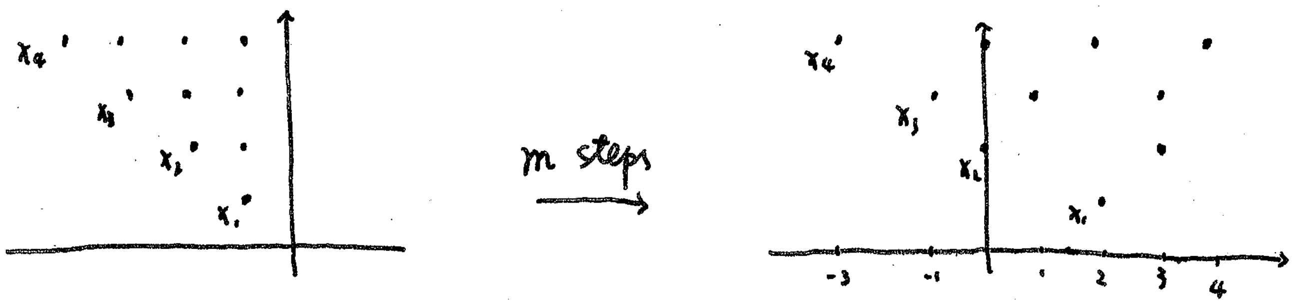



Question: if $x_k = -k$ ($k = 1, \cdots, n$) initially, what's the distribution of $x_n$ after $m$ steps?

Answer: the same as the distribution of $\lambda^{(n)}_n - n$, where $\lambda^{(n)} = (\lambda^{(n)}_1, \dots, \lambda^{(n)}_n)$ is in the Macdonald measure

$$MM(\underbrace{1, \dots, 1}_{n} ; \underbrace{p, \dots, p}_{m}) \qquad \text{(actually Schur measure)}$$




Initially, $(\lambda^{(1)}, \dots, \lambda^{(n)})$ are frozen to $\lambda^{(k)} = (\underbrace{0, \dots, 0}_{k})$, since they are in the distribution of the Macdonald process with $a_1 = \dots = a_n = 1$, $b = 0$.

After $m$ steps, $(\lambda^{(1)}, \dots, \lambda^{(n)})$ are in the distribution of the Macdonald process with $a_1 = \dots = a_n = 1$, $b_1 = \dots = b_m = p$ (after each step, one $p$ is added). (actually Schur process)

(The figures above show $\lambda^{(k)}_j - j$ to make dots not overlapped.)

4. Continuous limits.

It is straight forward to get the continuous limit of the particle model considered above, as $p \to 0_+$.

Each particle $x_k$ has an exponential clock that is working as long as its right neighbour site is not occupied by $x_{k-1}$, otherwise

the exponential clock is paused until $x_{k-1}$ moves away. When the clock clicks, $x_k$ moves to the right by one unit.
This is the celebrated totally asymmetric simple exclusive process (TASEP).
The distribution of $x_n$ after time $t$ is given by $\lambda^{(n)}_n - n$ where $\lambda^{(n)}$ is in the Macdonald / Schur measure $MM(\underbrace{1, \ldots, 1}_{n} ; \rho)$, and $\rho$ is the limiting Plancherel specialization depending on $t$.
A variation of TASEP: Each particle $x_k$ has an exponential clock that is slower if $x_{k-1}$ is near to it: The parameter is $(1 - q^{x_{k-1} - x_k - 1})$. (So if $x_{k-1}$ is at the right neighbour site to $x_k$, the exponential clock stops.) This is called the $q$-TASEP.
The distribution of $x_n$ after time $t$, suppose the initial condition is $x_k = -k$, is given again by $\lambda^{(n)}_n - n$, where $\lambda^{(n)}$ is in the Macdonald measure $MM(\underbrace{1, \ldots, 1}_{n} : \rho)$, but now the $t$ and $q$ parameters are $t \to 0$, $q = q$. This is the $q$-Whittaker measure.