Subject : Date:

1. Whittaker functions and Whittaker processes.

The class-one $gl_{\ell+1}$-Whittaker functions are extensively studied in representation theory. Here we use a formula by Givental to define them.

$$\psi_{\lambda_{\ell+1}}(x_{\ell+1}) = \int_{\mathbb{R}^{\ell(\ell+1)/2}} e^{F_\lambda(x)} \prod_{k=1}^{\ell}\prod_{i=1}^{k} dx_{k,i},$$

where $x_{\ell+1} := \{x_{\ell+1,1}, \dots, x_{\ell+1,\ell+1}\} \in \mathbb{R}^{\ell+1}$, $\lambda_{\ell+1} := \{\lambda_1, \dots, \lambda_{\ell+1}\} \in \mathbb{C}^{\ell+1}$, and ($x = (x_1, x_2, \dots, x_{\ell+1})$)

$$F_\lambda(x) := i\sum_{k=1}^{\ell+1} \lambda_k \left(\sum_{i=1}^{k} x_{k,i} - \sum_{i=1}^{k-1} x_{k-1,i} \right) - \sum_{k=1}^{\ell}\sum_{i=1}^{k} \left(e^{x_{k,i} - x_{k+1,i}} + e^{x_{k+1,i+1} - x_{k,i}} \right).$$

Remark: There is no order among $\{x_{k,i}\}$ or $\{\lambda_i\}$, and there is no interlacing between $\{x_{k,i}\}$ and $\{x_{k-1,i}\}$.

The Whittaker functions satisfy the orthogonal relations

$$\int_{\mathbb{R}^{\ell+1}} \overline{\psi_{\lambda_{\ell+1}}(x_{\ell+1})}\, \psi_{\mu_{\ell+1}}(x_{\ell+1})\, dx_{\ell+1} = \frac{1}{(\ell+1)!\, m_{\ell+1}(\lambda_{\ell+1})} \sum_{\sigma \in S_{\ell+1}} \delta(\lambda_{\ell+1} - \sigma(\mu_{\ell+1})),$$

$$\int_{\mathbb{R}^{\ell+1}} \overline{\psi_{\lambda_{\ell+1}}(x_{\ell+1})}\, \psi_{\lambda_{\ell+1}}(y_{\ell+1})\, m(\lambda_{\ell+1})\, d\lambda_{\ell+1} = \delta(x_{\ell+1} - y_{\ell+1}).$$

where $m_{\ell+1}(\lambda_{\ell+1})$ is the Skylanin measure

$$m_{\ell+1}(\lambda_{\ell+1}) := \frac{1}{(2\pi)^{\ell+1}(\ell+1)!} \prod_{j \neq k} \frac{1}{\Gamma(i\lambda_k - i\lambda_j)}.$$

The Whittaker functions are the limits of the q-Whittaker

Subject : Date:

functions as $q \to 1_-$, with proper scaling. Recall the q-Whittaker functions

$$\Psi_{z_{\ell+1}}(p_{\ell+1}) = \lim_{t \to 0} P_{p_{\ell+1}}(z_1, \ldots, z_{\ell+1}; q, t).$$

Let

$$q = e^{-\varepsilon}, \quad z_k = e^{i\varepsilon \nu_k}, \quad p_{\ell+1,k} = (\ell+2-2k)\, m(\varepsilon) + \varepsilon^{-1} x_{\ell+1,k}.$$

$$m(\varepsilon) = -[\varepsilon^{-1} \log \varepsilon]. \qquad A(\varepsilon) = -\frac{\pi^2}{6}\frac{1}{\varepsilon} - \frac{1}{2}\log\frac{\varepsilon}{2\pi}.$$

Then as $\varepsilon \to 0$,

$$\varepsilon^{\ell(\ell+1)/2}\, e^{\frac{\ell(\ell+3)}{2} A(\varepsilon)}\, \Psi_{z_{\ell+1}}(p_{\ell+1}) \to \psi_{\nu_{\ell+1}}(x_{\ell+1}).$$

Let $\tau > 0$ be a constant. The change of variables

$$q = e^{-\varepsilon}, \quad z_k = e^{-\varepsilon a_k}, \quad p_{\ell+1,k} = \tau \varepsilon^{-2} + (\ell+2-2k)\, m(\varepsilon) + \varepsilon^{-1} T_{\ell+1,k}$$

yields

$$\varepsilon^{\ell(\ell+1)/2}\, e^{\frac{\ell(\ell+3)}{2} A(\varepsilon) + \tau(\sum_{k=1}^{\ell+1} a_k)\varepsilon^{-1}}\, \Psi_{z_{\ell+1}}(p_{\ell+1}) \to \psi_{i a_{\ell+1}}(T_{\ell+1}).$$

