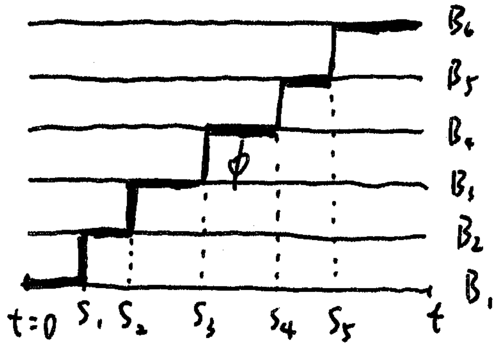
1. O'Connell-Yor semi-discrete directed polymer and Whittaker processes.



$N=6$ example. ϕ is a semi-discrete directed polymer. $0 < s_1 < s_2 < \cdots < s_{N-1} < t$.

B_i are Brownian motions with drift a_i.

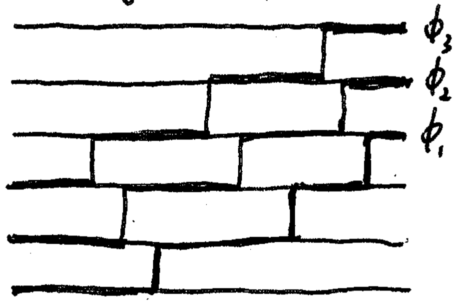
Energy of a path:

$$E(\phi) = B_1(s_1) + \big(B_2(s_2) - B_2(s_1)\big) + \cdots + \big(B_N(t) - B_N(s_{N-1})\big).$$

Partition function:

$$Z^N(t) = \int_{0<s_1<s_2<\cdots<s_{N-1}<t} e^{E(\phi)}\, d\phi. \quad \text{where } d\phi = ds_1\, ds_2 \cdots ds_{N-1}.$$

Hierarchy of partition functions:



$N=6,\ n=3$ Example. $\phi_1, \phi_2, \cdots, \phi_n$ are nonintersecting semi-discrete directed polymers.

$$Z^N_n(t) = \int_{\phi_1,\cdots,\phi_n \text{ nonintersecting}} e^{\sum_i^n E(\phi_i)}\, d\phi_1 \cdots d\phi_n.$$

Hierarchy of free energies.

$$F^N_n(t) = \log\left(\frac{Z^N_n(t)}{Z^N_{n-1}(t)}\right).$$

The array of stochastic processes $F_j^k(t): [0,\infty) \to \mathbb{R}$, $F_j^k(0)=0$

$$\begin{array}{ccccccc} F_1^N & & F_2^N & \cdots\cdots & & F_N^N \\ & F_1^{N-1} & & F_2^{N-1} & \cdots & F_{N-1}^{N-1} \\ & & \ddots & & ⋰ & \\ & & & F_1^2 \quad F_2^2 & & \\ & & & F_1^1 & & \end{array}$$

satisfy the recursive equations

$$dF_1^1 = dB_1$$

$$dF_1^k = dF_1^{k-1} + e^{F_2^k - F_1^{k-1}}dt$$

$$dF_2^k = dF_2^{k-1} + \left(e^{F_3^k - F_2^{k-1}} - e^{F_2^k - F_1^{k-1}}\right)dt$$

$$\cdots$$

$$dF_{k-1}^k = dF_{k-1}^{k-1} + \left(e^{F_k^k - F_{k-1}^{k-1}} - e^{F_{k-1}^k - F_{k-2}^{k-1}}\right)dt$$

$$dF_k^k = dB_k - e^{F_k^k - F_{k-1}^{k-1}}dt.$$

Comparing these equations with those satisfied by the Markov process of the evolution of the 2d-Whittaker growth model, we have that for all t, the distribution of $\{F_n^N\}_{1\le n\le N}$ is the same as that of $\{T_{N,n}\}$ in the Whittaker process $W(a_1,\cdots,a_N;t)$.

Remark: So F_1^N corresponds to $T_{N,1}$. We have got formulas for $T_{N,N}$. We note that there is a symmetry of the stochastic equations satisfied by $\{T_{N,n}\}$, which implies that $\{T_{N,N-n+1}\}_{1\le n\le N}$ satisfies the equations with a_i changed into $-a_i$. Therefore F_1^N also corresponds to $T_{N,N}$ in $W(-a;t)$.

2. Tracy-Widom distribution and the limit of free energies.

Goal of this talk: Find the asymptotics of F^N_1 as $N\to\infty$ in the special case that $a_1 = \cdots = a_N = 0$.