Recall the q-Whittaker measure with Plancherel specialization

$$MM_{(z_1, \ldots, z_{\ell+1}; \rho; q, t=0)}(p_{\ell+1}) = \frac{1}{\Pi(z_1, \ldots, z_{\ell+1}; \rho)}$$

$$\times \underbrace{P_{p_{\ell+1}}(z_1, \ldots, z_{\ell+1}; q, t=0)}_{\Psi_{z_{\ell+1}}(p_{\ell+1})}\, Q_{p_{\ell+1}}(\rho; q, t=0).$$

Subject : Date:

By the same change of variables,

$$e^{-(\ell+1)\tau\varepsilon^{-2}+\tau(\sum_{k=1}^{\ell+1}a_k)\varepsilon^{-1}}\Pi(z_1,\cdots,z_{\ell+1};\rho)\to e^{\tau\sum_{k=1}^{\ell+1}a_k^2/2}.$$

$$\varepsilon^{-(\ell+1)(\ell+2)/2}\,e^{-\frac{\ell(\ell-1)}{2}A(\varepsilon)-(\ell+1)\tau\varepsilon^{-1}}Q_{P_{\ell+1}}(\rho;q,t=0)\to\theta_\tau(T_{\ell+1}),$$

where

$$\theta_\tau(T_{\ell+1})=\int_{\mathbb{R}^{\ell+1}}\psi_{\lambda_{\ell+1}}(T_{\ell+1})\,e^{-\tau\sum_{k=1}^{\ell+1}\lambda_{\ell+1,k}^2/2}\,m_{\ell+1}(\lambda_{\ell+1})\,d\lambda_{\ell+1}.$$

Note that in the Plancherel specialisation ρ, there is a scalar parameter γ. In formulas above, we take $\gamma=\tau\varepsilon^{-2}$.

The conclusion is that by the limit above, the q-Whittaker measure becomes the Whittaker measure

$$WM_{(a;\tau)}(T_{\ell+1})=e^{-\tau\sum_{k=1}^{\ell+1}a_k^2/2}\,\psi_{ia}(T_{\ell+1})\,\theta_\tau(T_{\ell+1}).$$

More generally, the q-Whittaker process becomes the Whittaker process

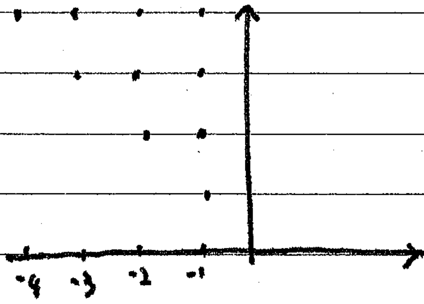
$$W_{(a;\tau)}(T_1,T_2,\cdots,T_{\ell+1})=e^{-\tau\sum_{k=1}^{\ell+1}a_k^2/2}\exp(F_{ia}(T))\,\theta_\tau(T_{\ell+1}).$$

2. Probability meaning of Whittaker processes.

As explained in last talk, the q-Whittaker processes with Plancherel specialisations are evolved from the

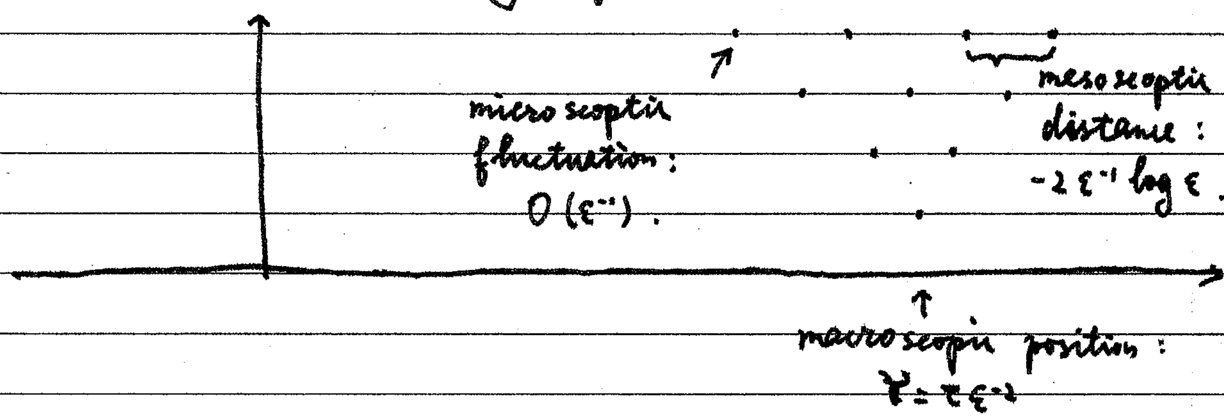
Subject : Date:

"frozen" initial state by a continuous time Markov process, where the parameter ν is the time.