Theorem: Denote the digamma function $\Psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Define $\bar{f}_\kappa = \inf_{t>0}(\kappa t - \Psi(t))$, and let $\bar{t}_\kappa$ denote the unique value of t at which the minimum is achieved. Finally, define the positive number, $\bar{g}_\kappa = -\Psi''(\bar{t}_\kappa)$. Then for all large enough κ (a technical assumption), and $t = \kappa N$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{-F^N_1(t) - N\bar{f}_\kappa}{N^{1/3}} \le r\right) = \lim_{N\to\infty} \mathbb{P}_{W(0;t)}\left(\frac{-T_{N,N} - N\bar{f}_\kappa}{N^{1/3}} \le r\right)$$
$$= F_{GUE}\left((\bar{g}_\kappa/2)^{-\frac{1}{3}} r\right).$$

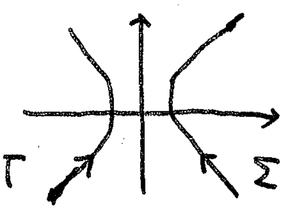
Here $F_{GUE}(x)$ is defined by the Fredholm determinant

$$F_{GUE}(x) = \det\left(1 - K_{Airy}(u,v)\right)_{L^2[x,\infty)}$$
$$= \det\left(1 - K_{Airy}(u+x, v+x)\right)_{L^2[0,\infty)}.$$

and

$$K_{Airy}(u,v) = \frac{-1}{2\pi i}\int_\Gamma dz \int_\Sigma dw \, \frac{e^{-\frac{z^3}{3} - uz}}{e^{-\frac{w^3}{3} - vw}} \frac{1}{z-w},$$

where



The proof of the theorem is based on an exact formula for the Whitteker process that we mentioned briefly in the end of last talk.

For all $u \in \mathbb{C} \setminus \mathbb{R}_-$,

$$\langle e^{-u e^{-T_{N,N}}} \rangle_{W(0;t)} = \det(I + K_u),$$

where $\det(I + K_u)$ is the Fredholm determinant of $K_u : L^2(\Gamma_N) \to L^2(\Gamma_N)$, for Γ_N a small enough counterclockwise contour enclosing 0, and

$$K_u(v, v') = \frac{i}{2} \int_{\Sigma_N} \frac{1}{\sin(\pi s)} \left(\frac{\Gamma(v)}{\Gamma(s+v)} \right)^N \frac{u^s e^{vts + ts^2/2}}{v + s - v'} \, ds. \qquad (*)$$

with Σ_N a contour from $-\infty i$ to ∞i, to the right of the imaginary axis.

Idea of the proof of the theorem:

Denote the function $f_N(x) = e^{-e^{N^{1/3} x}}$, and $f_N^r(x) = f_N(x-r)$ $= (e^{-e^{N^{1/3} x}}) e^{N^{1/3} r}$. Then

$$f_N^r\left(\frac{-T_{N,N} - N \bar{f}_K}{N^{1/3}}\right) = e^{-u e^{-T_{N,N}}}, \quad \text{where } u = u(N, r, K) = e^{-N \bar{f}_K - r N^{1/3}}.$$

A simple observation is that if we can prove

$$\lim_{N \to \infty} \left\langle f_N^r\left(\frac{-T_{N,N} - N \bar{f}_K}{N^{1/3}}\right) \right\rangle_{W(0;t)} = \lim_{N \to \infty} \langle e^{-u e^{-T_{N,N}}} \rangle_{W(0;t)} = F_{GUE}((\bar{g}_K/2)^{-1/3} r),$$

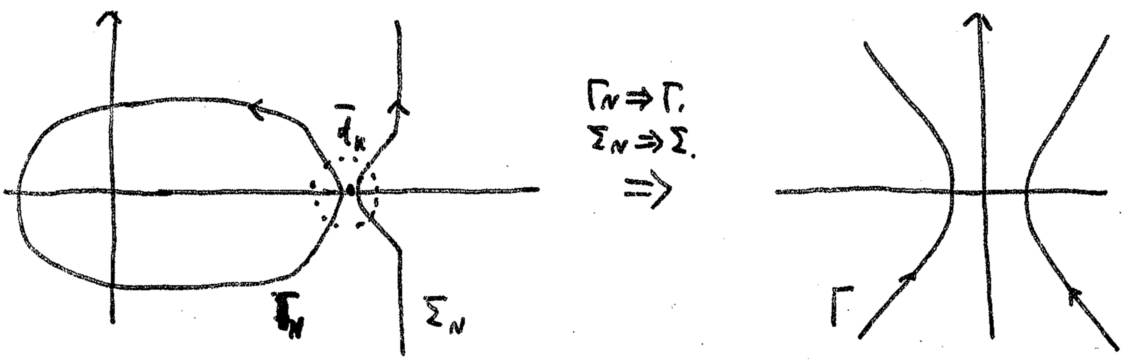
then

$$\lim_{N\to\infty} \mathbb{P}_{W(0;t)}\left(\frac{-T_{N,N} - N\bar{f}_K}{N^{1/3}} \le r\right) = F_{GUE}\left((\bar{g}_K/2)^{-1/3} r\right),$$

since $f_N^r(x)$ is a good approximation of the indicator function of $(-\infty, r)$:

$f_N(x) = f_N^0(x)$.

To prove the convergence of $\langle e^{-u e^{-T_{N,N}}} \rangle_{W(0,t)}$, we deform the contour Γ_N such that the integral operator $K_u(v, v')$ is concentrated at $\bar{t}_K$.



also we deform the contour Σ_N so that the integrand of $(*)$ is concentrated at $\bar{t}_K$.

Why $\bar{t}_K$ is so important: It is the saddle point of the integrand of $(*)$.

Now focus on the local parts of contours Γ_N and Σ_N around $\bar{t}_K$. after change of variables,

$$K_u(\tilde{v}, \tilde{v}') \approx \frac{1}{2\pi i}\int_\Sigma \frac{1}{\tilde{v} - \tilde{\xi}} \, \frac{e^{-\frac{\tilde{v}^3}{3} + (\bar{g}_K/2)^{-1/3} r \tilde{v}}}{e^{-\frac{\tilde{\xi}^3}{3} + (\bar{g}_K/2)^{-1/3} r \tilde{\xi}}} \, \frac{d\tilde{\xi}}{\tilde{\xi} - \tilde{v}'}.$$

Note that $\mathrm{Re}\,\tilde{\xi} > \mathrm{Re}\,\tilde{v}'$, and then

$$\frac{1}{\tilde{\xi}-\tilde{v}'} = \int_0^\infty e^{-x(\tilde{\xi}-\tilde{v}')}\,dx.$$

So for any $f(\tilde{v}) \in L^2(\Gamma)$,

$$\int_\Gamma K(\tilde{v},\tilde{v}')\, f(\tilde{v}')\, d\tilde{v}'$$

$$= \int_\Gamma \frac{1}{2\pi i}\int_\Sigma \frac{1}{\tilde{v}-\tilde{\xi}}\, \frac{e^{-\tilde{v}^3/3 + (\tilde{g}_\kappa/2)^{-1/3} r \tilde{v}}}{e^{-\tilde{\xi}^3/3 + (\tilde{g}_\kappa/2)^{-1/3} r \tilde{\xi}}} \int_0^\infty e^{x(\tilde{\xi}-\tilde{v}')}\,dx\,d\tilde{\xi}\, f(\tilde{v}')\, d\tilde{v}'$$

$$= \underbrace{\frac{1}{2\pi i}\int_\Sigma \frac{1}{\tilde{v}-\tilde{\xi}}\, \frac{e^{-\tilde{v}^3/3 + (\tilde{g}_\kappa/2)^{-1/3} r \tilde{v}}}{e^{-\tilde{\xi}^3/3 + (\tilde{g}_\kappa/2)^{-1/3} r \tilde{\xi}}}}_{C:\,L^2(\Sigma)\to L^2(\Gamma)} \underbrace{\int_0^\infty e^{-x\tilde{\xi}}}_{B:\,L^2(\mathbb{R}_+)\to L^2(\Sigma)} \underbrace{\int_\Gamma e^{x\tilde{v}'}}_{A:\,L^2(\Gamma)\to L^2(\mathbb{R}_+)} f(\tilde{v}')\, d\tilde{v}'\, dx\, d\tilde{\xi}$$

we decompose $K_u = CBA$.

Since $\det(I + CBA) = \det(I + ACB)$, and ACB is an operator on $L^2(\mathbb{R}_+)$, we show that $ACB(x,y) = -K_{Airy}\left(x + (\tilde{g}_\kappa/2)^{-1/3} r,\ y + (\tilde{g}_\kappa/2)^{-1/3} r\right)$ and finish the proof.

For any $f(y) \in L^2(\mathbb{R}_+)$,

$$\int_0^\infty ACB(x,y)\, f(y)\, dy$$

$$= \int_\Gamma e^{x\tilde{v}} \frac{1}{2\pi i}\int_\Sigma \frac{1}{\tilde{v}-\tilde{\xi}}\, \frac{e^{-\tilde{v}^3/3 + (\tilde{g}_\kappa/2)^{-1/3} r \tilde{v}}}{e^{-\tilde{\xi}^3/3 + (\tilde{g}_\kappa/2)^{-1/3} r \tilde{\xi}}} \int_0^\infty e^{-y\tilde{\xi}} f(y)\, dy\, d\tilde{\xi}\, d\tilde{v}$$

$$= \int_0^\infty \underbrace{\left(\frac{1}{2\pi i}\int_\Gamma d\tilde{v} \int_\Sigma d\tilde{\xi}\, \frac{1}{\tilde{v}-\tilde{\xi}}\, \frac{e^{-\tilde{v}^3/3 + ((\tilde{g}_\kappa/2)^{-1/3} r + x)\tilde{v}}}{e^{-\tilde{\xi}^3/3 + ((\tilde{g}_\kappa/2)^{-1/3} r + y)\tilde{\xi}}}\right)}_{ACB(x,y)\; =\; -K_{Airy}\left(x + (\tilde{g}_\kappa/2)^{-1/3} r,\ y + (\tilde{g}_\kappa/2)^{-1/3} r\right)} f(y)\, dy$$