$\nu = 0$. all $\lambda_j^{(k)}$ (in our notation, $P_{k,j}$) are in the smallest possible values. Dots are $\lambda_j^{(k)} - j$.

$\Downarrow$ after time $\nu = \tau \varepsilon^{-2}$.



The rescaled variables $T_{\ell+1,1}, \cdots, T_{\ell+1,\ell+1}, \cdots T_{k,j}, \cdots$ are the fluctuations of the globally fixed particles in the 2d-q-Whittaker growth model.

The Markov process of the evolution of the 2d-q-Whittaker growth model has a Whittaker-free description: Each of the coordinates $\lambda_j^{(k)}$ has its own independent exponential clock with rate

$$Z_m \frac{\left(1-q^{\lambda^{(m-1)}_{k-1}-\lambda^{(m)}_{k}}\right)\left(1-q^{\lambda^{(m)}_{k}-\lambda^{(m)}_{k+1}+1}\right)}{1-q^{\lambda^{(m)}_{k}-\lambda^{(m-1)}_{k}+1}}.$$

When the $\lambda^{(B)}_A$-clock rings, we find the longest string $\lambda^{(B)}_A = \lambda^{(B+1)}_A = \cdots = \lambda^{(B+C)}_A$ and move all the coordinates in this string to the right by one. Observe that if $\lambda^{(B)}_A = \lambda^{(B-1)}_{A-1}$ the the jump rate automatically vanishes.

As $\varepsilon \to 0$, by "standard methods of stochastic analysis", the evolution, in variables $T_{k,j}$ and time τ, becomes recursively

$$dT_{k,1} = dW_{k,1} + (a_k + e^{T_{k-1,1}-T_{k,1}})\,dt$$

$$dT_{k,2} = dW_{k,2} + (a_k + e^{T_{k-1,2}-T_{k,2}} - e^{T_{k,2}-T_{k-1,1}})\,dt.$$

$$\cdots$$

$$dT_{k,k-1} = dW_{k,k-1} + (a_k + e^{T_{k-1,k-1}-T_{k,k-1}} - e^{T_{k,k-1}-T_{k-1,k-2}})\,dt.$$

$$dT_{k,k} = dW_{k,k} + (a_k - e^{T_{k,k}-T_{k-1,k-1}})\,dt.$$

where $\{W_{k,j}\}$ are standard 1d Brownian motions.

Finally, we remark that all the contour integral formulas and Fredholm determinant formulas are

Subject : Date:

inherited by the Whittaker processes from the q-Whittaker ones. For example,

$$\left\langle e^{-k T_{N,N}} \right\rangle_{WM(a_1,\dots,a_N;\tau)} = \frac{(-1)^{k(N-1)}}{(2\pi i)^k} e^{k\tau/2}$$

$$\times \oint \cdots \oint \prod_{1\le A<B\le k} \frac{w_A - w_B}{w_A - w_B + 1} \prod_{j=1}^{k} \left(\prod_{m=1}^{N} \frac{1}{w_j + a_m} \right) e^{-\tau w_j}\, dw_j$$

where the w_j contour contains $\{w_{j+1}-1, \dots, w_k - 1, -a_1, \dots, -a_N\}$ and no other singularities.

$$\left\langle e^{-u e^{-T_{N,N}}} \right\rangle_{WM(a_1,\dots,a_N;\tau)} = \det(1 + K_u)$$

where

$$K_u(v, v') = \frac{1}{2\pi i} \int_{-i\infty+\delta_2}^{i\infty+\delta_2} ds\, \Gamma(-s)\Gamma(1+s) \prod_{m=1}^{N} \frac{\Gamma(v - a_m)}{\Gamma(s + v - a_m)} \frac{u^s e^{v\tau s + \tau s^2/2}}{v + s - v'}.$$

and we omit the domain of the integral operator.